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SEQUENTIAL RETRACTIVITIES AND REGULARITY ON INDUCTIVE LIMITS

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Abstract. In this paper we prove the following result: an inductive limit $(E, t) = \operatorname{ind}(E_n, t_n)$ is regular if and only if for each Mackey null sequence (x_k) in (E, t) there exists $n = n(x_k) \in \mathbb{N}$ such that (x_k) is contained and bounded in (E_n, t_n). From this we obtain a number of equivalent descriptions of regularity.

Keywords: inductive limits, regularity, sequential retractivities

MSC 2000: 46A13, 46M40

In this paper we keep the notations of [1].

Let $(E_n, t_n)_{n\in\mathbb{N}}$ be an inductive sequence of locally convex spaces (E_n, t_n), i.e. an increasing sequence of locally convex spaces with continuous inclusions $(E_n, t_n) \subset (E_{n+1}, t_{n+1})$. If the union $E = \bigcup\limits_{n=1}^{\infty} E_n$ is endowed with the strongest locally convex topology t such that all inclusions $i_n\colon (E_n, t_n) \to E$ are continuous, then (E, t) is called the locally convex inductive limit of the inductive sequence $(E_n, t_n)_{n\in\mathbb{N}}$ and it is denoted by $\operatorname{ind}(E_n, t_n)$. In general, $(E, t) = \operatorname{ind}(E_n, t_n)$ need not be Hausdorff even if every (E_n, t_n) is Hausdorff. Here we always assume that every (E_n, t_n) is Hausdorff and (E, t) is also Hausdorff. Recall that an inductive limit $(E, t) = \operatorname{ind}(E_n, t_n)$ is said to be

(a) sequentially retractive if, for each null sequence (x_k) in (E, t), there exists $n = n(x_k) \in \mathbb{N}$ such that (x_k) is a null sequence contained in (E_n, t_n);

(b) weakly sequentially retractive if, for each weak null sequence (x_k) in (E, t), there exists $n = n(x_k) \in \mathbb{N}$ such that (x_k) is a weak null sequence contained in (E_n, t_n);

(c) regular if, for each bounded set B in (E, t) there exists $n = n(B) \in \mathbb{N}$ such that B is contained and bounded in (E_n, t_n).

Sequentially retractive limits were introduced and studied thoroughly by Floret ([2]). He proved that sequential retractivity implies regularity. In [3], we showed that sequential retractivity implies weak sequential retractivity and the latter already implies regularity. In this paper we will consider various sequential retractivities and investigate the relation among them and regularity. We will find a very weak sequential retractivity, i.e. such that each Mackey null sequence in (E,t) is contained and bounded in some (E_n,t_n), which already implies regularity. From this we obtain a number of equivalent descriptions of regularity. Recall that a sequence (x_k) in (E,t) is said to be a Mackey null sequence if there exists an absolutely convex bounded set B such that the sequence (x_k) converges to 0 in E_B. Here E_B denotes the linear span of B endowed with the topology defined by the gauge of B (see [4], p. 151). First we give the following somewhat surprising result.

Theorem 1. *(E,t) is regular if and only if the following condition is satisfied: for each Mackey null sequence (x_k) in (E,t) there exists $n=n(x_k)\in\mathbb{N}$ such that (x_k) is contained and bounded in (E_n,t_n).*

Proof. Obviously each Mackey null sequence is bounded. Hence regularity implies that the condition is satisfied. Conversely, suppose that the condition is satisfied, we shall prove that (E,t) is regular. Let B be any bounded set in (E,t). Without loss of generality, we may assume that B is absolutely convex. First, we conclude that there exists $n\in\mathbb{N}$ such that B is contained in E_n. If it does not hold, then there exists $x_k\in B\setminus E_k$ for each $k\in\mathbb{N}$. Obviously $\frac{1}{k}x_k\xrightarrow{k}0$ in E_B and $(\frac{1}{k}x_k)_{k\in\mathbb{N}}$ is a Mackey null sequence in (E,t). By the hypothesis, there exists $n\in\mathbb{N}$ such that $(\frac{1}{k}x_k)_{k\in\mathbb{N}}\subset E_n$. This yields $x_k\in E_n$ for every k. In particular, $x_n\in E_n$. This contradicts the choice of x_k. For brevity, we assume that B is contained in E_1. We show below that B is bounded in some (E_n,t_n) by contradiction. Assume that B is not bounded in any (E_m,t_m) for every m. Then B is not bounded in $(E_m,\sigma(E_m,E'_m))$ for every m, since (E_m,t_m) and $(E_m,\sigma(E_m,E'_m))$ have the same bounded sets. Thus for each $m\in\mathbb{N}$ there exists $f_m\in E'_m$ such that $f_m(B):=\{f_m(x)\colon x\in B\}$ is an unbounded scalar set. Hence for each $m\in\mathbb{N}$ we may select a sequence $(x_k^{(m)})_{k\in\mathbb{N}}$ in B such that $|f_m(x_k^{(m)})|\geqslant(m+k)^2$ for $k=1,2,3,\ldots$. Repeating this process, we obtain a sequence of sequences as follows:

$$\begin{array}{c}
x_1^{(1)},x_2^{(1)},x_3^{(1)},\ldots\\
x_1^{(2)},x_2^{(2)},x_3^{(2)},\ldots\\
x_1^{(3)},x_2^{(3)},x_3^{(3)},\ldots\\
\ldots\ldots\ldots\ldots\ldots
\end{array}$$

By the diagonal process, we construct a sequence $(z_j)_{j\in\mathbb{N}}$ in B as follows: $z_1=x_1^{(1)}$, $z_2=x_2^{(1)}$, $z_3=x_1^{(2)}$, $z_4=x_3^{(1)}$, $z_5=x_2^{(2)}$, $z_6=x_1^{(3)},\ldots$. For each $j\in\mathbb{N}$, put

$\lambda_j = m + k$ when $z_j = x_k^{(m)}$. Obviously $j \geqslant \lambda_j = m + k$ for any $j \geqslant 3$. On the other hand, we can prove that $\frac{1}{2}\lambda_j(\lambda_j - 1) \geqslant j$ for any $j \in \mathbb{N}$ (see Appendix). Thus $j \to \infty$ if and only if $\lambda_j \to \infty$. Hence $\frac{1}{\lambda_j} z_j \xrightarrow{j} 0$ in E_B, or $(\frac{1}{\lambda_j} z_j)_{j\in\mathbb{N}}$ is a Mackey null sequence in (E, t). For any fixed m, $|f_m(\frac{1}{m+k} x_k^{(}m))| \geqslant m + k$ for $k = 1, 2, 3, \ldots$, hence f_m is unbounded on the set $\{\frac{1}{m+k} x_k^{(m)}\colon\ k \in \mathbb{N}\}$. Certainly f_m is unbounded on $\{\frac{1}{\lambda_j} z_j\colon\ j \in \mathbb{N}\}$. Thus $\{\frac{1}{\lambda_j} z_j\colon\ j \in \mathbb{N}\}$ is not bounded in (E_m, t_m) for every $m \in \mathbb{N}$. However, by the hypothesis, the Mackey null sequence $(\frac{1}{\lambda_j} z_j)_{j\in\mathbb{N}}$ in (E, t) must be contained and bounded in some (E_n, t_n), a contradiction. □

From Theorem 1, we immediately have the following implications.

Corollary 2. *Consider the following conditions:*

(1) *for each null sequence (x_k) in (E, t), there exists $n = n(x_k) \in \mathbb{N}$ such that (x_k) is a null sequence contained in (E_n, t_n), i.e. (E, t) is sequentially retractive;*

(2) *for each null sequence (x_k) in $(E, \sigma(E, E'))$, there exists $n = n(x_k) \in \mathbb{N}$ such that (x_k) is a null sequence contained in $(E_n, \sigma(E_n, E_n'))$, i.e. (E, t) is weakly sequentially retractive (see [3]);*

(3) *for each null sequence (x_k) in (E, t), there exists $n = n(x_k) \in \mathbb{N}$ such that (x_k) is a null sequence contained in $(E_n, \sigma(E_n, E_n'))$, see [1], p. 108. Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ regularity.*

P r o o f. (1) $\Rightarrow$ (2): See [3], Remark 1.

(2) $\Rightarrow$ (3): It is obvious.

(3) $\Rightarrow$ regularity: Obviously, condition (3) implies the condition in Theorem 1, and the latter is equivalent to regularity by Theorem 1. Thus condition (3) implies regularity. □

From Theorem 1 we can also obtain the following equivalent descriptions for regularity.

Theorem 3. *The following conditions are all equivalent and each of them is equivalent to regularity:*

(1) *for each null sequence (x_k) in $(E, \sigma(E, E'))$ there exists $n = n(x_k) \in \mathbb{N}$ such that (x_k) is contained and bounded in (E_n, t_n);*

(2) *for each null sequence (x_k) in (E, t) there exists $n = n(x_k) \in \mathbb{N}$ such that (x_k) is contained and bounded in (E_n, t_n);*

(3) *for each Mackey null sequence (x_k) in (E, t) there exists $n = n(x_k) \in \mathbb{N}$ such that (x_k) is contained and bounded in (E_n, t_n);*

(4) *for each Mackey null sequence (x_k) in (E, t) there exists $n = n(x_k) \in \mathbb{N}$ such that (x_k) is a null sequence contained in $(E_n, \sigma(E_n, E_n'))$;*

(5) *for each Mackey null sequence (x_k) in (E, t) there exists $n = n(x_k) \in \mathbb{N}$ such that (x_k) is a null sequence contained in (E_n, t_n);*

(6) *for each Mackey null sequence* (x_k) *in* (E,t) *there exists* $n=n(x_k)\in\mathbb{N}$ *such that* (x_k) *is a Mackey null sequence contained in* (E_n,t_n).

P r o o f. Obviously, (1) $\Rightarrow$ (2) $\Rightarrow$ (3). From Theorem 1, (3) $\Rightarrow$ regularity. Since any null sequence (x_k) in $(E,\sigma(E,E'))$ is bounded in $(E,\sigma(E,E'))$ and hence bounded in (E,t), it is clear that regularity $\Rightarrow$ (1). Now we have (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ regularity.

Obviously, (6) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (3). From Theorem 1, (3) $\Rightarrow$ regularity. We show below the implication regularity $\Rightarrow$ (6). For any Mackey null sequence (x_k) in (E,t) there exists an absolutely convex bounded set B in (E,t) such that $x_k\overset{k}{\to}0$ in E_B. Since (E,t) is regular, there exists $n\in\mathbb{N}$ such that B is contained and bounded in (E_n,t_n). Thus $(x_k)\subset E_B\subset E_n$ is a Mackey null sequence contained in (E_n,t_n). Now we have proved the implications (6) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ regularity $\Rightarrow$ (6). Therefore, all the conditions are equivalent and each of them is equivalent to regularity. $\square$

If we replace null sequences in the conditions in Theorem 3 by compact (or sequentially compact, or countably compact) sets, then we obtain the following description for regularity, whose proof is omitted.

Corollary 4. *Each of the following conditions is equivalent to regularity:*

(1) *for each Mackey compact set* K *in* (E,t) *there exists* $n=n(K)\in\mathbb{N}$ *such that* K *is contained and bounded in* (E_n,t_n);

(2) *for each (weakly) compact set* K *in* (E,t) *there exists* $n=n(K)\in\mathbb{N}$ *such that* K *is contained and bounded in* (E_n,t_n);

(3) *for each (weakly) sequentially compact set* K *in* (E,t) *there exists* $n=n(K)$ *such that* K *is contained and bounded in* (E_n,t_n);

(4) *for each (weakly) countably compact set* K *in* (E,t) *there exists* $n=n(K)\in\mathbb{N}$ *such that* K *is contained and bounded in* (E_n,t_n).

Appendix. Now we prove the inequality $\frac{1}{2}\lambda_j(\lambda_j-1)\geqslant j$ which appeared in the proof of Theorem 1. For each $n\in\mathbb{N}$, put $N_n=\{j\in\mathbb{N}\colon\ \lambda_j-1=n\}$, then $N=N_1\cup N_2\cup N_3\cup\ldots$. It is easy to see that there exist n numbers in N_n for each $n\in\mathbb{N}$. Let max N_n denote the greatest number in N_n. Obviously $N_1=\{1\}$, $N_2=\{2,3\}$, $N_3=\{4,5,6\},\ldots$. In general, $N_n=\{(\max N_{n-1})+1,\ (\max N_{n-1})+2,\ldots,(\max N_{n-1})+n\}$. By induction, we can prove that $\max N_n=1+2+3+\ldots+n=\frac{n(n+1)}{2}$. For any $j\in\mathbb{N}$ there exists a unique $n\in\mathbb{N}$ such that $j\in N_n$. Thus $j\leqslant\max N_n=\frac{n(n+1)}{2}=\frac{1}{2}\lambda_j(\lambda_j-1)$ and the proof is complete.

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