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ON THE INDIVIDUAL ERGODIC THEOREM IN D-POSETS OF FUZZY SETS

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Abstract. Calculus for observables in a space of functions from an abstract set to the unit interval is developed and then the individual ergodic theorem is proved.

Keywords: individual ergodic theorem, d-posets of fuzzy sets, state and observable

MSC 2000: 28E10, 58F11

1. Introduction

We will consider a measurable space $(\Omega, \mathscr{S})$, where $\Omega \in \mathscr{S}$ and $\mathscr{F}$ is the family of all measurable functions $f\colon \Omega \to \langle 0,1\rangle$. In our quantum mechanics model ([5], [3]) there are two basic notions: state and observable.

A state is a mapping $m\colon \mathscr{F} \to \langle 0,1\rangle$ satisfying the following conditions:

(i) $m(1_\Omega) = 1$.

(ii) If $f, g, h \in \mathscr{F}$, $f = g + h$, then $m(f) = m(g) + m(h)$.

(iii) If $f_n \in \mathscr{F}$ $(n = 1, 2, \ldots)$, $f_n \nearrow f$, $f \in \mathscr{F}$, then $m(f_n) \nearrow m(f)$.

Of course, by a theorem of Butnariu and Klement ([1]) there is a probability measure $\mu\colon \mathscr{S} \to \langle 0,1\rangle$ such that

$$m(f) = \int_\Omega f \,\mathrm{d}\mu$$

for all $f \in \mathscr{F}$, hence our model coincides with the classical one. On the other hand, the notion of an observable gives a new point of view, new possibilities and new problems.

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An observable is a mapping $x\colon \mathscr{B}(\mathbb{R}) \to \mathscr{F}$ (where $\mathscr{B}(\mathbb{R})$ is the σ-algebra of all Borel subsets of $\mathbb{R}$) satisfying the following conditions:

(i) $x(\mathbb{R}) = 1_\Omega$.

(ii) If $A, B \in \mathscr{B}(\mathbb{R})$, $A \cap B = \emptyset$, then $x(A \cup B) = x(A) + x(B)$.

(iii) If $A_n \in \mathscr{B}(\mathbb{R})$, $A_n \subset A_{n+1}$ $(n = 1, 2, \ldots)$, then

$$x\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigvee_{n=1}^{\infty} x(A_n).$$

As an example of an observable a random variable on a probability space $(\Omega, \mathscr{S}, \mu)$ can be considered. If $\xi\colon \Omega \to \mathbb{R}$ is a random variable, then $x\colon \mathscr{B}(\mathbb{R}) \to \mathscr{F}$, defined by the formula $x(A) = \chi_{\xi^{-1}(A)}$, is an observable.

Let $m\colon \mathscr{F} \to \langle 0, 1\rangle$ be a state, $x\colon \mathscr{B}(\mathbb{R}) \to \mathscr{F}$ an observable. Then the composite mapping $m_x = m \circ x\colon \mathscr{B}(\mathbb{R}) \to \langle 0, 1\rangle$ is a probability measure. This notion corresponds to the notion of the probability distribution $\mu_\xi\colon \mathscr{B}(\mathbb{R}) \to \langle 0, 1\rangle$ of a random variable $\xi\colon (\Omega, \mathscr{S}, \mu) \to \mathbb{R}$. Indeed, μ_ξ is defined by the formula

$$\mu_\xi(A) = \mu\left(\xi^{-1}(A)\right).$$

On the other hand (in this classical case)

$$\begin{aligned} m_x(A) &= m\left(x(A)\right) = m\left(\chi_{\xi^{-1}(A)}\right) \\ &= \int_\Omega \chi_{\xi^{-1}(A)} d\mu = \mu\left(\xi^{-1}(A)\right). \end{aligned}$$

By help of the probability distribution m_x the mean value $E(x)$ can be defined. Namely, in the case of a random variable $\xi\colon \Omega \to \mathbb{R}$ the mean value $E(\xi)$ is defined as the integral $\int_\Omega \xi \, \mathrm{d}\mu$. But by the integral transformation formula we have

$$E(\xi) = \int_\Omega \xi \, \mathrm{d}\mu = \int_\mathbb{R} t \, \mathrm{d}\mu_\xi(t).$$

Therefore we define

$$E(x) = \int_\mathbb{R} t \, \mathrm{d}m_x(t),$$

if the integral exists. In this case we say that the observable x is integrable.

There are some results concerning the probability theory for observables and states in the fuzzy quantum model (e.g., the strong law of large numbers in [4]). In this paper the individual ergodic theorem will be formulated and proved.

2. Formulation

First we recall the classical ergodic theorem ([9], Th. 1.5). Let (X,σ,P) be a probability space, $T\colon X\to X$ a measure preserving transformation (i.e., $A\in\sigma\Rightarrow T^{-1}(A)\in\sigma$ and $P\left(T^{-1}(A)\right)=P(A)$), let $\xi\colon X\to\mathbb{R}$ be an integrable observable. Then there is an integrable observable ξ^* such that the following conditions are satisfied:

(i) $E(\xi)=E(\xi^*)$.

(ii) $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=0}^{n-1}\xi\circ T^i=\xi^*$ P-almost everywhere.

We have defined the mean value of an observable. We must define a state preserving transformation, operations with observables and the m-almost everywhere convergence of a sequence of observables.

A mapping $\tau\colon\mathscr{F}\to\mathscr{F}$ is called an m-preserving transformation if the following conditions are satisfied:

(i) $\tau(1_\Omega)=1_\Omega$.

(ii) If $f,g,h\in\mathscr{F}$, $f=g+h$, then $\tau(f)=\tau(g)+\tau(h)$.

(iii) If $f_n\in\mathscr{F}$ $(n=1,2,\dots)$, $f\in\mathscr{F}$, $f_n\nearrow f$, then $\tau(f_n)\nearrow\tau(f)$.

(iv) $\tau(f)\cdot\tau(g)=\tau(f\cdot g)$ and $m\big(\tau(f)\big)=m(f)$ for all $f,g\in\mathscr{F}$.

By a theorem of [7] for every observables $x_1,\dots,x_n$ there exists a mapping $h_n\colon\mathscr{B}(\mathbb{R}^n)\to\mathscr{F}$ satisfying the following conditions:

(i) $h_n(\mathbb{R}^n)=1_\Omega$.

(ii) If $A,B\in\mathscr{B}(\mathbb{R}^n)$, $A\cap B=\emptyset$, then $x(A\cup B)=x(A)+x(B)$.

(iii) If $A_i\in\mathscr{B}(\mathbb{R}^n)$, $A_i\subset A_{i+1}$ $(i=1,2,\dots)$, then

$$h_n\left(\bigcup_{i=1}^{\infty}A_i\right)=\bigvee_{i=1}^{\infty}h_n(A_i).$$

(iv) $h_n(A_1\times\dots\times A_n)=x_1(A_1)\cdot\ldots\cdot x_n(A_n)$ for every

$$A_1,\dots,A_n\in\mathscr{B}(\mathbb{R}).$$

The function h_n is called the joint observable of observables $x_1,\dots,x_n$. By the help of the joint observable h_n some operations can be defined. If $g\colon\mathbb{R}^n\to\mathbb{R}$ is a Borel measurable function, then we define a mapping $g(x_1,\dots,x_n)\colon\mathscr{B}(\mathbb{R})\to\mathscr{F}$ by the formula

$$g(x_1,\dots,x_n)(A)=h_n\left(g^{-1}(A)\right),\quad A\in\mathscr{B}(\mathbb{R}).$$

The motivation is the following. If $(\xi_1, \dots \xi_n) = U$ is a random vector then $g(\xi_1, \dots, \xi_n) = g \circ U$ is a random variable and

$$(g \circ U)^{-1}(A) = U^{-1}\left(g^{-1}(A)\right).$$

In the general situation $U^{-1}\colon \mathscr{B}(\mathbb{R}^n) \to \mathscr{S}$ induces the joint distribution $h_n(A) = \chi_{U^{-1}(A)}$.

Finally, we shall define the m-almost everywhere convergence of a sequence of observables. If $(y_n)_n$ is a sequence of observables, we say that $\limsup\limits_{n\to\infty} y_n$ exists, if there exists an observable $\bar{y}$ such that

$$\begin{aligned} m\big(\bar{y}\big((-\infty,t)\big)\big) &= \lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} m\Big(\bigwedge_{n=k}^{k+i} y_n\Big(\Big(-\infty, t-\frac{1}{p}\Big\rangle\Big)\Big) \\ &= m\Big(\bigvee_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} y_n\Big(\Big(-\infty, t-\frac{1}{p}\Big\rangle\Big)\Big) \end{aligned}$$

for every $t \in \mathbb{R}$.

We say that $\liminf\limits_{n\to\infty} y_n$ exists, if there exists an observable $\underline{y}$ such that

$$\begin{aligned} m\big(\underline{y}\big((-\infty,t)\big)\big) &= \lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} m\Big(\bigvee_{n=k}^{k+1} y_n\Big(\Big(-\infty, t-\frac{1}{p}\Big\rangle\Big)\Big) \\ &= m\Big(\bigvee_{p=1}^{\infty}\bigwedge_{k=1}^{\infty}\bigvee_{n=k}^{\infty} y_n\Big(\Big(-\infty, t-\frac{1}{p}\Big\rangle\Big)\Big) \end{aligned}$$

for every $t \in \mathbb{R}$.

We say that a sequence $(y_n)_n$ of observables converges m-almost everywhere to an observable y, if $\limsup\limits_{n\to\infty} y_n = \bar{y}$ and $\liminf\limits_{n\to\infty} y_n = \underline{y}$ exist and

$$m((\underline{y}(-\infty,t))) = m((\bar{y}(-\infty,t))) = m((y(-\infty,t)))$$

for every $t \in \mathbb{R}$.

The main result of the paper is contained in the following theorem.

Theorem 1. *Let x be an integrable observable, τ an m-preserving transformation. Then there is an integrable observable x^* satisfying the following conditions:*

(i) $E(x^*) = E(x)$.

(ii) $\frac{1}{n}\sum\limits_{i=0}^{n-1} \tau^i \circ x \to x^*$ m-a.e.

3. Proof

The main idea of the proof consists in the construction of a probability space and an application of the classical individual ergodic theorem.

Let $x_n (n = 1, 2, \ldots)$ be the observable defined by the formula $x_n = \tau^{n-1} \circ x$. Let $\mathbb{N}$ be the set of all positive integers, $\emptyset \neq J \subset \mathbb{N}$, J finite, $J = \{j_1, \ldots, j_k\}$. Then we define a probability measure $P_J\colon \mathscr{B}(\mathbb{R}^{|J|}) \to \langle 0, 1\rangle$ determined by the formula

$$P_J(A_1 \times \ldots \times A_k) = m\left(x_{j_1}(A_1) \cdot \ldots \cdot x_{j_k}(A_k)\right),$$

$A_1, \ldots, A_n \in \mathscr{B}(\mathbb{R})$. It is not difficult to prove that the family

$$\{P_J;\ J \subset \mathbb{N},\ J \neq \emptyset,\ J \text{ finite}\}$$

is a consistent system of probability measures. That is, if $J_1 \subset J_2$, $J_1 \neq \emptyset$, J_2 is finite and $\pi_{J_2,J_1}\colon\ \mathbb{R}^{|J_2|} \to \mathbb{R}^{|J_1|}$ is the projection, then

$$P_{J_1}(A) = P_{J_2}\left(\pi^{-1}_{J_2,J_1}(A)\right)$$

for every $A \in \mathscr{B}(\mathbb{R}^{|J_1|})$. Therefore the Kolmogorov theorem is applicable. Denote by $\mathscr{V}$ the family of all cylinders $B \subset \mathbb{R}^{\mathbb{N}}$, i.e. the sets of the form

$$\pi_J^{-1}(A) = \big\{(x_n)_n;\ (x_{j_1}, \ldots, x_{j_k}) \in A\big\},$$

where $A \in \mathscr{B}(\mathbb{R}^{|J|})$, $J \neq \emptyset$, $J \subset \mathbb{N}$, J finite and $\pi_J\colon\ \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{|J|}$ is the projection. If $\sigma(\mathscr{V})$ is the σ-algebra generated by $\mathscr{V}$, then there exists exactly one probability measure P such that

$$P\left(\pi_J^{-1}(A)\right) = P_J(A)$$

for every $\pi_J^{-1}(A) \in \mathscr{V}$.

Proposition 1. *Let $T\colon\ \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ be the transformation defined by the formula $T\left((t_n)_n\right) = (s_n)_n$, where $s_n = t_{n+1}$ $(n = 1, 2, \ldots)$. Then T preserves the probability measure P, i.e. $P(A) = P\left(T^{-1}(A)\right)$ for every $A \in \sigma(\mathscr{V})$.*

P r o o f. It is sufficient to prove the equality $P(A) = P\left(T^{-1}(A)\right)$ for sets of the form $A = \pi_J^{-1}(B)$, where B is the product of $k = |J|$ sets of $\mathscr{B}(\mathbb{R})$. Let

$J = \{i_1, \dots, i_k\}$, $B = B_1 \times \dots \times B_k$, $J_1 = \{i_1 + 1, \dots, i_k + 1\}$. Then

$$\begin{aligned}
P\left(T^{-1}(A)\right) &= P\bigl(T^{-1}\left(\pi_J^{-1}(B_1 \times \dots \times B_k)\right)\bigr) \\
&= P\bigl(T^{-1}\left(\{(s_n)_n;\ s_{i_1} \in B_1, \dots, s_{i_k} \in B_k\}\right)\bigr) \\
&= P\bigl(\bigl\{(t_n)_n;\ t_{i_1+1} \in B_1, \dots, t_{i_k+1} \in B_k\bigr\}\bigr) \\
&= P_{J_1}\bigl(\bigl\{(t_{i_1+1}, \dots, t_{i_k+1});\ t_{i_1+1} \in B_1, \dots, t_{i_k+1} \in B_k\bigr\}\bigr) \\
&= m\left(x_{i_1+1}(B_1) \cdot \ldots \cdot x_{i_k+1}(B_k)\right) \\
&= m\bigl(\tau^{i_1}\left(x(B_1)\right) \cdot \ldots \cdot \tau^{i_k}\left(x(B_k)\right)\bigr) \\
&= m\bigl(\tau^{i_1-1}\left(x(B_1)\right) \cdot \ldots \cdot \tau^{i_k-1}\left(x(B_k)\right)\bigr) \\
&= m\left(x_{i_1}(B_1) \cdot \ldots \cdot x_{i_k}(B_k)\right) \\
&= P_J(B_1 \times \dots \times B_k) = P_J(B) \\
&= P\left(\pi_J^{-1}(B)\right) = P(A).
\end{aligned}$$

□

Proposition 2. *Let $\xi_n\colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ be the projection defined by the formula $\xi_n\left((t_i)_i\right) = t_n$. Then for every $n \in \mathbb{N}$, ξ_n is a random variable, $P\left(\xi_n^{-1}(A)\right) = m\left(x_n(A)\right)$, $A \in \mathscr{B}(\mathbb{R})$ and $P\Bigl(\Bigl\{(u_i)_i;\ \frac{1}{n}\sum\limits_{i=1}^{n} \xi_i(u) < t\Bigr\}\Bigr) = m\Bigl(\Bigl(\frac{1}{n}\sum\limits_{i=1}^{n} x_i\Bigr)\left((-\infty, t)\right)\Bigr)$, $t \in \mathbb{R}$.*

P r o o f. If $A \in \mathscr{B}(\mathbb{R})$, then $\xi_n^{-1}(A) = \bigl\{(t_i)_i;\ t_n \in A\bigr\} = \pi_{\{n\}}^{-1}(A) \in \mathscr{V} \subset \sigma(\mathscr{V})$, hence $\xi_n\colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is a random variable. Moreover,

$$\begin{aligned}
P\left(\xi_n^{-1}(A)\right) &= P\bigl(\bigl\{(t_i)_i;\ t_n \in A\bigr\}\bigr) = P\left(\pi_{\{n\}}^{-1}(A)\right) \\
&= P_{\{n\}}(A) = m\left(x_n(A)\right).
\end{aligned}$$

Finally, let $J_n = \{1, \dots, n\}$. Then

$$\begin{aligned}
P\left(\left\{(u_i)_i; \frac{1}{n}\sum_{i=1}^{n} \xi_i(u) < t\right\}\right) &= P_{J_n}\left(\left\{(u_1, \dots, u_n);\ \frac{1}{n}\sum_{i=1}^{n} u_i < t\right\}\right) \\
&= m \circ h_n\left(\left\{(u_1, \dots, u_n);\ \frac{1}{n}\sum_{i=1}^{n} u_i < t\right\}\right).
\end{aligned}$$

If we put $g(u_1, \dots, u_n) = \frac{1}{n}\sum\limits_{i=1}^{n} u_i$, then by definition

$$\frac{1}{n}\sum_{i=1}^{n} x_i = g(x_1, \dots, x_n) = h_n \circ g^{-1},$$

hence

$$P\Big(\Big\{(u_i)_i\,;\ \frac{1}{n}\sum_{i=1}^{n}\xi_i(u)<t\Big\}\Big)=P\big(g^{-1}((-\infty,t))\big)$$
$$=m\big(h_n\big(g^{-1}((-\infty,t))\big)\big)=m\Big(\Big(\frac{1}{n}\sum_{i=1}^{n}x_i\Big)((-\infty,t))\Big).$$

□

Proposition 3. *There exists an observable x^* such that $\frac{1}{n}\sum_{i=0}^{n-1}\tau^i\circ x\to x^*$ m-a.e.*

P r o o f. We have proved that $\xi_1\colon\ \mathbb{R}^{\mathbb{N}}\to\mathbb{R}$ is a random variable (with respect to $\sigma(\mathscr{V})$) and ξ_1 and x have the same probability distribution defined by the formula

$$P_{\xi_1}(A)=P\left(\xi_1^{-1}(A)\right)=m\left(x_1(A)\right)=m\left(x(A)\right)=m_x(A).$$

Since x is integrable (i.e. $\int_{\mathbb{R}}t\,\mathrm{d}m_x(t)$ exists), ξ_1 is integrable, too. Therefore, by the individual ergodic theorem, there exists an integrable, invariant random variable ξ^* such that

$$\frac{1}{n}\sum_{i=0}^{n-1}\xi_1\circ T^i\to\xi^*\quad P\text{-a.e.}$$

Of course, $\xi_1\circ T^i=\xi_{i+1}$, hence

$$(*)\qquad \frac{1}{n}=\sum_{j=1}^{n}\xi_j\to\xi^*\quad P\text{-a.e.}$$

Theorem 3 of [8] states that P-a.e. convergence of the sequence $(g_n(\xi_1,\dots\xi_n))_n$ implies m-a.e. convergence of the sequence $(g_n(x_1,\dots,x_n))_n$ to an observable x^* and

$$P\Big(\big\{u\in\mathbb{R}^{\mathbb{N}}\,;\ \limsup_{n\to\infty}g_n\left(\xi_1(u),\dots,\xi_n(u)\right)<t\big\}\Big)$$
$$=m\Big(\limsup_{n\to\infty}g_n(x_1,\dots,x_n)(-\infty,t)\Big)$$
$$=m\left(\liminf_{n\to\infty}g_n(x_1,\dots,x_n)(-\infty,t)\right)=m\big(x^*\left((-\infty,t)\right)\big)$$

for every $t\in\mathbb{R}$. Put

$$g_n(u_1,\dots,u_n)=\frac{1}{n}\sum_{i=1}^{n}u_i.$$

Then by $(*)$ and Proposition 2 we have

$$\frac{1}{n}\sum_{i=0}^{n-1}\tau^i\circ x=\frac{1}{n}\sum_{i=1}^{n}x_i\to x^*\quad m\text{-a.e.}$$

□

Proposition 4. *Let x^* be the observable introduced in Proposition 3. Then x^* is integrable and $E(x^*) = E(x)$.*

Proof. Since ξ_1 is an integrable random variable, by the individual ergodic theorem ξ^* is integrable, too and

$$E(\xi_1) = E(\xi^*),$$
$$P\left(\{u;\ \xi^*(u) < t\}\right) = P\left(\left\{u;\ \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \xi_1 \circ T^i(u) < t\right\}\right)$$

for every $t \in \mathbb{R}$. By Theorem 3 of [8]

$$\begin{aligned} m_{x^*}\big((-\infty,t)\big) &= m\big(x^*\left((-\infty,t)\right)\big) \\ &= P\big(\{u;\ \xi^*(u) < t\}\big) = P_{\xi^*}\left((-\infty,t)\right). \end{aligned}$$

Therefore $m_{x^*} = P_{\xi^*}$. Since also $m_x = P_{\xi_1}$, we have

$$\begin{aligned} E(x) &= \int_{\mathbb{R}} t\,\mathrm{d}m_x(t) = \int_T t\,\mathrm{d}P_{\xi_1} = E(\xi_1) = E(\xi^*) \\ &= \int_{\mathbb{R}} t\,\mathrm{d}P_{\xi^*}(t) = \int_{\mathbb{R}} t\,\mathrm{d}m_{x^*}(t) = E(x^*). \end{aligned}$$

□

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