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INTEGRAL TRANSFORMS FOR DIVISORS IN $\mathbf{P}_n(\mathbb{C})$ AND SOLUTIONS OF SYSTEMS OF PDE'S*

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1. Introduction

The subject of the article is a study of two integral transformations in complex analysis—the analytic Radon and the Andreotti-Norguet transforms for compact effective analytic divisors (or cycles of codimension 1) in open subsets $Y \subset \mathbf{P}_n(\mathbb{C})$. Both these transformations had been studied intensively before (see e.g. [A-N2], [O2], [O4], [O5], [G-H]) but most of the results are known only for linear cycles. For $Y = \mathbf{P}_n(\mathbb{C}) \setminus \{\text{point}\}$, this corresponds to compact hyperplanes contained in Y and the situation is very simple since the kernel and the image are trivial: for instance, the Radon transformation is injective and the image of the Andreotti-Norguet transformation is the space of all analytic functions defined on the subset of the Grasmannian $G(n+1, n)$ parametrizing the compact hyperplanes contained in Y (cf. Prop. 0). This can be generalized to a linearly concave open subset (i.e. union of the compact hyperplanes included in it) Y of $\mathbf{P}_n(\mathbb{C})$: for the Radon transformation, it was done in [G-H], for the second transformation, we prove it in Prop. 1, using some well-known results of [O1] and [O7].

The aim of the paper is to characterize the image of the transforms essentially in the case of compact hypersurfaces (not always smooth) of arbitrary fixed degree contained in a linearly concave open subset of $\mathbf{P}_n(\mathbb{C})$. We want to show that the image is the set of all solutions of a suitable system of partial differential equations. This is the main result of the paper (see Prop. 6). Let Y denote a linearly concave open subset and $C^d_{n-1}(Y)$ the space of compact analytic cycles of dimension $n-1$ and degree d. An important tool used in the proof is the fact (proved in Prop. 5) that

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analytic functions on $C^d_{n-1}(Y)$, which are solutions of the system of PDE's mentioned above, are completely determined by their restrictions to the space of linear cycles, i.e. to $C^1_{n-1}(Y)$. This property makes it possible to use effectively results known for linear cases and injectivity results proved before for the general case, so that the proof of the main result is reasonably simple.

On $\mathbf{P}_n(\mathbb{C})$, the Andreotti-Norguet transformation ϱ^0 coincides for linear cycles with the Penrose transformation. For homogeneous manifolds, Bailey, Baston, Eastwood and Singer have developed a powerful method of description of the (generalized) Penrose transform by means of suitable spectral sequences (see [B-E], [B-E-S], [W-W]) when everything is smooth.

Since we want to study hypersurfaces of arbitrary degree, we are no more in the situation where everything is smooth; for instance even for $n = d = 2$ the space $C^d_{n-1}(Y) = C^2_1(Y)$ contains couples of lines (and double lines) which are singular along the intersection set. In particular, it is no more true that the projections π and π' in the corresponding double fibrations (see below) are submersions and the methods mentioned above are no more applicable. Even so, it would be interesting to see what type of results can be found by applying the above mentioned method to smaller subsets of smooth cycles of higher degree. As far as we know, such results are not yet available.

In the last part of the paper, we extend our results beyond the analytic category (see Corollary 7). Throughout the whole paper, we are describing local results, i.e. we study cycles in certain linearly concave domains.

Some of the results can be obtained from the statements of [HE], but the therein ideas are entirely different from those in the present work. Here, we give complete computable proofs, using very simple tools, which allow us to understand more precisely the situation. For instance, we describe exactly which properties of the integral transformations result from the differential structure and which from the analytical structure. Moreover, to give a better and more concrete understanding of the situation, we illustrate the problems with some examples where all the computations are explicitly done.

2. Description of the problem

Integral transformations of Radon type are always based on a geometrical situation characterized by a suitable double fibration. It consists of three topological spaces A, B, C together with two maps π, and π' from B to A, respectively to C:

$$\begin{array}{ccc} B & \xrightarrow{\pi} & A \\ {\scriptstyle \pi'}\downarrow & & \\ C & & \end{array}$$

We will always assume that the maps π and π' are open. The transformations themselves send certain analytic data defined on A to their counterparts defined on C via the applications π and π'.

In the context of analytic geometry and its integral transformations, the space A is a complex manifold, the space C is a suitable subset of the space of all analytic cycles on A and the upper space $B \subset A \times C$ is the corresponding incidence set, i.e. $B = \{(x, c) \in A \times C \mid x \in c\}$. The analytic transforms map sections of suitable holomorphic bundles on A to sections of bundles on C. The general Penrose transformation, the analytic Radon transformation and the Andreotti-Norguet (or ϱ^0) transformation belong to this scheme. We will discuss below the properties of the last two transforms.

2.1. The space of hypersurfaces in $\mathbf{P}_n(\mathbb{C})$. In this paper we are interested in the space C of all compact analytic cycles of codimension 1 in suitable open subsets of $\mathbf{P}_n(\mathbb{C})\setminus\{O\}$, where O is a point in $\mathbf{P}_n(\mathbb{C})$. The manifold $Y_O = \mathbf{P}_n(\mathbb{C})\setminus\{O\}$ is actually the simplest model of a variety where there are "a lot" of differential $\bar{\partial}$-closed forms of type $(n-1, n-1)$ (because it is $(n-1)$-complete) and "a lot" of compact $(n-1)$-cycles (because it is pseudoconcave). More precisely, the spaces $H^{n-1}(Y_O, \Omega^{n-1})$ and $H^{n-1}(Y_O, \Omega^n)$ are infinite dimensional vector spaces and the space $C = C^d_{n-1}(Y_O)$ of all cycles of degree d in Y_O is parametrized by $C = \mathbb{C}^N \subset \mathbf{P}_N(\mathbb{C})$ for a suitable $N > 0$. This is consequence of a very general result (cf. [A-N1]) but will be shown easily below by a direct computation.

2.2. The Analytic Radon Transformation. Firstly, let Y be a quasi-algebraic manifold, let ψ' be a semi-meromorphic differential form with a simple pole on a smooth hypersurface c, which is smooth and d-closed on $Y \setminus c$. For such a form, there is residue-class $\dot{\mathrm{res}}_{\,\mathrm{c}}\, \psi' \in H^*(c, \mathbb{C})$ corresponding to it. The Gelfand-Leray algorithm allows us to compute explicitly a representant of this d-cohomology class. Let $\xi_U = 0$

be a (local) equation of c in an open set U and let φ'_U be a smooth differential form such that $d\xi_U \wedge \varphi'_U = \xi_U \psi'$. The local forms φ'_U define together a cohomological class in $H^*(c, \mathbb{C})$ which is precisely $\dot{\mathrm{res}}_c \psi'$ (cf. [L] for the precise construction).

Now we can define the analytic Radon transformation R (or A.R.T.). Let ψ be a $\bar{\partial}$-closed differential form of type $(n, n-1)$ on Y. Suppose that f is a meromorphic function with a simple pole on a compact analytic hypersurface c; since ψ/f is a $\bar{\partial}$-closed $(n, n-1)$-form, it is d-closed in $Y \setminus c$. If c is smooth, we set $R(\psi)(f) = \int_c \mathrm{res}_c(\psi/f)$, where $\mathrm{res}_c(\psi/f)$ is any differential form belonging to the class $\dot{\mathrm{res}}_c(\psi/f)$.

Except for some "pathological" situations (and certainly in the situation we study), for any compact cycle c it is possible to find a neighbourhood V of c (in $C_{n-1}(Y)$) s.t. $V \setminus \{c\}$ is everywhere dense in V (and in particular is nonempty) and every cycle belonging to $V \setminus \{c\}$ is smooth; by continuity we extend the definition to any cycle. Alternatively, a direct construction of the A.R.T. for all cycles (including those with singularities) was given in [O5] using the theory of residue-currents of Coleff-Herrera.

Since R is invariant when we replace ψ by any form in the same $\bar{\partial}$-coholomogy class on Y, we can define the A.R.T. on $H^{n-1}(Y, \Omega^n)$. Then this transformation takes its values in the space of the sections of a holomorphic line bundle E' defined over some flag space (cf. [O5] for the complete construction).

It is possible, as we will do here, to simplify the problem by restricting the construction to meromorphic functions with a fixed polar set. More precisely, let us consider the case $Y \subset \mathbf{P}_n(\mathbb{C}) \setminus \{O\}$ and $\tilde{h}^\infty$ a fixed linear form with the zero set h^∞, which is a compact hyperplane in Y.

Then we define the Analytic Radon Transform $\mathcal{R}^1(\tilde{h}^\infty)$ by

$$\mathcal{R}^1(\tilde{h}^\infty)\dot{\psi}(F) = \int_c \mathrm{res}_c(\psi(\tilde{h}^\infty)^d/F)$$

where F is a homogeneous polynomial of degree d whose zero set c is compact in Y. Then $\mathcal{R}^1(\tilde{h}^\infty)$ is a morphism sending $H^{n-1}(Y, \Omega^n)$ to $H^0(C_{n-1}(Y), \mathcal{O}(-1))$ where $\mathcal{O}(-1)$ is the sheaf of germs of homogeneous functions of order -1. Moreover, since the residue of an analytic function is trivially zero, $\mathcal{R}^1(\tilde{h}^\infty)$ takes its values in $H^0(C_{n-1}(Y), \mathcal{O}_{(h^\infty)}(-1))$, the space of sections vanishing at the point $h^\infty \in C_{n-1}(Y)$. For an analytic subspace $W \in C_{n-1}(Y)$, we can naturally define a "restriction" $\mathcal{R}^1_W(\tilde{h}^\infty)$: $H^{n-1}(Y, \Omega^n) \to H^0(W, \mathcal{O}_{(h^\infty)}(-1))$ by restricting $\mathcal{R}^1(\tilde{h}^\infty)$ to the polynomials whose zero set belongs to W. For $W = C^d_{n-1}(Y)$ (i.e. for the space of all effective divisors of degree d in Y), we will use notation $\mathcal{R}^1_d(\tilde{h}^\infty)$. Moreover, when the linear form h^∞ is fixed once for all, we will omit it in the notation of the Radon transform, and for instance we will write $\mathcal{R}^1_d$ instead of $\mathcal{R}^1_{C^d_{n-1}(Y)}(\tilde{h}^\infty)$.

Remark. The term cycle can be ambiguous; it denotes either the point c in the cycle space $C_{n-1}(Y)$ or the set $|c|$ of all points of Y belonging to it. To simplify notation (when no confusion is possible), we will use c instead of $|c|$. In the above definition of the A.R.T., it is important to make the distinction between the two notions. The sections of $\mathcal{O}_{(h^\infty)}(-1)$ do not vanish at points of h^∞ but at the point $h^\infty \in C_{n-1}(Y)$ (since here $\mathcal{O}$ is a sheaf on $C_{n-1}(Y)$).

2.3. The Andreotti-Norguet Transformation. This transformation (also called the ϱ^0 transformation or the transformation of integration on cycles) is the most natural transformation to be defined on the space of differential forms of a given degree. Let φ be a smooth $(n-1, n-1)$ form on Y (where smooth means of class $\mathcal{C}^k$, $k \in \mathbb{N} \setminus \{0\}$ or $k = \{\infty\}$). We define $\check{\varrho}^0\varphi$ to be the function on $C_{n-1}(Y)$ given by integration $\check{\varrho}^0\varphi(c) = \int_c \varphi$. The function $\check{\varrho}^0\varphi$ is smooth on $C_{n-1}(Y)$ and if φ is $\bar{\partial}$-closed, it is an analytic function (cf. [A-N1]). Moreover, in this case, it depends only on the $\bar{\partial}$-cohomology class of φ in $H^{n-1}(Y, \Omega^{n-1})$ and hence induces an application $\varrho^0\colon\ H^{n-1}(Y, \Omega^{n-1}) \to H^0(C_{n-1}(Y), \mathcal{O})$. As for the A.R.T., for any subspace $W \subset C_{n-1}(Y)$ we can naturally define a restriction $\varrho^0_W\colon\ H^{n-1}(Y, \Omega^n) \to H^0(W, \mathcal{O})$. In particular, for $W = C^d_{n-1}(Y)$ we denote the application ϱ^0_W by ϱ^0_d. Using the double fibration with $A = Y$, $C = C_{n-1}(Y)$ and $B = \{(x, c) \in Y \times C_{n-1}(Y);\ x \in c\}$, we can easily obtain ϱ^0 by the formula $\varrho^0(\varphi) = \pi^*\pi'_*\varphi$ where π'_* denotes the direct image in the sense of currents (in some pathological situations this formula does not always hold (cf. [O7]), but generally and especially in the case we are interested in, it is true). This transformation was often studied (cf. [O6] for a short survey) but mostly in the case when W is included in a set of the hyperplane sections.

The use of inverse and direct images is not restricted to the Andreotti-Norguet transform. A more general transformation $\varrho^{r,s}$ was defined in [O7]; it acts on $H^{n-1+s}(Y, \Omega^{n-1+r})$ and has values in the space of $\bar{\partial}$-closed currents of type (r, s) on $C_{n-1}(Y)$. In particular, for $s = 0$ and $r = 1$ we can define an application ϱ^1 on $H^{n-1}(Y, \Omega^n)$ by setting $\varrho^1\psi = \pi^*\pi'_*\psi$ and ϱ^1_d is then the restriction of ϱ^1 to $C^d_{n-1}(Y)$; when $C^d_{n-1}(Y)$ is a manifold, ϱ^1_d takes its values in $H^0(C_{n-1}(Y), \Omega^1)$. The result is a transformation closely related to the Radon transformation (cf. [O4] for the details). In all these cases, the above constructions show that these integral transformations consist of integration of differential forms on fibers of a suitable bundle.

2.4 Explicit examples. Before entering a detailed discussion of the problem, it is worth while to illustrate the situation on simple examples. As stated above, the transform ϱ^0_1 is surjective on the space of hyperplanes in $Y_O = \mathbf{P}_n(\mathbb{C}) \setminus \{O\}$. For the transform ϱ^0_d defined on the space of higher order hypersurfaces, this is no more true. We have to expect that functions in the image will be solutions of a suitable system of differential equations. The value of the integral transform can be in many special cases computed by an explicit formula. It is instructive to see what type of polynomials are in the image and, at the same time, how the process of integration along fibres looks like. For illustration, it is sufficient to consider the case of dimension 2.

Any hypersurface c_A of degree d in $\mathbf{P}_2(\mathbb{C})$ with homogeneous coordinates $[z^0, z^1, z^2]$ is given by a homogeneous equation

$$F(A,z) \equiv \sum_{|I|\leqslant d} A_I (z^0)^{d-|I|}(z^1)^{i_1}(z^2)^{i_2} = 0; \quad I = (i_1, i_2).$$

It can be also written as

$$P_0(z^0)^d + P_1(z^0)^{d-1} + \ldots + P_{d-1}z^0 + P_d = 0, \tag{1}$$

where

$$P_j = P_j(A_I, z^1, z^2); \tag{2}$$

$$P_0 = A_{00},\ P_1 = A_{10}z^1 + A_{01}z^2,\ P_2 = A_{20}(z^1)^2 + A_{11}z^1z^2 + A_{02}(z^2)^2,\ \ldots$$

The hypersurface does not contain the point $O = [1, 0, 0]$ iff $A_{00} \neq 0$. Hence the space $C_1^d(Y_O)$ is just $\mathbb{C}^N$, $N = \frac{1}{2}d(d+3)$, with nonhomogeneous coordinates $a_I = A_I/A_{00}$.

(i) The ϱ^0 transform

Let φ be a 2-form on Y_O. Let us first discuss how to compute the integral $\int_{c_A}\varphi$ for a given hypersurface c_A (A fixed). First, we need a parametrization of an open dense subset of c_A. Let us consider nonhomogeneous coordinates

$$x^0 = z^0/z^2; \quad x^1 = z^1/z^2.$$

Let $\Omega \subset \mathbb{C}$ be the set of all x^1 such that the equation (1) has d different solutions x_j^0; $j = 1, \ldots, d$. We can suppose that Ω is dense in $\mathbb{C}$ (otherwise, the polynomial $F(A, .)$, A fixed, can be split into irreducible components including multiplicities and the same procedure can be applied to each of them; the value of the transform is then the sum of individual contributions). Then $\partial F/\partial x^0(x_j^0, x^1, 1) \neq 0$ for every j. In a neighbourhood of such points, the equation (1) can be used to express x_j^0 as a function of x^1 for $j = 1, \ldots, d$. The functions x_j^0 are defined only up to a permutation of indices but symmetric functions of them are well defined on the whole set Ω. The same is true for the form $\Big(\sum\limits_{j=1}^{d}(X_j^0)^*\varphi\Big)$, where $X_j^0(x_1) = (x_j^0(x_1), x_1)$. Then we have

$$\int_{c_A}\varphi = \int_\Omega \Big(\sum_{j=1}^{d}(X_j^0)^*\varphi\Big). \tag{3}$$

This procedure leads immediately to many examples of functions $\varphi(A)$ which belong to the image of the ϱ^0 transform.

Let $\tilde{\omega}$ denote the invariant volume form on $\mathbf{P}_1(\mathbb{C})$ (usually called the Fubini metric). In the nonhomogeneous coordinate $x^1 \in \mathbb{C}$, it has a form

$$\tilde{\omega} = \frac{\mathrm{d}x^1 \wedge \mathrm{d}\overline{x^1}}{(1+|x^1|^2)^2}.$$

Let π be the projection of Y_O onto $\mathbf{P}_1(\mathbb{C})$ given by the projection to the last two homogeneous coordinates, and let $\omega = \pi^*\tilde{\omega}$ be the corresponding form on Y_O.

1) Take first $\varphi = \omega$. Then $\tilde{\omega} = (X_j^0)^*\omega$ and

$$[\varrho_d^0(\varphi)](A) = d\int_{\mathbb{C}} \tilde{\omega}$$

is a finite, positive number which does not depend on A. Hence the constant functions are contained in the image of ϱ_d^0.

2) Let

$$f = \frac{z^0\overline{z^2}}{(|z^2|^2+|z^1|^2)}, \quad g = \frac{z^0\overline{z^1}}{(|z^2|^2+|z^1|^2)}.$$

Then the forms $\varphi_f = f\omega$, $\varphi_g = g\omega$ are manifestly $\bar{\partial}$-closed. Moreover, we know that

$$\sum_j x_j^0(x^1) = -(1/A_{00})[A_{10}x^1 + A_{01}]$$

and we get (using the fact that the integral of an odd function over $\mathbb{C}$ vanishes)

$$\begin{aligned}[\varrho_d^0(\varphi_f)](A) &= \int_\Omega \frac{\sum_j x_j^0(x^1)}{(1+|x^1|^2)}\,\tilde{\omega} = (1/A_{00})\int_\Omega \frac{A_{10}x^1 + A_{01}}{(1+|x^1|^2)^3}\,\mathrm{d}x^1\wedge\mathrm{d}\bar{x}^1\\ &= k\cdot A_{01}/A_{00},\ k\in\mathbb{R}.\end{aligned}$$

Similarly

$$\begin{aligned}[\varrho_d^0(\varphi_g)](A) &= \int_\Omega \frac{(\sum_j x_j^0(x^1))\overline{x^1}}{(1+|x^1|^2)}\,\tilde{\omega} = (1/A_{00})\int_\Omega \frac{(A_{10}x^1 + A_{01})\overline{x^1}}{(1+|x^1|^2)^3}\,\mathrm{d}x^1\wedge\mathrm{d}\bar{x}^1\\ &= k'\cdot A_{10}/A_{00},\ k'\in\mathbb{R}.\end{aligned}$$

3) In a similar way, we can find higher order polynomials in A which belong to the image of the transform ϱ_d^0. For that, let us first recall a few well known facts on symmetric polynomials in d variables (for details see e.g. [HU], Chap.13). Let us consider the elementary symmetric functions

$$\sigma_k^d(x_1,\ldots,x_d) = \sum_{1\leqslant i_1<\ldots<i_k\leqslant d} x_{i_1}\ldots x_{i_k};\ k=1,\ldots,d$$

in d variables of order k. Then any symmetric polynomial in $x_1, \ldots, x_d$ can be expressed as a polynomial in elementary symmetric polynomials. In particular, for any integer l, there are universal polynomials s_l^d of d variables such that

$$(x_1)^l + \ldots + (x_d)^l = s_l^d(\sigma_1^d, \ldots, \sigma_d^d). \tag{4}$$

For example, $s_1^d = \sigma_1^d, s_2^d = (\sigma_1^d)^2 - 2\sigma_2^d$. In general, we have

$$s_l^d(\sigma_1^d, 0, \ldots, 0) = (\sigma_1^d)^l. \tag{5}$$

Note also that if x_j^0, $j = 1, \ldots, d$ are solutions of the equation (1), then

$$P_j/P_0 = (-1)^j \sigma_j^d(x_1^0, \ldots, x_d^0);\ j = 1, \ldots, d; \quad P_0 = A_{00}.$$

Let us now denote

$$f_k^l = \frac{(z^0)^l (\overline{z^1})^k (\overline{z_2})^{l-k}}{(|z^2|^2 + |z^1|^2)^l}; \quad 0 \leqslant k \leqslant l.$$

Then the forms $\varphi_k^l = f_k^l \omega$ are again manifestly $\overline{\partial}$-closed and can be (almost) explicitly integrated. We get

$$\begin{aligned}\int_{c_A} \varphi_k^l &= \int_{\mathbb{C}} \left[\sum_1^d (x_j^0)^l (x^1)\right] (\overline{x^1})^k (1 + |x^1|^2)^{-l} \omega \\ &= \int_{\mathbb{C}} s_l^d(P_1/P_0, \ldots, P_l/P_0)(\overline{x^1})^k (1 + |x^1|^2)^{-l} \omega.\end{aligned}$$

The expression

$$s_l^d(-(1/A_{00})P_1(x^1, 1), \ldots, (-1)^l (1/A_{00}) P_l(x^1, 1))$$

is a polynomial of order l in x^1 with coefficients depending on the variables A_I. To compute the value of the integral, it is hence sufficient to know the values of the integrals

$$I_{jk} = \int_{\mathbb{C}} \frac{(x^1)^j (\overline{x^1})^k}{(1 + |x^1|^2)^l} \omega; \quad j, k = 0, \ldots, l.$$

The presence of the form ω implies that all these integrals are finite. Moreover, for $j = k$, the result is clearly a positive number, while for $j \neq k$, integration in polar coordinates gives immediately that the integrals vanish. Hence

$$I_{jk} = a_j \delta_{jk}; \quad j, k = 0, \ldots, l,$$

where a_j are (explicitly computable) positive numbers. So, whenever we are able to compute explicitly the form of the universal polynomials s^d_l, we get also explicit formulae for images of the forms φ^l_k.

Let us consider, for example, the case $l = 2$. We have $\sigma^d_1 = -(1/A_{00})P_1$, $\sigma^d_2 = (1/A_{00})P_2$, hence we get

$$s^d_2 = (\sigma^d_1)^2 - 2\sigma^d_2 = (1/A^2_{00})[A_{10}x^1 + A_{01}]^2 - (2/A_{00})[A_{20}(x^1)^2 + A_{11}x^1 + A_{02}]$$

and for $k = 0, 1, 2$, we get (up to a multiplicative constant)

$$\begin{aligned}
[\varrho^0_d(\varphi^2_0)](A) &= (1/A_{00})^2[(A_{01})^2 - 2A_{02}A_{00}];\\
[\varrho^0_d(\varphi^2_1)](A) &= (1/A_{00})^2[A_{01}A_{10} - 2A_{11}A_{00}];\\
[\varrho^0_d(\varphi^2_2)](A) &= (1/A_{00})^2[(A_{10})^2 - 2A_{20}A_{00}].
\end{aligned}$$

Note that the first terms in the brackets (the only ones which do not depend on A_{00}) form a basis of quadratic polynomials in A_{10}, A_{01}. Due to (5), the same behaviour of the leading orders is true for higher orders as well. We get that the images $[\varrho^0_d(\varphi^l_k)](A)$ will have a form (up to a constant)

$$(1/A_{00})^l[A^{l-k}_{01}A^k_{10} + A_{00}(\ldots)].$$

A formula for lower order terms can be computed explicitly for $d = 2$, because in this case we have an explicit and simple form for all universal polynomials s^2_l. For bigger d, these formulae will be as explicit as is our knowledge of the universal polynomials s^d_l. Note also that all polynomials in the image are homogeneous of degree 0, as expected, because they do not depend on choice of the equation for a hypersurfaces. They are pullbacks of functions on the corresponding subset of the projective space.

Question now is how broad our class of examples is. We can find sufficient information when studying the case of lines ($d = 1$). It is possible to prove that the images $\varrho^0_1(\varphi^l_k)$; $0 \leqslant k \leqslant l$ are dense in $H^0(C^1_{n-1}(Y_O), \mathcal{O})$. Hence we can expect that their images under ϱ^0_d will be dense in the image as well and that it is possible to get full information on the image from these examples. In the paper, we will not do it. Instead, we shall use results already known for the kernel of the transforms to make proofs simpler.

As a sideremark, note that the forms φ^l_k and their images described above have a typical interpretation in terms of representation theory. The projective space itself is a homogeneous space of the group $G = SU(3, \mathbb{C})$. Let K denote the Levi part of isotropy subgroup ($K = SU(2)$) of a point in $\mathbf{P}_2(\mathbb{C})$. The cohomology group

$H^1(Y_O, \Omega^1)$ is then a (g, K)-module, where g is the Lie algebra of G. The forms used above are examples of K-finite vectors in this module. The ϱ^0 transform commutes with the (g, K) action, hence the image is also a (g, K)-module and the described images of forms are K-finite elements in the image. The elements of the same homogeneity belongs to the same K-type. Moreover, the span of all these elements is dense in the whole image.

(ii) The analytic Radon transform

A similar computation as in the case of ϱ_d^0 can be done also for $\mathcal{R}_d^1$. This gives also a possibility to illustrate how the definition of the Radon transform given above can be worked out in some explicitly computable cases. As for the ϱ^0 transform, we will consider here several simple examples of $\overline{\partial}$-closed forms of type $(2,1)$ and we will compute their image under the analytic Radon trasform. They all will be of the form

$$\tau = \frac{(z^0)^l (\overline{z^1})^k (\overline{z^2})^{l+1-k}\, \mathrm{d}z^0}{(|z^2|^2 + |z^1|^2)^{l+1}} \wedge \omega,$$

hence in nonhomogeneous coordinates

$$\tau = \frac{(x^0)^l (\overline{x^1})^k\, \mathrm{d}x^0}{(1 + |x^1|^2)^{l+1}} \wedge \frac{\mathrm{d}x^1 \wedge (\mathrm{d}\overline{x^1})}{(1 + |x^1|^2)^2}.$$

In general, if

$$\tau = f(x^0, x^1)\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}\overline{x^1},$$

then we have to compute $\operatorname{res}\mu$; $\mu = ((x^0)^d / F_A))\tau$ and to integrate it over the hypersurface c_A given by the homogeneous equation $F_A = 0$.

Let us first see how to find the above residue. Locally, it is given by division of forms. More precisely, if $F_A(x) = 0$ is the equation of c_A, then the residue $\operatorname{res}\mu$ (i.e. a representant of the cohomology class) is a form ν satisfying the equation

$$\nu \wedge \frac{\mathrm{d}F_A}{F_A} = \mu = (1/F_A) \frac{(x^0)^{l+d} (\overline{x^1})^k\, \mathrm{d}x^0}{(1 + |x^1|^2)^{l+1}} \wedge \frac{\mathrm{d}x^1 \wedge \mathrm{d}\overline{x^1}}{(1 + |x^1|^2)^2}.$$

Using

$$\mathrm{d}F_A = \frac{\partial F_A}{\partial x^0}\, \mathrm{d}x^0 + \frac{\partial F_A}{\partial x^1}\, \mathrm{d}x^1,$$

we get

$$\operatorname{res}\mu = \left(1 \Big/ \frac{\partial F_A}{\partial x^0}\right) \frac{(x^0)^{l+d} (\overline{x^1})^k}{(1 + |x^1|^2)^{l+1}} \wedge \frac{\mathrm{d}x^1 \wedge \mathrm{d}\overline{x^1}}{(1 + |x^1|^2)^2}.$$

This leads to a similar integration procedure as in the case of the ϱ^0 transform. There is, however, an important difference which needs a bit of care. The main point making it possible to compute values of the ϱ^0 transform explicitly was the formula expressing the sum $\sum_{j=1}^{d}(x_j^0(x^1))^l$ of roots of the polynomial (1) using (4) as polynomials in P_i/A_{00}.

Here everything is divided by $\frac{\partial F_A}{\partial x^0}$. Hence we get now a more complicated rational function

$$\sum_{j=1}^{d} \frac{(x_j^0)^{l'}}{\frac{\partial F_A}{\partial x^0}(x_j^0)} \tag{6}$$

where

$$\frac{\partial F_A}{\partial x^0}(x_j^0) = A_{00}\Pi_{k\neq j}(x_j^0 - x_k^0).$$

An important fact to notice is that it is again just a polynomial! After a reduction to a common denominator (which is the discriminant $\Pi_{j<k}(x_j^0 - x_k^0)$), the denumerator vanishes whenever $x_j^0 = x_k^0$, hence all factors in the denominator will cancel. It is clear that (6) vanishes for all $l' < d-1$, because the degree of the denumerator is smaller then the degree of the denominator. For the next few cases, it is possible to have an explicit answer. For example

$$\sum_{j=1}^{d} \frac{(x_j^0)^{d-1}}{\frac{\partial F_A}{\partial x^0}(x_j^0)} = 1/A_{00}; \qquad \sum_{j=1}^{d} \frac{(x_j^0)^{d}}{\frac{\partial F_A}{\partial x^0}(x_j^0)} = (1/A_{00})\sum_{j=1}^{d} x_j^0;$$
$$\sum_{j=1}^{d} \frac{(x_j^0)^{d+1}}{\frac{\partial F_A}{\partial x^0}(x_j^0)} = (1/A_{00})\left[\left(\sum_j x_j^0\right)^2 - \sum_{j<k} x_j^0 x_k^0\right].$$

In general, after cancelation of the discriminant, we have again a symmetric polynomial in d variables x_j^0, hence there exist polynomials $\tilde{s}_l^d$ in d variables such that for positive integers l,

$$\sum_{j=1}^{d} \frac{{x_j^0}^{d+l}}{\frac{\partial F_A}{\partial x^0}(x_j^0)} = \tilde{s}_l^d(\sigma_1^d,\ldots,\sigma_d^d). \tag{7}$$

Hence the evaluation of the analytic Radon transform is going on exactly as for the ϱ^0 transform and is as explicit as our knowledge of the polynomials $\tilde{s}_l^d$ is. We are now able again to compute a few explicit examples.

1) Let us first consider two $\bar\partial$-closed $(2,1)$-forms $\varphi_1 = \alpha_1 \wedge \omega, \varphi_2 = \alpha_2 \wedge \omega$, where ω is the pullback of the Fubini form from $\mathbf{P}_1(\mathbb{C})$ to Y_O and

$$\alpha_1 = \frac{\overline{z^2}\,\mathrm{d}z^0}{(|z^2|^2 + |z^1|^2)}, \quad \alpha_2 = \frac{\overline{z^1}\,\mathrm{d}z^0}{(|z^2|^2 + |z^1|^2)}.$$

We know that

$$\sum_j \frac{(x_j^0(x^1))^d}{A_{00}\Pi_{k\neq j}(x_j^0(x^1) - x_k^0(x^1))} = -(1/(A_{00})^2)[A_{10}x^1 + A_{01}]$$

and we get

$$[\mathcal{R}_d^1(\varphi_1)](A) = -(1/(A_{00})^2)\int_{\mathbf{C}} \frac{A_{10}x^1 + A_{01}}{(1+|x^1|^2)^3}\,\mathrm{d}x^1 \wedge d\bar{x}^1 = k \cdot A_{01}/(A_{00})^2, \quad k \in \mathbb{R}.$$

Similarly

$$[\mathcal{R}_d^1(\varphi_2)](A) = -(1/(A_{00})^2)\int_{\mathbf{C}} \frac{(A_{10}x^1 + A_{01})\overline{x}^1}{(1+|x^1|^2)^3}\,\mathrm{d}x^1 \wedge d\bar{x}^1 = k \cdot A_{10}/(A_{00})^2, \quad k \in \mathbb{R}.$$

2) Let us now denote

$$f_k^l = \frac{(z^0)^l(\overline{z^1})^k(\overline{z_2})^{l+1-k}}{(|z^2|^2 + |z^1|^2)^{l+1}}; \quad 0 \leqslant k \leqslant l.$$

Then the forms $\varphi_k^l = f_k^l\,\mathrm{d}z^0 \wedge \omega$ are again manifestly $\overline{\partial}$-closed.

In general, we do not have an explicit formula for polynomials $\tilde{s}_l^d$, but we know again that

$$\tilde{s}_l^d(\sigma_1^d, 0, \ldots, 0) = (\sigma_1^d)^l$$

and we get

$$\mathcal{R}_d^1(\varphi_k^l) = h(1/A_{00})^{l+1}[A_{01}^{l-k}A_{10}^k + A_{00}(\ldots)], \ h \in \mathbb{R}.$$

For $l = 2$, $k = 0, 1, 2$ we get (up to a multiplicative constant)

$$\begin{aligned}
[\mathcal{R}_d^1(\varphi_0^2)](A) &= (1/A_{00})^3[(A_{01})^2 - A_{02}A_{00}];\\
[\mathcal{R}_d^1(\varphi_1^2)](A) &= (1/A_{00})^3[A_{01}A_{10} - A_{11}A_{00}];\\
[\mathcal{R}_d^1(\varphi_2^2)](A) &= (1/A_{00})^3[(A_{10})^2 - A_{20}A_{00}].
\end{aligned}$$

From these explicit examples, it is possible to see a connection among images under ϱ^0 and $\mathcal{R}_d^1$. It is clear that in these examples we have got that $\mathcal{R}_d^1(\mathrm{d}\omega)$ is equal to the coefficient at dA_{00} of the value of the de Rham differential $\mathrm{d}(\varrho_d^0(\omega))$. More on such relations among the ϱ^0 and ϱ^1 transforms and the analytic Radon transform can be found in [O7].

In this section, we introduce the notation and sumarize results for the transforms on hyperplanes, which will be needed later on.

3.1. Notation. Let $[z^0, \dots, z^n]$ be a system of homogeneous coordinates on $\mathbf{P}_n(\mathbb{C})$, O a fixed point in $\mathbf{P}_n(\mathbb{C})$ with homogeneous coordinates $[1, 0, \dots, 0]$ and $Y_O = \mathbf{P}_n(\mathbb{C}) \setminus \{O\}$. For any subset $S \subset \mathbf{P}_n(\mathbb{C})$, we denote by $C^d_{n-1}(S)$ the space of analytic compact cycles of codimension 1 and degree d in S. For any m-tuple $K = (k_1, \dots, k_m)$ of integers, $|K| = k_1 + \dots + k_m$ denotes its length.

We will define an order on the set of the m-tuples of integers in the following way:

$$
\begin{aligned}
&K < K' \text{ if either } |K| < |K'| \\
&\qquad K < K' \text{ or } |K| = |K'| \text{and } K \text{ is strictly smaller than } K' \\
&\qquad\qquad\qquad\qquad \text{for the anti-lexicographic order}
\end{aligned}
$$

(i.e. $(k_1, \dots, k_m) < (k'_1, \dots, k'_m)$ if $10^{m-1}k_m + \dots + 10k_2 + k_1 < 10^{m-1}k'_m + \dots + 10k_2 + k_1$). Moreover, we write $K \leqslant K'$ iff $K = K'$ or $K < K'$. This order is compatible with the natural order on monomials (in the sense that $z^K < z^J$ iff $\deg(z^K) < \deg(z^J)$ or if $\deg(z^K) = \deg(z^J)$ and K is strictly smaller than J for the anti-lexicographic order). It is moreover clearly compatible with the (partial) natural order on n-tuples defined by $K = (k_1, \dots, k_m)$ is smaller than $J = (j_1, \dots, j_m)$ if there is $h \in \{1, \dots, m\}$ such that $k_h < j_h$ and $k_l \leqslant j_l$ for all other $l \in \{1, \dots, m\}$, $l \neq h$.

For $K = (k^0, \dots, k^m)$, $z' = (z^1, \dots, z^m)$ and $K' = (k^1, \dots, k^m)$, we denote $z^K = (z^0)^{k_0} \dots (z^m)^{k_m}$ and $z'^{K'} = (z^1)^{k'_1} \dots (z^m)^{k'_m}$. With this notation, the homogeneous equation of $c \in C^d_{n-1}(\mathbf{P}_n(\mathbb{C}))$ can be written as $\sum\limits_{|K|=d} A_K z^K = \sum\limits_{k_0=0}^{d} \sum\limits_{|K|=d} A_{K'} z^K = \sum\limits_{|K'|=d-k_0} A_{K'} (z^0)^{k_0} z'^{K'} = 0$.

Using the previous order, the system $[A_K]$ (or $[A_{K'}]$) allows us to identify $C^d_{n-1}(\mathbf{P}_n(\mathbb{C}))$ and $\mathbf{P}_N(\mathbb{C})$ (with $N = \binom{n+d}{n} - 1$) and gives a system of homogeneous coordinates on the space $C^d_{n-1}(\mathbf{P}_n(\mathbb{C}))$. From now on, we will use this order on the m-tuples of integers.

Let $l(K)$ (respectively $l'(K')$) denote the position of K (K') in $\{1, \dots, N\}$. Since for any $h \in \{1, \dots, N\}$ there is exactly one K (K') such that $l(K) = h$ ($l'(K') = h$), we can define $l^{-1}(h)$ (${l'}^{-1}(h)$) as well. Since no confusion is possible, we will denote by l both the applications l and l'.

The subset $C\{O\}$ of hypersurfaces in $\mathbf{P}_n(\mathbb{C})$ containing the point $\{O\}$ is then given in $C^d_{n-1}(\mathbf{P}_n(\mathbb{C}))$ (using the previous homogeneous coordinates) by the equation

$A_{0,\dots,0} = 0$, i.e. it is a hyperplane in $\mathbf{P}_N(\mathbb{C})$, so that $C^d_{n-1}(Y_O) = C^d_{n-1}(\mathbf{P}_n(\mathbb{C})) \setminus C\{O\}$ can be identified with $\mathbb{C}^N$ equipped with the system of coordinates $(a_{i_1,\dots,i_n} = A_{i_1,\dots,i_n}/A_{0,\dots,0})$. We fix once for all $h^\infty \subset Y_O$ to be the hyperplane at infinity given in $\mathbf{P}_n(\mathbb{C})$ by the equation $\tilde{h}^\infty(z) = z^0 = 0$, so we shall use the notation $\mathcal{R}^1_d$ instead of $\mathcal{R}^1_d(\tilde{h}^\infty)$. Let U_0 be the open subset $U_0 = \mathbf{P}_N(\mathbb{C}) \setminus h^\infty$; O belongs to this affine subset and we can define a system of coordinates in U_0 by $x^i = z^i/z^0, i \in \{1,\dots,n\}$ such that O is the origin $\{0\}$ in $U_0 \cong \mathbb{C}^n$.

For any integer $k \leqslant n$, the symbols $\mathbf{p}$ (respectively $\mathbf{p}_k$) will be used for the n-tuple, all elements of which are equal to p (respectively to the n-tuple with k^{th} element equals to p and all other 0); for instance we have $\mathbf{0} = (0,\dots,0)$ and $\mathbf{p} = \mathbf{p}_1 + \dots + \mathbf{p}_n$; moreover we write A_k instead of $A_{\mathbf{1}_k}$. For any $I' \subset I$, the symbol $I(\hat{I}')$ will be used for the index $I'' = (j)_{j \in I \setminus I'}$ of length $(|I| - |I'|)$. Note that for the order we defined previously, we have $1_h < I = (i_1,\dots,i_n)$ for any n-tuple such that $\max_k(i_k) > 1$.

For any open subset U of Y_O, we will use notation $C^d_{n-1}(U) = \{c \in C^d_{n-1}(Y_O);\ c \subset U\}$. Note that for U open, the set $C^d_{n-1}(U)$ is also open in $C^d_{n-1}(Y_O)$ and thus has a natural analytic structure.

Let us recall that an open subset Y in $\mathbf{P}_n(\mathbb{C})$ is called linearly concave (in the sense of Martineau [M]), if Y is the union of all compact hyperplanes contained in Y. From now on, Y will denote an open proper linearly concave subset in $\mathbf{P}_n(\mathbb{C})$. Since Y_0 is trivially linearly concave, we can assume without loss of generality Y is included in Y_0.

3.2. Integral transforms on hyperplanes. As usual, the open subset $Y_O = \mathbf{P}_n(\mathbb{C}) \setminus \{O\}$ is a good model for an open subset of $\mathbf{P}_n(\mathbb{C})$ with some hypothesis of convexity-concavity. For the convenience of the reader, we first recall here well-known properties of the Radon and the Andreotti-Norguet transformations on Y_O.

Properties of integral transforms on hyperplanes are summarized in the following propositions.

Proposition 0.

i) *The Radon transform* $\mathcal{R}^1_1\colon H^{n-1}(Y_O,\Omega^n) \to H^0(C^1_{n-1}(Y_O),\mathcal{O}_{(h^\infty)}(-1))$ *is bijective.*

ii) *The sequence*

$$H^{n-1}(Y_O,\Omega^{n-2}) \xrightarrow{\mathrm{d}} H^{n-1}(Y_O,\Omega^{n-1}) \xrightarrow{\varrho^0_1} H^0(C^1_{n-1}(Y_O),\mathcal{O}) \longrightarrow 0$$

is exact.

P r o o f. These properties are well-known (cf. for instance [G-H], [O1], [O4] and [B]). The quickest way to see them, is to consider the long cohomological dual sequences:

$$\begin{aligned}
\mathbb{C}^{n-k} \cong H^{n-1}(\mathbf{P}_n(\mathbb{C}),\Omega^k) &\longrightarrow H^{n-1}(Y_O,\Omega^k) \xrightarrow{\delta_k} H^n_{\{0\}}(\mathbf{P}_n(\mathbb{C}),\Omega^k) \\
&\longrightarrow H^n(\mathbf{P}_n(\mathbb{C}),\Omega^k) \cong \mathbb{C}^{k+1-n}, \\
\mathbb{C}^{n-k} \cong H^1(\mathbf{P}_n(\mathbb{C}),\Omega^{n-k}) &\longleftarrow H^1_{CO}(Y_O,\Omega^{n-k}) \xleftarrow{\partial_k} H^0_{CO}(\{O\},\Omega^{n-k}) \\
&\longleftarrow H^0_{CO}(\mathbf{P}_n(\mathbb{C}),\Omega^{n-k}) \cong \mathbb{C}^{k+1-n}
\end{aligned}$$

with $k \in \{n, n-1\}$ where Γ_{CO} denotes the functor of sections with compact supports (cf. [O2] for details).

Let $(u_1,\ldots,u_n)$ be a system of coordinates in a neighbourhood of $O=(0,\ldots,0)$; by duality we get

$$H^n_{\{O\}}(\mathbf{P}_n(\mathbb{C}),\Omega^n) \cong (H^0(\{O\},\mathcal{O}_{\mathbf{P}_n(\mathbb{C})}))' = \left\{\sum_{|I|} C_I \tilde{\mathrm{d}}^I\,;\ \lim_{|I|\to\infty}(C_I)^{1/|I|} < \infty\right\}$$

where $I=(i_1,\ldots,i_n)\in\mathbb{N}^n$ and $\tilde{\mathrm{d}}^I$ is the I-th differential of the Dirac current $\tilde{\mathrm{d}}$ in O. It is easy to show that the values of the Radon transform $R^1_1(\dot{\psi})$ at c with $\dot{\psi}\in H^{n-1}(Y_O,\Omega^n)$ (respectively $\varrho^0_1(\dot{\varphi})(c)$ with $\dot{\varphi}\in H^{n-1}(Y_O,\Omega^{n-1})$) are precisely (up to a constant) the pairing $\langle\delta_n\dot{\psi},1/F\rangle$ ($\langle\delta_{n-1}\dot{\varphi},\,\mathrm{d}F/F\rangle$) where $F=a_0+\sum_{i=1}^n a_i u^i=0$ is an equation of c. Let ω be the Fubini $(1,1)$-form on $\mathbf{P}_n(\mathbb{C})$.

i) For $k=n-1$ the kernel of δ_{n-1} is the complex vector space generated by ω^{n-1} and clearly we obtain $\ker\varrho^0_1=\mathrm{d}H^{n-1}(Y_O,\Omega^{n-2})$; the surjectivity of δ_{n-1} implies immediately the one of ϱ^0_1.

ii) For $k=n$ and $a_0\neq 0$ we have (up to a constant) $\langle\tilde{\mathrm{d}}^I,1/F\rangle=(a')^I/a_0$ where $a'_i=a_i/a_0$ $(i=1,\ldots,n)$, $a'=(a'_1,\ldots,a'_n)$ and $(a')^I=(a'_1)^{i_1}\ldots(a'_n)^{i_n}$. The injectivity of δ_n implies the one of $\mathcal{R}^1_1$ and from the surjectivity of δ_n modulo the (vector) line generated by ω, the surjectivity follows. □

The previous proposition can be extended to linearly concave open subsets in $\mathbf{P}_n(\mathbb{C})$. First, we prove the following (easy) lemma:

Lemma 3.1. *Let U be an open subset in $\mathbf{P}_n(\mathbb{C})$. If $H^1(\mathbf{P}_n(\mathbb{C})\setminus U)=0$ (i.e. the first cohomology group of $\mathbf{P}_n(\mathbb{C})\setminus U$ is zero) the following equality holds:*

$$\ker(H^{n-1}(Y,\Omega^{n-1})\xrightarrow{\mathrm{d}}H^{n-1}(Y,\Omega^n)) = \mathbb{C}\dot{\omega}^{n-1}\oplus dH^{n-1}(Y,\Omega^{n-2}).$$

P r o o f. Let us note by K the kernel of the application $\mathrm{d}\colon H^{n-1}(Y,\Omega^{n-1})\to H^{n-1}(Y,\Omega^n))$ and $X=\mathbf{P}_n(\mathbb{C})\setminus Y$.

i) First, since ω is d-closed, the inclusion $\mathbb{C}\dot{\omega}^{n-1} \oplus \mathrm{d}H^{n-1}(Y, \Omega^{n-2}) \subset K$ is trivial.

ii) Let φ be a $\bar{\partial}$-closed $(n-1, n-1)$-differential form in Y and $\dot{\varphi}$ its $\bar{\partial}$-cohomology class in $H^{n-1}(Y, \Omega^{n-1})$; if $\dot{\varphi}$ belongs to K we can find an $(n, n-2)$-differential form α_1 s.t.

$$\mathrm{d}\varphi = \partial\varphi = \bar{\partial}\alpha_1. \tag{1}$$

Since $H^n(Y, \Omega^{n-1}) = 0$ (its dual space is $H^0_{CO}(Y, \Omega^1)$), we can find an $(n-1, n-2)$-differential form α_2 s.t.

$$\alpha_1 = \partial\alpha_2. \tag{2}$$

iii) From (1) and (2) we get $\partial\varphi = \partial\bar{\partial}\alpha_2$ or $\partial(\varphi - \bar{\partial}\alpha_2) = 0$ and for $\varphi_1 = \varphi - \bar{\partial}\alpha$ we have $\partial\varphi_1 = 0, \bar{\partial}\varphi_1 = 0$, thus φ_1 induces a class $\tilde{\varphi}_1$ of d-cohomology in $H^{2n-2}(Y)$.

The well-known exact sequence $H^1(K) \to H^2_{CO}(\mathbf{P}_n(\mathbb{C}) \operatorname{Mod} K) \to H^2(\mathbf{P}_n(\mathbb{C}))$ and the isomorphism $H^2_{CO}(\mathbf{P}_n(\mathbb{C}) \operatorname{Mod} K) \cong H^2_{CO}(Y)$ proves, when $H^1(K) = 0$, that the space $H^2_{CO}(Y)$ is generated by the restriction of $\dot{\omega}$ to Y. By duality we obtain that $H^{2n-2}(Y)$ is generated by the class of d-cohomology defined by ω^{n-1}, thus there is $\lambda \in \mathbb{C}$ s.t. $\varphi_1 = \lambda\omega^{n-1} + \mathrm{d}\beta$; for $\varphi_2 = \varphi_1 - \lambda\omega^{n-1}$ we have $\varphi_2 = \mathrm{d}\Big(\sum\limits_{i=0}^{3} \beta_i\Big)$ where the β_i are the differential $(n-i, n-3+i)$-forms.

iv) As in ii) we have $\beta_0 = \partial\alpha_0$ and $\beta_3 = \bar{\partial}\alpha_3$ with α_0 and α_3 respectively of type $(n-1, n-3)$ and $(n-3, n-1)$, thus

$$\varphi_2 = \partial(\beta_1 - \bar{\partial}\alpha_0) + \bar{\partial}\beta_1 + \partial\beta_2 + \bar{\partial}(\beta_2 - \partial\alpha_3) \tag{3}$$

and using the type of the forms, we obtain $\partial(\beta_1 - \bar{\partial}\alpha_0) = \bar{\partial}(\beta_3 - \partial\beta_3) = 0$; setting $\beta = \beta_1 - \bar{\partial}\alpha_0$ and $\beta' = \beta_2 - \partial\alpha_3$, (3) gives $\varphi_2 = \bar{\partial}\beta + \partial\beta' = d(\beta + \beta')$. Since φ_2 is $\bar{\partial}$-cohomologuous to $\varphi - \lambda\omega^{n-1}$ this proves the lemma. □

Proposition 3.2. *Under the hypotheses*

1. *Y is linearly-concave,*
2. *for every $y \in Y$ the set of the compact hyperplanes containing y is contractible (in $C^1_{n-1}(Y)$),*

then

i) *the application $\mathcal{R}^1_1\colon\ H^{n-1}(Y, \Omega^n) \longrightarrow H^0(C^1_{n-1}(Y), \mathcal{O}_{(h^\infty)}(-1))$ is bijective;*

ii) *if moreover $H^1(\mathbf{P}_n(\mathbb{C}) \setminus Y) = 0$, then the sequence*

$$H^{n-1}(Y, \Omega^{n-2}) \xrightarrow{\mathrm{d}} H^{n-1}(Y, \Omega^{n-1}) \xrightarrow{\varrho^0_1} H^0(C^1_{n-1}(Y), \mathcal{O}) \longrightarrow 0$$

is exact.

P r o o f. i) The bijectivity of $\mathcal{R}_1^1$ is exactly the particular case of codimension 1 of [G-H].

ii) Let $\dot\varphi \in H^{n-1}(Y,\Omega^{n-1})$ be in $\ker \varrho_d^0$; by theorem IV.4.4 and lemma I.2.2 of [O4] the form $\mathrm{d}\dot\varphi$ belongs to $\ker \mathcal{R}_1^1$; the injectivity of $\mathcal{R}_1^1$ implies

$$\dot\varphi \in \ker(H^{n-1}(Y,\Omega^{n-1}) \xrightarrow{\mathrm{d}} H^{n-1}(Y,\Omega^n))$$

and from the above lemma, there is $\lambda \in \mathbb{C}$ such that $\dot\varphi - \lambda\dot\omega^{n-1} \in \mathrm{d}H^{n-2}(Y,\Omega^{n-1})$; Lelong's theorem gives the inclusion $\mathrm{d}H^{n-2}(Y,\Omega^{n-1}) \subset \ker \varrho_1^0$. But $\varrho_d^0(\dot\omega^{n-1})$ is identically constant equal to d, thus $\lambda = 0$ and finally we obtain $\ker \varrho_1^0 = \mathrm{d}H^{n-2}(Y,\Omega^{n-1})$. The proof of the surjectivity of ϱ_1^0 is similar. $\square$

Definition. Once the hyperplane h^∞ at infinity is fixed, we can set

$$C_{n-1}^{d,\infty}(Y) = \{(d-1)h^\infty + h;\ h \in C_{n-1}^1(Y)\}.$$

Remark. The subspace $C_{n-1}^{d,\infty}(Y)$ in $C_{n-1}^d(Y)$ can be canonically identified with $C_{n-1}^1(Y)$; this identification allows us to define an inclusion i_d of $C_{n-1}^1(Y)$ into $C_{n-1}^d(Y)$; for any subset C of continuous functions on $C_{n-1}^d(Y)$, i_d induces an application i_d^* from C into the space of continuous functions on $C_{n-1}^1(Y)$. In the case that there is no confusion possible, we will use notation $i_d^*\varrho_d^0$ (respectively $i_d^*\mathcal{R}_d^1$) for the application $I\colon H^{n-1}(Y,\Omega^{n-1}) \to H^0(C_{n-1}^{d,\infty}(Y),\mathcal{O})$ ($J\colon H^{n-1}(Y,\Omega^n) \to H^0(C_{n-1}^{d,\infty}(Y),\mathcal{O}_{(h^\infty}(-1)))$ defined by $I(\dot\varphi) = \varrho_d^0(\dot\varphi)\circ i_d$ (resp. $J(\dot\psi) = \mathcal{R}_d^1(\dot\psi)\circ i_d$).

Corollary 3.3. *The morphism i_d^* from $\operatorname{Im}\varrho_d^0$ to $\operatorname{Im}\varrho_1^0$ (or from $\operatorname{Im}\mathcal{R}_d^1$ to $\operatorname{Im}\mathcal{R}_1^1$) is an isomorphism.*

P r o o f. Since i_d is an inclusion, the surjectivity is trivial. Let f be in $\operatorname{Im}\mathcal{R}_d^1$ (respectively g in $\operatorname{Im}\varrho_d^0$) with $i_d^*(f) = 0$ (resp. $i_d^*(g) = 0$; from the first part of the previous proposition, $f = \mathcal{R}_d^1(\dot 0)$ and $f = 0$. From the second part there exists $\dot\gamma \in H^{n-1}(Y,\Omega^{n-2})$ s.t. $g = \varrho_d^0(\dot{d\gamma})$; let γ denote an $(n-2,n-1)$-differential $\bar\partial$-closed form belonging to $\dot\gamma$ (through Dolbeault isomorphism); since by Lelong's theorem the integration on cycles defines a d-closed current, for any $c \in C_{n-1}(Y)$ we have $\int_c \mathrm{d}\gamma = 0$ and the corollary is proved. $\square$

4. A suitable system of PDE's

In this section we only need to be in the real smooth category. Let F be a smooth function defined on an open subset of $\mathbb{R}^{n+1}\times\mathbb{R}^N$ and G the hypersurface given by the equation $F=0$; let $(x,a)=(x^1,\dots,x^{n+1},a_1,\dots,a_N)$ be a system of coordinates on $\mathbb{R}^{n+1}\times\mathbb{R}^N$. Let $W=U\times U'\times V\subset\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^N$ be an open subset where $\partial F/\partial x_{n+1}\neq 0$ on W and $W\cap G$ is given by a local equation $x^{n+1}=y(x^1,\dots,x^n,a_1,\dots,a_N)$. So we can use (x',a) as a system of coordinates on $W\cap G$ (where x' stands for $(x^1,\dots,x^n)$).

Lemma 4.1. *Suppose that a hypersurface G is given by an equation $F=0$ and that G is the graph of a function $y(x',a)$ in $W=U\times U'\times V$, as described above. Let g be a smooth function on $U\times U'$ and let $f^{(q)}$ $(q\in\{0,\dots,N+n\})$ be smooth functions on $U\times V$, defined respectively by $f^{(0)}(x',a)=g(x',y(x',a))$, $f^{(m)}(x',a)=g(x',y(x',a))(\frac{\partial y}{\partial a_m}(x',a))$ and $f^{(N+r)}(x',a)=g(x',y(x',a))(\frac{\partial y}{\partial x^r}(x',a))$ with $m\in\{1,\dots,N\}$, $r\in\{1,\dots,n\}$. Suppose further that for chosen four indices $\{i,j,h,k\}$, the following conditions are satisfied:*

(i) *For any x and for any a_l fixed ($l\in\{1,\dots,N\}\setminus\{i,j,h,k\}$), the function F is affine with respect to (a_i,a_j,a_h,a_k);*

(ii) *$c_ic_j=c_hc_k$ where c_s is the coefficient of a_s in F ($s\in\{i,j,k,h\}$).*

Then for any $q\in\{0,\dots,N+n\}$, the functions $f^{(q)}$ satisfy the differential equation

$$\partial^2 f^{(q)}/\partial a_i\partial a_j=\partial^2 f^{(q)}/\partial a_h\partial a_k. \tag{$a_{i,j,h,k}$}$$

P r o o f. To simplify the notation, let us replace coordinates (x',x_{n+1},a) on W by

$$(z_1,\dots,z_n,y,z_{n+1},\dots,z_{n+N})=(x^1,\dots,x^n,x_{n+1},a_1,\dots,a_N,).$$

For any function $T=T(z,y)$ and any multiindex $L=(l_1,\dots,l_s)\subset\{1,\dots,N+n\}^s$, we will use the notation $T_L=\partial^sT/\partial z_L=\partial^sT/\partial z_{l_1}\dots\partial z_{l_s}$; $T_{y^s}=\partial^sT/\partial y\dots\partial y$, where S is the s-tuple $(y,\dots,y)$. Let us further consider the map $Z\colon U\times V\to W$ given by $Z(x',a)=(x',y(x',a),a)$. Then for any function T on W we have

$$Z^*T_i+(Z^*T_y)y_i=(Z^*T)_i. \tag{α}$$

1) Since $Z^*F\equiv 0$ for $p\in\{1,\dots,N+n\}$, we can write

$$Z^*F_p+(Z^*F_y)y_p=0. \tag{$*^p$}$$

In particular for $p,l\in\{1,\dots,N+n\}$ we get

$$y_py_l=Z^*(F_pF_l/(F_y)^2). \tag{1}$$

Differentiating $(*^p)$ with respect to z_l and applying (α), we obtain

$$
\begin{aligned}
(2) \qquad 0 &= (Z^*F_p)_l + ((Z^*F_y)y_p)_l \\
&= Z^*F_{p,l} + Z^*F_{y,p}y_l + Z^*F_{y,l}y_p + Z^*F_{y^2}y_py_l + Z^*F_yy_{p,l}.
\end{aligned}
$$

This implies

$$Z^*F_{p,l} - Z^*(F_{y,p}F_l/F_y) - Z^*(F_{y,l}F_p/F_y) + Z^*F_yy_{p,l} + Z^*(F_{y^2}F_pF_l/F_y^2) = 0$$

(since $y_p = -Z^*(F_p/F_y), y_l = -Z^*(F_l/F_y)$) and

$$(3) \qquad y_{p,l} = Z^*((F_pF_l)_yF_y - F_{y^2}F_pF_l - F_{p,l}F_y^2)/F_y^3).$$

Since F is affine with respect to a_i and a_j, $F_{i,j} = 0$ for $i, j \leqslant N$, and (3) becomes

$$(4) \qquad y_{i,j} = Z^*((F_iF_j)_yF_y - F_{y^2}F_iF_j/F_y^3).$$

2) We define $y_0 = 1$ and denote by $\tilde{Z}$ the application $\tilde{Z}(x', a) = (x', y(x', a))$ (thus $f^{(q)} = (\tilde{Z}^*g)y_q$ for $q \in \{0, \dots, N+n\}$). Differentiating f with respect to a_i and a_j we obtain $f_i^{(q)} = \tilde{Z}^*g_yy_iy_q + \tilde{Z}^*gy_{q,i}$ and $f_{i,j}^{(q)} = (\tilde{Z}^*g_y)_jy_iy_q + \tilde{Z}^*g_yy_{i,j}y_q + \tilde{Z}^*g_yy_iy_{q,j} + (\tilde{Z}^*g)_jy_{q,i} + \tilde{Z}^*gy_{q,i,j} = \tilde{Z}^*g_{y^2}y_jy_iy_q + \tilde{Z}^*g_yy_{i,j}y_q + \tilde{Z}^*g_yy_iy_{q,j} + \tilde{Z}^*g_yy_jy_{q,i} + \tilde{Z}^*gy_{q,i,j}$.

Since the application F is affine with respect to (a_i, a_j, a_h, a_k), we have $\frac{\partial F}{\partial a_s} = c_s$ for $s = i, j, h, k$. The hypothesis and the equality (1) give $\tilde{Z}^*g_{y^2}y_jy_iy_q = \tilde{Z}^*g_{y^2}y_hy_ky_q$, $\tilde{Z}^*g_yy_iy_{q,j} + \tilde{Z}^*g_yy_jy_{q,i} = \tilde{Z}^*g_y(y_iy_j)_q = \tilde{Z}^*g_y(y_hy_k)_q$, from (4) we obtain $\tilde{Z}^*g_yy_{i,j}y_q = \tilde{Z}^*g_yy_{h,k}y_q$ and since $y_{q,i,j} = (y_{i,j})_q = (y_{h,k})_q$,we have $\tilde{Z}^*gy_{q,i,j} = \tilde{Z}^*gy_{q,h,k}$. The above relations give

$$(a_{i,j,h,k}) \qquad \partial^2 f^{(q)}/\partial a_i\partial a_j = \partial^2 f^{(q)}/\partial a_h\partial a_k$$

for any $q \in \{0, \dots, N\}$ and the lemma is proved. □

The above lemma is all what is needed to show that the functions in the image of the transformations have satisfy a system of PDE's. Let us describe them now. Recall that $C_{n-1}^d(Y_0) \simeq \mathbb{C}^N$. We will use homogeneous coordinates A_I on $C_{n-1}^d(Y)$, the set W of all of them is given by $W = \mathbb{C}^{N+1} \setminus \{A_0 = 0\}$. Holomorphic functions on $C_{n-1}^d(Y)$ are represented by 0-homogeneous holomorphic functions on W.

Let us now consider the space $\mathcal{O}(W)$ of all holomorphic functions on W (not necessarily homogenous) and let us define S_d to be the following system of partial differential equations for $f \in \mathcal{O}(W)$:

$$(A_{H,K}) \qquad \left[\frac{\partial^2}{\partial A_0\partial A_I} - \frac{\partial^2}{\partial A_H\partial A_K}\right] f = 0$$

where I, H, K describe n-tuples in $\{0,\dots,d\}^n$ of length $\leqslant d$ with $I = H + K$. The space of all holomorphic functions of degree k on W satisfying the system S_d will be denoted by $\mathcal{S}_d(k)$. We will also use the symbol $\mathcal{S}_{d,h^\infty}(k)$ for the space of functions f vanishing at the point $d\cdot h^\infty \in C^d_{n-1}(Y)$.

Let U be an open subset in Y; let us define G_U by $G_U = \{(x,c)\in U\times C^d_{n-1}(Y);\ x\in c\subset U\}$ and let i_{G_U} be the canonical injection of G_U into $U\times C^d_{n-1}(U)$ and π_U (respectively π'_U) the restriction of π (π') to G_U. To simplify the notation, we set $V = C^d_{n-1}(U)$. We will denote the solutions of S_d belonging to $\mathcal{O}_V$ (respectively to $\mathcal{O}_V(k)$, to $\mathcal{O}_V(k)$ and vanishing at h^∞) by $\mathcal{S}_U$ ($\mathcal{S}_U(k)$, $\mathcal{S}_{U,h^\infty}(k)$).

Notations. As usual if $U = Y$, we omit the index U and V in the above symbols.

Corollary 4.2. *The following inclusions holds:* $\operatorname{Im}\varrho^0_U \subset \mathcal{S}_U(0)$ *and* $\operatorname{Im}\mathcal{R}^1_U \subset \mathcal{S}_{U,h^\infty}(-1)$.

Proof. Let $(x^1,\dots,x^n)$, $x_i = z_i/z_0$ be the system of coordinates in $\mathbf{P}_n(\mathbb{C})\setminus h^\infty \cong \mathbb{C}^n$, then any differential form φ of type $(n-1,n-1)$ can be written as $\varphi = \sum\limits_{i,j=1}^{n} \varphi_{i,j}\,\mathrm{d}x^{\hat{\imath}}\wedge \mathrm{d}\bar{x}^{\hat{\jmath}}$, where $\varphi_{i,j}$ are smooth functions on U and the hat means that the corresponding differential is missing. For any $c_A\in V$, we denote by $F_A(x) = \sum\limits_{|I|\leqslant d}^{N} A_I x^I = 0$ a polynomial defining $c_A\cap(\mathbf{P}_n(\mathbb{C})\setminus h^\infty)$.

Let us first assume that c_A has no multiple component. Let us consider the set

$$W' = \{(x,A)\in G_U; (\partial F_A/\partial x^n)(x,A)\neq 0\},$$

where A stands for $(A_0,\dots,A_N)$; there is a covering of W' by open polydiscs (W_α) such that in every W_α we have $x\in c_A$ iff $x^n = y(x^1,\dots,x^{n-1},A)$; let W'_α be the projection of W_α onto the first $(n-1)$ coordinates. The restriction of the projection p to $c_A\cap W'_\alpha$ is injective, let us denote its inverse by h_A. Let us also choose a partition of unity (ψ_α) for (W_α).

i) In every W'_α, we can write

$$h^*_A\varphi = G\cdot \mathrm{d}x^1\wedge\ldots\wedge \mathrm{d}x^{n-1}\wedge \mathrm{d}\bar{x}^1\wedge\ldots\wedge \mathrm{d}\bar{x}^{n-1},$$

where

$$G = \Big[\sum_{i=1,j}^{n-1}(-1)^{2n-i-j}(h^*_A\varphi_{i,j})(\partial y/\partial x^i)(\partial\bar{y}/\partial\bar{x}^j) + \sum_{i=1}^{n-1}(-1)^{n-i}h^*_A\varphi_{i,n}(\partial y/\partial x^j)$$
$$+\sum_{j=1}^{n-1}(-1)^{n-i}h^*_A\varphi_{n,j}(\partial\bar{y}/\partial\bar{x}^j) + h^*_A\varphi_{n,n}\Big].$$

The polynomial F_A is linear with respect to A and for any 4-tuple $(I, J, H, K) \in \{0, \ldots, N\}^4$, the coefficients of A_I, A_J, A_H, A_K are respectively (x^I, x^J, x^H, x^K) and the hypotheses of Lemma 1 are fulfided for $I + J = H + K$. Let us denote by g_α the function defined on V by $g_\alpha(A) = \int_{c_A \cap \pi(W'_\alpha)} \psi_\alpha \varphi$; by applying Lemma 1 and differentiating under the integral we obtain

$$\partial^2 g_\alpha / \partial A_I \partial A_J = \partial^2 g_\alpha / \partial A_H \partial A_K$$

for $I + J = H + K$. By linearity, this is still true for the function $A \to \int_{c_A \cap \pi(W')} \varphi$; since $U \setminus \pi(W')$ is negligible this is still true for the function $\varrho_d^0 \varphi(c_A) = \int_{c_A} \varphi$. By continuity this is true for any cycle $c \in V$ (even with a multiple component) and the first part of the corollary is proved.

ii) The case of the Radon transformation is similar. Let Φ be a $\bar{\partial}$-closed $(n, n-1)$-form representing the corresponding class of cohomology. It can be written as $\Phi = \Phi' \wedge \mathrm{d}x_n$, where Φ' is an $(n-1, n-1)$-form. According to the definition, we have first to compute the residue $\mathrm{res}_c(\Phi(\tilde{h}^\infty)^d / F_A$. It means that we have to find a form ν satisfying the relation

$$\mathrm{d}F_A / F_A \wedge \nu = ((x^0)^d \Phi' \wedge \mathrm{d}x_n) / F_A.$$

We can take $\nu = ((x^0)^d \Phi') / (\partial F_A / \partial x_n)$. The function $1/(\partial F_A / \partial x_n)$ can be (locally) expressed as $1/(\partial F_A / \partial x_n) = -(\partial y / \partial A_I) \cdot (1/(\partial F_A / \partial A_I))$, where $1/(\partial F_A / \partial A_I)$ is independent of A and we are then reduced to the case i). $\square$

Remark. In the proof of the corollary we obtained a system consisting of the equations $(A_{I,J,H,K})$ with $I + J = H + K$; but as we will see, there are many redundant equations and this system is in fact equivalent to S_U.

With the same notation we have

Proposition 4.3. *Let f be in $\mathcal{S}_U(k)$; if V is connected, then f is completely determined by its values on the subspace $V^\infty = C_{n-1}^{d,\infty}(Y) \cap V = \{c \in V;\ c = (d-1)h^\infty + h, h \in C_{n-1}^1(U)\}$. More precisely, if the restriction of f to V^∞ is zero then f is identically equal to zero.*

Remark. Roughly speaking, the proposition states that for any f, g in $\mathcal{S}_U(k)$ such that their restrictions on $C_{n-1}^1(Y)_{|U}$ of compact hyperplanes on V are equal then they are equal everywhere (this is exact modulo the identification $C_{n-1}^1(Y)$ and $C_{n-1}^{d,\infty}(Y) \subset C_{n-1}^d(Y)$ given by the application i_d as defined in Section 3.2).

Proof. Ordering the indices I by the anti-lexicographic order, the subspace $C_{n-1}^{d,\infty}(Y)$ is given by the equations

$$A_I = 0, \quad I \neq 0, 1_1, \ldots, 1_n. \tag{1}$$

We will use the ordering on the n-tuples defined in 3.1.

1) First let us assume that V is a polydisc in $\mathbb{C}^N$ centered at 0 of polyradius $(r,\dots,r)$. Since f is an entire homogeneous function of degree k in homogeneous coordinates on $C^d_{n-1}(Y)$, we can write $f(A)=\sum_\alpha C_\alpha A^\alpha/A_{\mathbf{0}}^{|\alpha|-k}$ where α is a multi-index in $\mathbb{N}^N$ and the coefficients $C_\alpha = C_{\alpha_1,\dots,\alpha_N}\in\mathbb{C}$ verify

$$\lim_{|\alpha|\to\infty}(|C_\alpha|)^{1/|\alpha|}<1/r. \tag{2}$$

We note that an element of V belongs to V^∞ iff it verifies (1) and (2), thus V^∞ is again a polydisc in $C^{d,\infty}_{n-1}(Y)$.

The proof is a simple recursivity on α ordered by the anti-lexicographic order; we have to show that if $C_\alpha=0$ for any $\alpha<\mathbf{1}_{n+1}$ then $C_\alpha=0$ for any α. First, we can write

$$\frac{\partial^2 f}{\partial A_0\partial A_I}=\sum_{\alpha_I\geqslant 1}(k-|\alpha|)\alpha_I C_\alpha\frac{A^{\alpha-\mathbf{1}_I}}{A_0^{|\alpha|-k+1}}$$

and for $H,K\neq\mathbf{0}$ we get if $H\neq K$

$$\frac{\partial^2 f}{\partial A_H\partial A_K}=\sum_{\alpha_H,\alpha_K\geqslant 1}\alpha_H\alpha_K C_\alpha\frac{A^{\alpha-\mathbf{1}_H-\mathbf{1}_K}}{A_0^{|\alpha|-k}}$$

and if $H=K$

$$\frac{\partial^2 f}{\partial A_H^2}=\sum_{\alpha_H\geqslant 2}\alpha_H(\alpha_H-1)C_\alpha\frac{A^{\alpha-\mathbf{2}_H}}{A_0^{|\alpha|-k}}.$$

When f belongs to $\mathcal{S}_d(k)$ we have for any $\alpha\geqslant 1_I$: if $H\neq K$ and $I=H+K$ (as n-tuples)

$$C_\alpha=\big((\alpha_H+1)(\alpha_K+1)/((k-|\alpha|)\alpha_I)\big)C_{\alpha-\mathbf{1}_I+\mathbf{1}_H+\mathbf{1}_K} \tag{1}$$

if $I=H+H=2H$ (as n-tuples)

$$C_\alpha=\big((\alpha_H+1)\alpha_H/((k-|\alpha|)\alpha_I)\big)C_{\alpha-\mathbf{1}_I+\mathbf{2}_H}. \tag{2}$$

Now if $l(I)>n$ (i.e. $I>\mathbf{1}_n$) we can find $H,K\in\mathbb{N}^n\setminus\{\mathbf{0}\}$ s.t. $I=H+K$ and clearly for the order defined above, we have $H<I$ and $K<I$; for the anti-lexicographic order, we have $\beta=\alpha-\mathbf{1}_I+\mathbf{1}_H+\mathbf{1}_K<\alpha$ and $\gamma=\alpha-\mathbf{1}_I+\mathbf{2}_H<\alpha$; then by recursivity we can compute C_α from the coefficients C_δ with $\delta<\mathbf{1}_{n+1}$. Since C_α in (1) and (2) depends linearly of C_β and C_γ, when all the coefficients C_δ vanish, so are C_α for any α and the proposition is proved for V being a polydisc.

2) Let V be open in $\mathbb{C}^N$, up to a translation we can always assume $0\in V$, and V contains a polydisc V' centered at 0 of polyradius $(r,\dots,r)$ for r small enough. In V' if $f_{|V'\cap V^\infty}=0$ then $f=0$ on all V'; since V is connected it vanishes identically on V and the proposition is proved. □

Proposition 5.1. *Let $Y \subset \mathbf{P}_n(\mathbb{C})$ be a proper open linearly concave subset in $\mathbf{P}_n(\mathbb{C})$ such that for every $y \in Y$ the set of the compact hyperplanes in Y containing y is contractible in $C^1_{n-1}(Y)$.*

Then:

i) *The application $\mathcal{R}^1_d : H^{n-1}(Y, \Omega^n) \longrightarrow \mathcal{S}_{d,h^\infty}(-1)$ is an isomorphism.*

ii) *If moreover $H^1(\mathbf{P}_n(\mathbb{C}) \setminus Y) = 0$, then the sequence*

$$H^{n-1}(Y, \Omega^{n-2}) \xrightarrow{\mathrm{d}} H^{n-1}(Y, \Omega^{n-1}) \xrightarrow{\varrho^0_d} \mathcal{S}_d \longrightarrow 0$$

is exact.

Proof. 1) By additivity, for any $\dot\varphi \in H^{n-1}(Y, \Omega^{n-1})$, $\dot\psi \in H^{n-1}(Y, \Omega^n)$ and any hyperplane $h \in C^1_{n-1}(Y)$, we have $i^*_d \varrho^0_d \varphi(h) = \varrho^0_1 \varphi(h) + (d-1)\varrho^0_d \dot\varphi(h^\infty)$ and

$$\int_{i_d(h)} \mathrm{res}_{i_d(h)}\Big(\psi \cdot \frac{(z^0)^d}{l \cdot (z^0)^{d-1}}\Big) = \int_h \mathrm{res}_h\Big(\psi \cdot \frac{z^0}{l}\Big) + \mathrm{res}_{h^\infty}(\psi) = \int_h \mathrm{res}_h\Big(\psi \cdot \frac{z^0}{l}\Big),$$

where l is a homogeneous equation of h. We obtain $i^*_d \mathcal{R}^1_d = \mathcal{R}^1_1$ and $i^*_d \varrho^0_d \dot\varphi = \varrho^0_d \dot\varphi + C(\dot\varphi)$, where $C(\dot\varphi)$ is a constant (independent of the cycle $c \in C^d_{n-1}(Y)$).

2) i) From the first part of Proposition 3.2, we have $\mathcal{R}^1_1 = i^*_d \mathcal{R}^1_d$ is injective and $\mathcal{R}^1_d$ is determined by its restriction to $C^{d,\infty}_{n-1}(Y)$.

ii) Let $\dot\varphi \in H^{n-1}(Y, \Omega^{n-1})$ be in $\ker \varrho^0_d$; from Theorem IV.4.4 and Lemma I.2.2 of [O4] the form $d\dot\varphi$ belongs to $\ker \mathcal{R}^1_d$; the injectivity of $\mathcal{R}^1_1$ and $\mathcal{R}^1_d$ implies

$$\dot\varphi \in \ker(H^{n-1}(Y, \Omega^{n-1}) \xrightarrow{d} H^{n-1}(Y, \Omega^n))$$

and by Lemma 3.1, there is $\lambda \in \mathbb{C}$ such that $\dot\varphi - \lambda\dot\omega^{n-1} \in d(H^{n-1}(Y, \Omega^{n-2})$; Lelong's theorem gives the inclusion $\mathrm{d}H^{n-1}(Y, \Omega^{n-2}) \subset \ker \varrho^0_d \subset \ker \varrho^0_1$. But $\varrho^0_d(\dot\omega^{n-1})$ is constant equal to d, thus $\lambda = 0$ and finally we obtain $\ker \varrho^0_1 = \ker \varrho^0_d = \mathrm{d}H^{n-1}(Y, \Omega^{n-2})$. This proves the exactness of the first part of the sequence of ii).

3) i) Corollary 4.2 implies in particular the inclusion $\mathrm{Im}\, \mathcal{R}^1_d \subset \mathcal{S}_{d,h^\infty}(-1)$. Conversely, let f be a function in $\mathcal{S}_{d,h^\infty}(-1)$ and f^1 its restriction to $C^1_{n-1}(Y)$; since $\mathcal{R}^1_1$ is bijective (Proposition 3.2), there exists $g^1 \in H^0(C^1_{n-1}(Y), \mathcal{O}_{(h^\infty)}(-1)) = \mathrm{Im}\, \varrho^0_1$ s.t. $f^1 = g^1$; but g^1 belongs to $\mathrm{Im}\, \varrho^0_1$, so there is $g \in \mathrm{Im}\, \varrho^0_d$ s.t. $g^1 = i^*_d g$. Now i^*_d is an isomorphism (Corollary 3.3) and we obtain $g = f$. This proves the surjectivity of $\mathcal{R}^1_d$.

ii) The proof of the surjectivity of ϱ^0_d is the same as for $\mathcal{R}^1_d$. ☐

In this last section, we will see how it is possible to expand the previous construction beyond the analytical category. As we showed in [O2], the natural object for the Andreotti-Norguet Transformation are the spaces of $\partial\bar{\partial}$-cohomology. More precisely, we introduce

Notations. Let $\mathcal{A}^{*,*}$ be the sheaves of germs of $\mathcal{C}^\infty$ differential forms on Y; we define the $\partial\bar{\partial}$-cohomology spaces by

$$V^{*,*}(Y) = \frac{\ker(\mathcal{A}^{*,*}(Y) \xrightarrow{\partial\bar{\partial}} \mathcal{A}^{*,*}(Y))}{\partial\mathcal{A}^{*,*}(Y) + \bar{\partial}\mathcal{A}^{*,*}(Y)} \quad \text{and} \quad \Lambda^{*,*}(Y) = \frac{\ker(\mathcal{A}^{*,*}(Y) \xrightarrow{d} \mathcal{A}^{*,*}(Y))}{\partial\bar{\partial}\mathcal{A}^{*,*}(Y)}.$$

The same definitions hold if we replace the sheaves $\mathcal{A}^{*,*}$ by the sheaves of germs of currents or hyperforms and thus we can define $\partial\bar{\partial}$-cohomology spaces with compact supports or with support in a closed subset of Y.

Since the transformation ϱ_d^0 commutes with the anti-automorphism $z \to \bar{z}$, by symmetry for a $\partial\bar{\partial}$-closed differential form $\varphi \in \mathcal{A}^{n-1,n-1}(Y)$, the application $\check{\varrho}^0\varphi(c) = \int_c \varphi$ $(c \in C_{n-1}^d(Y))$ is a pluriharmonic function on $C_{n-1}^d(Y)$. Using once more Lelong's theorem we get $\check{\varrho}^0\varphi = 0$ when φ is $\partial-$ or $\bar{\partial}$-closed; so we can extend the definition of ϱ_d^0 to a transformation $\tilde{\varrho}_d^0\colon V^{n-1,n-1}(Y) \to H^0(C_{n-1}^d(Y), \mathcal{H})$ where $\mathcal{H}$ is the sheaf of germs of pluriharmonic functions (for the precise construction see [O2]).

First of all we need a relation between the $\bar{\partial}$ and $\partial\bar{\partial}$-cohomologies.

Lemma 6.1. *For any connected open subset U in $\mathbf{P}_n(\mathbb{C})$ the natural morphism*

$$H^{n-1}(U, \Omega^{n-1}) \oplus H^{n-1}(U, \bar{\Omega}^{n-1}) \to V^{n-1,n-1}(U)$$

is surjective.

P r o o f. First, let us note that if $U = \mathbf{P}_n(\mathbb{C})$, then $V^{n-1,n-1}(U) = H^{n-1,n-1(U})$ (cf. [O2]) and the result is trivial. We can thus suppose U is proper (i.e. noncompact) in $\mathbf{P}_n(\mathbb{C})$.

1) Let X be the (compact) set $X = \mathbf{P}_n(\mathbb{C}) \setminus U$. The long cohomology sequence with compact supports for the constant sheaf $\mathbb{C}$ gives

$$0 \longrightarrow H^0(\mathbf{P}_n(\mathbb{C}), \mathbb{C}) \longrightarrow H^0(X, \mathbb{C}) \longrightarrow H^1_{CO}(\mathbf{P}_n(\mathbb{C}) \operatorname{Mod} X, \mathbb{C}) \longrightarrow H^1(\mathbf{P}_n(\mathbb{C}), \mathbb{C}).$$

Since $\dim H^0(\mathbf{P}_n(\mathbb{C}), \mathbb{C}) = \dim H^0(X, \mathbb{C}) = 1$ the first morphism is an isomorphism and the second is injective, thus $H^1_{CO}(\mathbf{P}_n(\mathbb{C}) \operatorname{Mod} X, \mathbb{C}) = H^1(\mathbf{P}_n(\mathbb{C}), \mathbb{C}) = 0$; by duality and the natural indentification $H^*(\mathbf{P}_n(\mathbb{C})\,\mathrm{Mod}\,X, \mathbb{C}) = H^*(U, \mathbb{C})$ we obtain

$$H^{2n-1}(U, \mathbb{C}) = 0. \tag{1}$$

2) Following [A-N2] or [O2], we have a resolution $(\tilde{\mathcal{A}}^i, \tilde{\mathrm{d}}_i)$ (the Bigolin resolution) of $\mathcal{H}$ where $\tilde{\mathcal{A}}^i = \overline{\Omega}^{i+1} \oplus \bigoplus_{j=0}^{i} \mathcal{A}^{j,i-j} \oplus \Omega^{i+1}$ for $0 \leqslant i \leqslant n-2$, $\tilde{\mathcal{A}}^{n-1+i} = \bigoplus_{j=i}^{n-1} \mathcal{A}^{j,i-j+n-1}$ and $\tilde{\mathrm{d}}_i$ is the natural morphism from $\tilde{\mathcal{A}}^i$ to $\tilde{\mathcal{A}}^{i+1}$ induced by ∂, $\bar{\partial}$ and the inclusion $\Omega^i \subset \mathcal{A}^{i,0}$. The resolution can be written as the following exact sequence of sheaves:

$$\begin{aligned} 0 \to \mathcal{H} \longrightarrow \tilde{\mathcal{A}}^0 \xrightarrow{\tilde{\mathrm{d}}_0} \tilde{\mathcal{A}}^1 \ldots \longrightarrow \tilde{\mathcal{A}}^{2n-3} &= \mathcal{A}^{n-2,n-1} \oplus \mathcal{A}^{n-1,n-2} \xrightarrow{\partial+\bar{\partial}} \tilde{\mathcal{A}}^{2n-2} \\ &= \mathcal{A}^{n-1,n-1} \xrightarrow{\partial\bar{\partial}} \mathcal{A}^{n,n}. \end{aligned}$$

3) Let us denote $\tau^i = \ker \tilde{\mathrm{d}}_i$; in particular, we have $\tau^0 = \ker \tilde{\mathrm{d}}_0 = \mathcal{H}$ and $\tau^{2n-3} = \ker \tilde{d}_{2n-3} = \ker \mathcal{A}^{n-2,n-1} \oplus \mathcal{A}^{n-1,n-2} \xrightarrow{\partial+\bar{\partial}} \mathcal{A}^{n-1,n-1}$. Since $\tilde{\mathcal{A}}^{2n-3}$, $\tilde{\mathcal{A}}^{2n-2}$ and $\mathcal{A}^{n,n}$ are soft sheaves, we have

$$V^{n-1,n-1}(U) = H^1(U, \tau^{2n-3}) \tag{2}$$

and since for $i \geqslant n-1$ the sheaves $\tilde{\mathcal{A}}^i$ are soft,

$$V^{n-1,n-1}(U) = H^1(U, \tau^{2n-3}) = H^{n-1}(U, \tau^{n-1}) \tag{3}$$

and from $\tilde{\mathcal{A}}^{n-2} = \overline{\Omega}^{n-1} \oplus \mathcal{A}^{0,n-2} \ldots \oplus \mathcal{A}^{n-2,0} \oplus \Omega^{n-1}$, we obtain an exact cohomology sequence

$$\begin{aligned} H^{n-1}(U, \Omega^{n-1}) \oplus H^{n-1}(U, \overline{\Omega}^{n-1}) &\longrightarrow H^{n-1}(U, \tau^{n-1}) \\ = V^{n-1,n-1}(U) &\longrightarrow H^n(U, \tau^{n-2}). \end{aligned} \tag{4}$$

Now, for any $j \in \mathbb{N}$ and $k \geqslant n$, $H^k(U, \Omega^j) = H^k(U, \overline{\Omega}^j) = 0$ (U is noncompact) hence $H^k(U, \tilde{\mathcal{A}}^{n-2}) = 0$ ($k \geqslant n$) and we have

$$H^n(U, \tau^{n-2}) = H^{2n-2}(U, \mathcal{H}). \tag{5}$$

For the same reason $H^k(U, \mathcal{O}) = H^k(U, \overline{\mathcal{O}}) = 0$ ($k \geqslant n$) and we have an exact sequence of sheaves

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O} \oplus \overline{\mathcal{O}} \longrightarrow \mathcal{H} \longrightarrow 0. \tag{6}$$

For $n \geqslant 2$, we obtain that the natural morphism $H^{2n-2}(U, \mathcal{H}) \longrightarrow H^{2n-1}(U, \mathbb{C})$ is injective, and (1) gives $H^{2n-2}(U, \mathcal{H}) = 0$; by equality (4) we can complete the proof of the lemma.

For $n = 1$ we have $V^{0,0}(U) = H^0(U, \mathcal{H})$ and (6) gives the exactness of the sequence $H^0(U, \mathcal{O}) \oplus H^0(U, \overline{\mathcal{O}}) \longrightarrow H^0(U, \mathcal{H}) \longrightarrow H^1(U, \mathbb{C})$ and we can again complete the proof using (1). $\square$

Let $\tilde{S}_d = (S_d, \overline{S}_d)$ be the system of partial differential equations

$$(\tilde{A}_{H,K}) \qquad \frac{\partial^2}{\partial A_{\mathbf{0}}\partial A_I} - \frac{\partial^2}{\partial A_H \partial A_K} = \frac{\partial^2}{\partial \overline{A}_{\mathbf{0}}\partial \overline{A}_I} - \frac{\partial^2}{\partial \overline{A}_H \partial \overline{A}_K} = 0.$$

We denote by $\tilde{\mathcal{S}}_d$ the set of pluriharmonic functions on $C^d_{n-1}(Y)$ verifying the system $\tilde{S}_d$.

Corollary 6.2. *The transformation $\tilde{\varrho}^0_d$: $V^{n-1,n-1}(Y) \to \tilde{\mathcal{S}}_d = \mathcal{S}_d + \overline{\mathcal{S}}_d$ is an isomorphism.*

Proof. Let θ be an $(n-1, n-1)$-differential $\partial\bar{\partial}$-closed form on Y and $\tilde{\theta}$ its class in $V^{n-1,n-1}(Y)$. The above lemma implies that there exist two $(n-1, n-1)$-differential $\bar{\partial}$-closed forms φ_1, phi_2 on Y s.t. $\varphi_1 + \overline{\varphi}_2$ belongs to $\tilde{\theta}$.

i) From Proposition 5.5 we have $\operatorname{Im}\tilde{\varrho}^0_d \subset \operatorname{Im}\varrho^0_d \oplus \operatorname{Im}\bar{\varrho}^0_d = \mathcal{S}(C^d_{n-1}(Y)) + \overline{\mathcal{S}}(C^d_{n-1}(Y))$ and since the inverse inclusion is trivial the application $\operatorname{Im}\tilde{\varrho}^0_d \to \mathcal{S}(C^d_{n-1}(Y)) + \overline{\mathcal{S}}(C^d_{n-1}(Y))$ is surjective. Moreover, since $C^d_{n-1}(Y) = C^d_{n-1}(\mathbf{P}_n(\mathbb{C})) \setminus C^d_{n-1}(X)$ and $C^d_{n-1}(\mathbf{P}_n(\mathbb{C})) = \mathbf{P}_N(\mathbb{C})$, if $H^1(C_{n-1}(X)) = 0$, then the exact d-cohomology sequence implies $H^{2n-2}(C^d_{n-1}(Y)) = 0$ which in turn gives the equality $H^0(C^d_{n-1}(Y), \mathcal{H}) = (H^0(C^d_{n-1}(Y), \mathcal{O}) \oplus H^0(C^d_{n-1}(Y), \overline{\mathcal{O}})$ (as in the proof of Lemma 3.1). This immediately gives $\tilde{S}_d \subset \mathcal{S}(C^d_{n-1}(Y)) + \overline{\mathcal{S}}(C^d_{n-1}(Y))$ and since the inverse inclusion is trivial the equality is proved.

ii) We need to prove the injectivity of the morphism $\operatorname{Im}\tilde{\varrho}^0_d \to \mathcal{S}(C^d_{n-1}(Y)) + \overline{\mathcal{S}}(C^d_{n-1}(Y))$. Let us suppose θ belongs to $\ker\tilde{\varrho}^0_d$, then $\varrho^0_d\varphi_1 + \overline{\varrho^0_d}\varphi_2$ is identically equal to zero; since $\varrho^0_d\varphi_1$ (respectively $\overline{\varrho^0_d}\varphi_2$ is holomorphic (anti-holomorphic), both functions are constant and we can find $\lambda \in \mathbb{C}$ s.t. $\varphi_1, \varphi_2 \in \pm\lambda\dot{\omega}^{n-1} + \ker\varrho^0_d$; since $\tilde{\omega}^{n-1}$ does not belong to $\ker\tilde{\varrho}^0_d$ ($\tilde{\varrho}^0_d\tilde{\omega}^{n-1}$ is identically equal to d on $C^d_{n-1}(Y)$) we have $\lambda = 0$. The above part α) gives $\ker\tilde{\varrho}^0_d \subset \ker\varrho^0_d \oplus \ker\overline{\varrho^0_d} \subset dH^{n-1}(Y, \Omega^{n-2}) \oplus dH^{n-1}(Y, \overline{\Omega}^{n-2})$. Since the inverse inclusion is trivial the result is proved. $\square$

Remark. We have obtained partial differential equations satisfied by the integral transforms in homogeneous coordinates since they are simpler. Nevertheless, it is easy to get equations in $C^d_{n-1}(Y) \cong \mathbb{C}^{\binom{n+d}{n}-1}$ using the inhomogeneous coordinates $a_I = A_I/A_{\mathbf{0}}$, $I > \mathbf{0}$. From the Euler formula for homogeneous functions of degree 0 we have

$$\partial/\partial A_{\mathbf{0}} = -\sum_{I>\mathbf{0}} a_I \partial/\partial a_I$$

and the systems $\mathcal{S}_d$ and $\tilde{\mathcal{S}}_d$ are given respectively by

$$\frac{\partial}{\partial a_I} + \frac{\partial^2}{\partial a_H \partial a_K} + \sum_{J>\mathbf{0}} \frac{\partial^2}{\partial a_I \partial a_J} = 0$$

and

$$\frac{\partial}{\partial a_I} + \frac{\partial^2}{\partial a_H \partial a_K} + \sum_{J>\mathbf{0}} \frac{\partial^2}{\partial a_I \partial a_J} = \frac{\partial}{\partial \bar{a}_I} + \frac{\partial^2}{\partial \bar{a}_H \partial \bar{a}_K} + \sum_{J>\mathbf{0}} \frac{\partial^2}{\partial \bar{a}_I \partial \bar{a}_J} = 0$$

for $|I|$, $|H|$, $|K| \leqslant d$ and $H + K = I$.

Remark. The methods used here can be easily extended to other situations in two different ways.

1. Since Lemma 3 of 3.3 is true without any assumption of analyticity, we can extend the sufficient condition for the integral transform to the differential case. We can extend naturally the transform ϱ^0 to $\mathcal{C}^k$ differential forms ($k \in \{1, \ldots\} \cup \{\infty\}$) by setting $\check{\varrho}^0\varphi(c) = \int_c \varphi$ for any $c \in C^d_{n-1}(Y)$. Then $\check{\varrho}^0\varphi$ is again a function of class $\mathcal{C}^k$ on $C^d_{n-1}(Y)$. The condition that f is a solution of a system of differential equations $\tilde{S}_d$ is still a necessary condition for f to be in $\operatorname{Im} \check{\varrho}^0$.

2. Moreover, as usual the situation of $\mathbf{P}_n(\mathbb{C}) \setminus \{O\}$ is a good model for an open subset of $\mathbf{P}_n(\mathbb{C})$ with some hypothesis of convexity-concavity. Let us recall that an open subset Y in $\mathbf{P}_n(\mathbb{C})$ is called linearly concave (in the sense of Martineau ([M]),) if Y is the union of all compact hyperplanes contained in Y. Then let Y be such a domain verifying, for any point $y \in Y$, that the fiber $\pi^{-1}(y)$ is contractible (i.e. the set of compact hyperplanes in Y containing y is a contractible subset in $C^d_{n-1}(Y)$). Corollary 4 is local and the linearly concavity of Y immediately implies $C^1_{n-1}(Y)$ is connected; since $i_d(C^1_{n-1}(Y))$ intersects any connected component of $C^d_{n-1}(Y)$ (for Y is linearly concave), the corollary can be extended to this case. Since the claim i) in Proposition 1 is true (cf. [G-H]), so is the claim i) in Proposition 6.

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