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COMMUTATIVITY OF RINGS THROUGH A STREB'S RESULT

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Abstract. In this paper we investigate commutativity of rings with unity satisfying any one of the properties:

$$\{1-g(yx^m)\}\ [yx^m - x^r f(yx^m)\ x^s, x]\{1-h(yx^m)\} = 0,$$
$$\{1-g(yx^m)\}\ [x^m y - x^r f(yx^m) x^s, x]\{1-h(yx^m)\} = 0,$$
$$y^t[x, y^n] = g(x)[f(x), y]h(x) \text{ and } [x, y^n]\ y^t = g(x)[f(x), y]h(x)$$

for some $f(X)$ in $X^2\mathbb{Z}[X]$ and $g(X)$, $h(X)$ in $\mathbb{Z}[X]$, where $m \geqslant 0$, $r \geqslant 0$, $s \geqslant 0$, $n > 0$, $t > 0$ are non-negative integers. We also extend these results to the case when integral exponents in the underlying conditions are no longer fixed, rather they depend on the pair of ring elements x and y for their values. Further, under different appropriate constraints on commutators, commutativity of rings has been studied. These results generalize a number of commutativity theorems established recently.

Keywords: commutators, division rings, factorsubrings, polynomial identities, torsion-free rings

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1. Introduction

Throughout the paper, R will represent an associative ring (maybe without unity), $N = N(R)$ the set of nilpotent elements of R, $Z = Z(R)$ the center of R, $C = C(R)$ the commutator ideal of R, and $U = U(R)$ the group of units of R. For any x, y in R, $[x, y]$ denotes the commutator $xy - yx$. As usual, $\mathbb{Z}[X]$ is the set of polynomials in X with coefficients in $\mathbb{Z}$, the ring of integers. Consider the following ring properties:

(I) For all x, y in R there exist polynomials $f(X)$ in $X^2\mathbb{Z}[X]$ and $g(X)$, $h(X)$ in $X\mathbb{Z}[X]$ such that

$$\{1-g(yx^m)\}\ [yx^m - x^r f(yx^m)x^s, x]\{1-h(yx^m)\} = 0$$

where $m \geqslant 0$, $r \geqslant 0$, $s \geqslant 0$ are fixed integers.

(I)′ For all x, y in R there exist integers $m \geqslant 0$, $r \geqslant 0$, $s \geqslant 0$ and polynomials $f(X)$ in $X^2\mathbb{Z}[X]$ and $g(X)$, $h(X)$ in $X\mathbb{Z}[X]$ such that

$$\{1-g(yx^m)\}\ [yx^m - x^r f(yx^m)x^s, x]\{1-h(yx^m)\} = 0.$$

(II) For all x, y in R there exist polynomials $f(X)$ in $X^2\mathbb{Z}[X]$ and $g(X), h(X)$ in $X\mathbb{Z}[X]$ such that

$$\{1-g(yx^m)\}\ [x^m y - x^r f(yx^m)\ x^s, x]\{1-h(yx^m)\} = 0$$

where $m \geqslant 0$, $r \geqslant 0$, $s \geqslant 0$ are fixed integers.

(II)′ For all x, y in R there exist integers $m \geqslant 0$, $r \geqslant 0$, $s \geqslant 0$ and polynomials $f(X)$ in $X^2\mathbb{Z}[X]$ and $g(X)$, $h(X)$ in $X\mathbb{Z}[X]$ such that

$$\{1-g(yx^m)\}\ [x^m y - x^r f(yx^m)x^s, x]\ \{1-h(yx^m)\} = 0.$$

(III) For every x in R there exist polynomials $f(X)$ in $X^2\mathbb{Z}[X]$ and $g(X), h(X) \in \mathbb{Z}[X]$ such that

$$y^t[x, y^m] = g(x)[f(x), y]h(x) \text{ and } y^t[x, y^n] = g(x)[f(x), y]\ h(x)$$

for all $y \in R$, where $t \geqslant 1$, $m \geqslant 1$, $n \geqslant 1$ are fixed integers with $(m, n) = 1$.

(III)′ For every $x, y \in R$ there exist integers $t \geqslant 1$, $m \geqslant 1$, $n \geqslant 1$ with $(m, n) = 1$ and polynomials $f(X)$ in $X^2\mathbb{Z}[X]$ and $g(X), h(X) \in \mathbb{Z}[X]$ such that

$$y^t[x, y^m] = g(x)[f(x), y]h(x) \text{ and } y^t[x, y^n] = g(x)[f(x), y]h(x).$$

(IV) For every x in R there exist polynomials $f(X)$ in $X^2\mathbb{Z}[X]$ and $g(X), h(X) \in \mathbb{Z}[X]$ such that

$$[x, y^m]y^t = g(x)[f(x), y]\ h(x) \text{ and } [x, y^n]y^t = g(x)[f(x), y]h(x)$$

for all $y \in R$, where $t \geqslant 1$, $m \geqslant 1$, $n \geqslant 1$ are fixed integers with $(m, n) = 1$.

(IV)$'$ For all x, y in R there exist integers $t \geqslant 1$, $m \geqslant 1$, $n \geqslant 1$ with $(m, n) = 1$ and polynomials $f(X) \in X^2\mathbb{Z}[X]$ and $g(X), h(X) \in \mathbb{Z}[X]$ such that

$$[x, y^m]y^t = g(x)[f(x), y]h(x) \text{ and } [x, y^n]y^t = g(x)[f(x), y]h(x).$$

(V) For every x, y in R there exist $f(T)$, $g(T)$ in $T^2\mathbb{Z}[T]$ such that $[x - f(x), y - f(y)] = 0$.

Searčoid and MacHale [8] proved the commutativity of any ring satisfying the condition $(xy)^{n(x,y)} = xy$ with $n(x, y) > 1$. Tominaga and Yaqub [10, Theorem 2] established that if R is a ring such that either $xy = p(xy)$ or $xy = p(yx)$, where $p(X)$ belongs to $X^2\mathbb{Z}[X]$, then R is commutative. The author jointly with Bell and Quadri [1, Theorem 2] obtained the commutativity of the rings with unity 1 satisfying polynomial identitites of the form $[xy - p(xy), x] = 0$ and $[xy - q(xy), x] = 0$, where $p(X), q(X)$ are in $X^2\mathbb{Z}[X]$. Our first aim is to investigate commutativity of rings with unity 1 satisfying either one of the properties (I) or (II). Further, we shall consider the properties (I)$'$ and (II)$'$, where the integral exponents are allowed to vary with the pair of rings elements x, y and the ring also satisfies Chacron's condition (V). Our second goal is to establish commutativity of rings with unity 1 satisfying any one of the properties (III), (IV), (III)$'$ and (IV)$'$. There are several results in the existing literature concerning commutativity of rings with unity 1 satisfying certain special cases of these conditions (cf. [7, Theorems 1, 2]).

In the present note we will confine our attention mainly to the case when polynomials in the underlying conditions are varying with the pair of ring's elements x, y which offer simultaneous extensions of these results to rings with unity 1. Lately, some related cases of conditions (III) and (IV) have been considered and commutativity of rings has been investigated under appropriate torsion restrictions on commutators. The idea of the proofs presented here is based on some iteration techniques developed by Tong [11].

2. Preliminary results

In order to be able to prove our theorem, let us first consider the following types of rings:

(i)$_l$ $\begin{pmatrix} GF(p) & GF(p) \\ 0 & 0 \end{pmatrix}$, p a prime, where $\begin{pmatrix} GF(p) & GF(p) \\ 0 & 0 \end{pmatrix}^3 := \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} | a, b \in GF(p)\}$;

(i)$_r$ $\begin{pmatrix} 0 & GF(p) \\ 0 & GF(p) \end{pmatrix}$, p a prime;

(i) $\begin{pmatrix} GF(p) & GF(p) \\ 0 & GF(p) \end{pmatrix}$, p a prime;

(ii) $M_\sigma(F) = \left\{ \begin{pmatrix} a & b \\ 0 & \sigma(a) \end{pmatrix} \mid a, b \in F \right\}$, where F is a finite field with a non-trivial automorphism σ;

(iii) a non-commutative division ring;

(iv) $S = \langle 1 \rangle + T$, T is a non-commutative radical subring of S;

(v) $S = \langle 1 \rangle + T$, T is a non-commutative subring of S such that $T[T,T] = [T,T]T = 0$.

In 1989, Streb [9] classified non-commutative rings, which has been used effectively as a tool by several authors to prove a number of commutativity theorems (cf. [5], [6] and [10]). From the proof of [9, Corollary 1], it is trivial to see that if R is a non-commutative ring with unity 1, then there exists a factorsubring of R which is of type (i), (ii), (iii), (iv) or (v). This gives us the following result of [9] that plays a vital role in our subsequent discussion.

Meta Theorem. *Let P be a ring property which is inherited by factorsubrings. Suppose no rings of type* (i), (ii), (iii), (iv) *or* (v) *satisfy P. Then every ring with unity* 1 *satisfying P is commutative.*

The proofs of the following lemmas can be found in [4], [3], [6, Corollary 1] and [11, Lemma 1].

Lemma 1. *Let f be a polynomial in n non-commuting indeterminates $x_1, x_2, \ldots, x_n$ with relatively prime integral coefficients. Then the following assertions are equivalent.*

(a) *For any ring R satisfying the polynomial identity $f = 0$, C is a nil ideal.*

(b) *For every prime p, $(GF(p))_2$ fails to satisfy $f = 0$.*

(c) *Every semiprime ring satisfying $f = 0$ is commutative.*

Lemma 2. *If R is a non-commutative ring satisfying* (V), *then there exists a factorsubring of R which is of type* (i) *or* (ii).

Lemma 3. *Let R be a ring in which for all x, y in R, there exists a polynomial $f(X)$ in $X^2\mathbb{Z}[X]$ such that $[x - f(x), y] = 0$. Then R is commutative.*

Lemma 4. *Let R be a ring with unity* 1, *and put $I_0^k(x) = x^k$. If $p \geqslant 1$, let $I_p^k(x) = I_{p-1}^k(x+1) - I_{p-1}^k(x)$ for all x in R. Then $I_{k-1}^k(x) = \frac{1}{2}(k-1)k! + k!x$; $I_k^k = k!$* and *$I_j^k(x) = 0$ for $j > k$.*

3. Main results

Theorem 1. *Let R be a ring with unity* 1 *satisfying any one of the conditions* (I) *and* (II). *Then R is commutative.*

Theorem 2. *Let R be a ring with unity* 1 *satisfying any one of the conditions* (III) *and* (IV). *Then R is commutative.*

We prove the assertion by a step-by-step reduction from division rings to the rings considered above.

Step 1. Let R be a division ring satisfying any one of the properties (I) and (II). Then R is commutative.

Proof. Let R satisfy (I). If u is a unit in R, then for every y in R choose polynomials $f(X)$ in $X^2\mathbb{Z}[X]$ and $g(X), h(X)$ in $X\mathbb{Z}[X]$ such that

$$\{1- g(yu^{-m}u^m)\}[yu^{-m}u^m - u^r f(yu^{-m}u^m)u^s, u]\{1 - h(yu^{-m}u^m)\} = 0$$

or

$$\{1 - g(y)\}[y - u^r f(y)u^s, u]\{1 - h(y)\} = 0.$$

This shows that either $1 - g(y) = 0$, or $1 - h(y) = 0$, or $[y - u^r f(y)u^s, u] = 0$. In the first two cases we get $y - yg(y) = 0, y - yh(y) = 0$, and R is commutative by Lemma 3. So, we may assume that

$$(1) \qquad [y - u^r f(y)u^s, u] = 0, \text{ where } f(X) \in X^2\mathbb{Z}[X]$$

for a unit u in U and arbitrary y in R. Next, choose a polynomial $f(X)$ in $X^2\mathbb{Z}[X]$ such that $[y - u^{-r} f(y)u^{-s}, u^{-1}] = 0$. This implies that $[y - u^{-r} f(y)u^{-s}, u] = 0$, and

$$(2) \qquad [u, f(y)] = u^r[u, y]u^s.$$

Now in view of (1), one can choose a polynomial $p(X) \in X^2\mathbb{Z}[X]$ such that $[f(y) - u^r p(f(y))u^s, u] = 0$, hence for $q(X) = p(f(X)) \in X^2\mathbb{Z}[X]$, we find that

$$(3) \qquad [u, f(y)] = u^r\ [u, q(y)]u^s.$$

From (2) and (3) we obtain $u^r[u, y]u^s = u^r[u, q(y)]u^s$. However, $u \in U$, thus $[y - q(y), u] = 0$ for $q(X) \in X^2\mathbb{Z}[X]$. So, again by Lemma 3 R is commutative.

Suppose that R satisfies (II). Then let u be a unit in R, i.e. $u \in U$, and for an arbitrary element y in R we obtain polynomials $f(X)$ in $X^2\mathbb{Z}[X]$ and $g(X), h(X) \in X\mathbb{Z}[X]$ such that

$$\{1- g(yu^{-m}u^m)\}\ [u^m yu^{-m} - u^r f(yu^{-m}u^m)u^s, u]\{1 - h(yu^{-m}u^m)\} = 0$$

or

$$\{1-g(y)\}[u^m y u^{-m} - u^r f(y) u^s, u]\{1-h(y)\} = 0.$$

This shows that either $1-g(y)=0$, or $1-h(y)=0$, or $[u^m y u^{-m} - u^r f(y) u^s, u] = 0$. In the first two cases R is commutative by Lemma 3. Next, we will assume the remaining possiblity that $[u^m y u^{-m} - u^r f(y) u^s, u] = 0$. Then

$$u^m[u,y] = u^r[u, f(y)]u^{m+s} \tag{4}$$

where $f(X) \in X^2\mathbb{Z}[X]$. Further, one can choose a polynomial $f(X) \in X^2\mathbb{Z}[X]$ such that $u^{-m}[u^{-1}, y] = u^{-r}[u^{-1}, f(y)]u^{-(m+s)}$, which gives

$$u^r[u,y]u^{m+s} = u^m[u, f(y)]. \tag{5}$$

By (4), one can get a polynomial $p(X) \in X^2\mathbb{Z}[X]$ such that $u^m[u, f(y)] = u^r[u, p(f(y))]\ u^{m+s}$. Hence for $q(X) = p(f(X))$ in $X^2\mathbb{Z}[X]$, (5) gives $u^r[u,y]u^{m+s} = u^r[u, q(y)]u^{m+s}$. But, since u in U, we get $[u, y - q(y)] = 0$. So Lemma 3 yields the required result. □

Step 2. Suppose that $k \geqslant 1$, $t \geqslant 1$ are fixed integers and R is a ring with unity 1 in which for every x in R there exist polynomials $f(X)$ in $X^2\mathbb{Z}[X]$ and $g(X), h(X)$ in $\mathbb{Z}[X]$ such that either

$$y^t[x, y^k] = g(x)\ [f(x), y]\ h(x) \text{ or } [x, y^k]y^t = g(x)[f(x), y]h(x)$$

for all y in R. Then $C \subseteq N$.

Proof. Let R satisfy $y^t[x, y^k] = g(x)\ [f(x), y]h(x)$. Set $1+y$ for y in the given condition, to obtain $(1+y)^t[x, (1+y)^k] = y^t[x, y^k]$.

Now $e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $y = e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ fail to satisfy the above polynomial identity in $(GF(p))_2$, p a prime. Thus by Lemma 1, R has its commutator ideal nil, i.e. $C \subseteq N$.

A similar argument can be used to obtain the result if R satisfies the condition $[x, y^k]y^t = g(x)[f(x), y]h(x)$. □

We are now well equipped to prove our theorems.

Proof of Theorem 1. Suppose that R is a ring of the type (i). Let R satisfy (I). Then in $(GF(p))_2$, p a prime, we get

$$\{1-\ g(e_{12}e_{22}^m)\}\ [e_{12}e_{22}^m - e_{22}^r f(e_{12}e_{22}^m)e_{22}^s, e_{22}]\{1-h(e_{12}e_{22}^m)\} = e_{12} \neq 0$$

for all $f(X) \in X^2\mathbb{Z}[X]$ and $g(X),\ h(X) \in X\mathbb{Z}[X]$. If R satisfies (II), then by taking $x = e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $y = e_{12}$ one gets

$$\{1 - \ g(e_{12}e_{11}^m)\}\ [e_{11}^m e_{12} - e_{11}^r f(e_{12}e_{11}^m)e_{11}^s, e_{11}]\ \{1 - h(e_{12}e_{11}^m)\} = e_{12} \neq 0$$

for all $f(X) \in X^2\mathbb{Z}[X]$ and $g(X), h(X) \in X\mathbb{Z}[X]$.

Hence, in both the cases we get a contradiction and therefore, no rings of type (i) satisfy (I) and (II).

Further, consider the ring $M_\sigma(F)$. Let R satisfy (I). Then take $x = \begin{pmatrix} \alpha & 0 \\ 0 & \sigma(\alpha) \end{pmatrix}$, $(\alpha \neq \sigma(\alpha))$, and $y = e_{12}$ such that

$$\{1 - g(yx^m)\}\ [yx^m - x^r f(yx^m)x^s, x]\{1 - h(yx^m)\} = (\alpha - \sigma(\alpha))\sigma^m(\alpha)e_{12} \neq 0$$

for all $f(X) \in X^2\mathbb{Z}[X]$ and $g(X),\ h(X)$ in $X\mathbb{Z}[X]$.

Next, if R satisfies (II), then with the same choice of x and y we get

$$\{1 - g(yx^m)\}\ [x^m y - x^r f(yx^m)x^s, x]\{1 - h(yx^m)\} = \alpha^m(\alpha - \sigma(\alpha))e_{12} \neq 0.$$

Thus in neither case R cannot be of type (ii). Also if R is of type (iii), then by Step 1 we get a contradiction.

Let R be of type (iv). If R satisfies either of the properties (I) or (II), then a careful scrutiny of the proof of Step 1 gives that there exist u in U and y in R such that either $y - yg(y) = 0$, or $y - yh(y) = 0$ or $[u, y - q(y)] = 0$ for all $q(X) \in X^2\mathbb{Z}[X]$ and $g(X), h(X) \in X\mathbb{Z}[X]$. Let $t_1, t_2 \in T$. Then $u = 1 + t_1$ is a unit and there exist $q(X) \in X^2\mathbb{Z}[X]$ and $g(X), h(X)$ in $X\mathbb{Z}[X]$ such that either $t_2 - t_2 g(t_2) = 0$, or $t_2 - t_2 h(t_2) = 0$ or $[t_2 - q(t_2), 1 + t_1] = 0$. Thus, in every case T is commutative by Lemma 3, a contradiction.

Further, let R be of type (v). Let $t_1, t_2 \in T$ be such that $[t_1, t_2] \neq 0$. Suppose that R satisfies (I). Then there exist polynomials $f(X) \in X^2\mathbb{Z}[X]$ and $g(X), h(X) \in X\mathbb{Z}[X]$ such that

$$\begin{aligned}
\{1 - g(t_2(1+t_1)^m)\}[t_2(1+t_1)^m - (1+t_1)^r f(t_2(1+t_1)^m)(1+t_1)^s, 1+t_1]& \\
\times\{1 - h(t_2(1+t_1)^m)\} &= 0 \\
\{1 - g(t_2(1+t_1)^m)\}[t_2(1+t_1)^m, 1+t_1]\{1 - h(t_2(1+t_1)^m)\} &= 0 \\
\{1 - g(t_2(1+t_1)^m)\}[t_2, t_1]\{1 - h(t_2(1+t_1)^m)\} &= 0 \\
[t_2, t_1] &= 0,
\end{aligned}$$

a contradiction.

In the same way we get a contradiction if R satisfies (II).

Hence we observe that no rings of type (i), (ii), (iii), (iv) or (v) satisfy (I) and (II), and by Meta Theorem, R is commutative. □

P r o o f of Theorem 2. By virtue of Lemma 1 and Step 2, R cannot be of type (iii) or (iv). Next, if R is assumed to be of type (i), then choosing $x = e_{12}$ and $y = e_{11}$ in $(GF(p))_2, p$ a prime, we get

$$e_{11}^t[e_{12}, e_{11}^n] = g(e_{12})[f(e_{12}), e_{11}]\ h(e_{12}) = e_{12} \neq 0$$

for all $f(X) \in X^2\mathbb{Z}[X]$ and $g(X), h(X) \in \mathbb{Z}[X]$. Hence in both cases we get a contradiction.

Now, consider the ring $M_\sigma(F)$, a ring of type (ii). If R satisfies (III), then note that $N(M_\sigma(F)) = Fe_{12}$. Hence for any $a \in N(M_\sigma(F))$ and an arbitrary unit $u \in U$ there exist polynomials $f(X) \in X^2\mathbb{Z}[X]$ and $g(X), h(X) \in \mathbb{Z}[X]$ such that $u^t[a, u^n] = g(a)[f(a), u]h(a) = 0$. Since $a^2 = 0$ and u is a unit, thus $[a, u^n] = 0$. Similarly we can obtain that $[a, u^m] = 0$. But $(m, n) = 1$, so we get $[a, u] = 0$. Thus for a non-central element $a = e_{12}$ and an arbitrary unit u one gets $[e_{12}, u] = 0$ which leads to a contradiction with the fact that e_{12} is central.

By a similar argument we obtain a contradiction if R satisfies (IV).

Finally, let R be a ring of type (v). Suppose R satisfies (III). Let $t_1, t_2 \in T$ be such that $[t_1, t_2] \neq 0$. Then there exist polynomials $f(X) \in X^2\mathbb{Z}[X]$ and $g(X), h(X) \in \mathbb{Z}[X]$ such that

$$n[t_2, t_1] = (1 + t_1)^t\ [t_2, (1 + t_1)^n] = g(t_2)\ [f(t_2), 1 + t_1]h(t_2) = 0.$$

One can similarly prove that $m[t_2, t_1] = 0$. This shows that $[t_2, t_1] = 0$ yields a contradiction.

Analogously, we can obtain a contradiction if R satisfies (IV).

Hence no rings of type (i), (ii), (iii), (iv) or (v) satisfy (III) and (IV) and R is commutative by Meta Theorem. $\square$

From the proofs of Theorem 1 and Theorem 2 we conclude that if R satisfies any one of the conditions (I)$'$, (II)$'$, (III)$'$ and (IV)$'$, then R has no factorsubrings of type (i) or (ii). Thus combining this fact together with Lemma 2, we obtain

Theorem 3. *Suppose that R is a ring with unity 1 satisfying (V). If R satisfies any one of the properties (I)$'$ and (II)$'$, then R is commutative (and conversely).*

Theorem 4. *Let R be a ring with unity 1 satisfying (V). Suppose further that R satisfies any one of the conditions (III)$'$ and (IV)$'$. Then R is commutative (and conversely).*

The following example demonstrates that in the hypothesis of Theorem 2, the conditions in the properties (III) and (IV) are not superfluous (even if the ring R has unity 1).

Example 1. Consider $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} | a, b, c, d \in GF(2) \right\}$. Then R is a non-commutative ring with unity satisfying the condition $y^t[x, y^m] = x^r[x^n, y]x^s$ for the fixed polynomial $f(x) = x^4$ and $m = 4$ where r, s, t, n maybe any non-negative integers.

4. Commutativity of torsion-free rings

In view of Example 1, it is natural to look for additional conditions sufficient for the commutativity of the ring R if we simply assume

$$y^t[x, y^m] = g(x)[f(x), y]h(x) \text{ and } [x, y^m]\ y^t = g(x)[f(x), y]h(x)$$

in the properties (III) and (IV), respectively. Finally, it is tempting to conjecture that an m-torsion free ring with unity 1 satisfying any one of the above properties must be commutative (under certain appropriate constraints on the commutators involved in the underlying conditions). We can prove some results in the interesting cases of the conjecture. In fact, we shall consider the following ring properties:

(VI) For every y in R there exist polynomials $f(X)$, $g(X)$, $h(X)$ in $\mathbb{Z}[X]$ such that $[x^n, y^r] = g(x)[f(x), y]h(x)$ for all x in R where $r > 1$, $n \geqslant 1$ are fixed integers.

(VII) For every x in R there exist polynomials $f(X)$, $g(X)$, $h(X)$ in $\mathbb{Z}[X]$ such that either

$$y^r[x^n, y] = g(x)[f(x), y]h(x) \text{ or } [x^n, y]y^r = g(x)[f(x), y]h(x)$$

for all x in R where $r \geqslant 1$, $n \geqslant 1$ are fixed integers.

(VIII) For all y in R there exist polynomials $f(X)$, $g(X)$, $h(X)$ in $\mathbb{Z}[X]$ such that either

$$x^n[x, y^r] = g(y)[x, f(y)]h(y) \text{ or } [x, y^r]x^n = g(y)[x, f(y)]h(y)$$

for all $x \in R$, where $r \geqslant 1$, $n \geqslant 1$ are fixed integers.

To prove the commutativity of a ring R with the above properties we need some extra conditions on commutators in R, such as the condition

$$Q(m)\ m[x, y] = 0 \text{ implies } [x, y] = 0 \text{ for all } x, y \text{ in } R \text{ (m is a positive integer).}$$

Our method of the proof uses some iteration techniques, which is based on the Lemma 4 due to Tong [11].

Theorem 5. *Let R be a ring with unity 1 satisfying any one of the properties* (VI), (VII), (VIII). *If R satisfies also $Q((\max\{r,n\})!)$, then R is commutative.*

Proof. Let R satisfy (VI). Then we shall first use induction on y^r. From Lemma 4 we have $I_k(x) = I_k^r(x)$, for $k \geqslant 0$. Then condition (VI) can be written as

$$(1) \qquad [x^n, I_0(y)] = g(x)[f(x), y]h(x).$$

Replacing y by $y+1$ in (1) and using Lemma 4, we get

$$[x^n, I_0(y) + I_1(y)] = g(x)[f(x), y]h(x).$$

Again using (1) we get

$$(2) \qquad [x^n, I_1(y)] = 0 \text{ for all } x, y \text{ in } R.$$

Putting $y+1$ instead of y and using Lemma 4, we get $[x^n, I_1(y+1)] = [x^n, I_1(y) + I_2(y)] = 0$. Again by (2) we get $[x^n, I_2(y)] = 0$. Hence one can observe that replacing y by $y+1$ and iterating $(r-1)$-times, we get $[x^n, I_{r-1}(y)] = 0$, i.e. $r![x^n, y] = 0$. At last, replacing x by $x+1$ and using similar technique as above we obtain $r!n![x,y] = 0$. The property Q $((\max\{r,n\})!)$ yields the commutativity of R.

Let R satisfy (VII). Then using the same techniques we get that either

$$I_0(y)[x^n, y] = g(x)[f(x), y]h(x)$$

or

$$[x^n, y]I_0(y) = g(x)[f(x), y]h(x).$$

Replacing y by $y+1$ and using Lemma 4 we obtain that either $I_1(y)[x^n, y] = 0$ or $[x^n, y]I_1(y) = 0$. Proceeding along the same line, we finally obtain $I_r(y)[x^n, y] = 0$ or $[x^n, y]I_r(y) = 0$. Thus in both cases we get $r![x^n, y] = 0$. Next, using the same way of replacing x by $x+1$ and iterating $(n-1)$-times we get that $r!n![x,y] = 0$, and the property Q $((\max\{r,n\})!)$ gives the commutativity of R.

Similarly we can prove that R is commutative if R satisfies (VIII). □

We close our discussion with the following

Conjecture. *Let R be a ring with unity 1 in which for every y there exists a polynomial $f(\lambda) \in \lambda^2\mathbb{Z}[\lambda]$ such that $y^r[x^m, y] = x^t[f(x), y]y^s$, where $m \geqslant 1$, r, s, t are non-negative integers. If the commutators in R are m-torsion free, then R is commutative.*

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