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ON COPIES OF c_0 IN THE BOUNDED LINEAR OPERATOR SPACE

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Abstract. In this note we study some properties concerning certain copies of the classic Banach space c_0 in the Banach space $\mathscr{L}(X,Y)$ of all bounded linear operators between a normed space X and a Banach space Y equipped with the norm of the uniform convergence of operators.

Keywords: Banach space, basic sequence, copy of c_0, copy of ℓ_∞

MSC 2000: 46E40, 46B25

1. Preliminaries

Throughout the paper X will denote a normed space and Y a Banach space. As usual $\mathscr{L}(X,Y)$ will stand for the Banach space of all continuous linear mappings from X into Y provided with the norm of the uniform convergence of operators. If X is infinite-dimensional and Y contains a copy of c_0, then $\mathscr{L}(X,Y)$ does contain a copy of ℓ_∞ [the argument given in the seminal paper [10, proof of Thm. 6] whenever X is a separable Banach space may be easily extended, see for instance [9, Remark 1]]. On the other hand, according to [9, Theorem 1], if $\mathscr{L}(X,Y)$ contains a copy of c_0 then Y contains a copy of c_0 or $\mathscr{L}(X,Y)$ contains a copy of ℓ_∞. Consequently, if X is infinite-dimensional then $\mathscr{L}(X,Y)$ does contain a copy of c_0 if and only if it contains a copy of ℓ_∞. Since $\mathscr{L}(\mathbb{K},c_0)$ is topologically isomorphic to c_0, the previous statement is not true if X is finite-dimensional. In [1] several conditions are given for $\mathscr{L}(X,Y)$ to contain a copy of c_0. Other results concerning copies of c_0 and ℓ_∞ in some spaces of linear operators can be found for example in [4], [5], [8] and [6]. In what follows we investigate the presence of certain copies of c_0 in $\mathscr{L}(X,Y)$ in relation with copies of c_0 and ℓ_∞ in the domain or range spaces. Much of our

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inspiration comes from [9], but we must also mention [5, proof of Lemma 4], which contains the seed of some techniques used in this paper.

2. Copies of c_0 in $\mathscr{L}(X,Y)$

Given a subset A of a Banach space E we denote by $\langle A\rangle$ [by $[A]$] the [closed] linear span of A, and for every infinite set $N \subseteq \mathbb{N}$ we denote by $\mathscr{P}_\infty(N)$ the class of all infinite subsets of N. We shorten by wuC the expression "weak unconditionally Cauchy". We start by noting that $\mathscr{L}(X,Y)$ may contain a copy of c_0 while Y fails to contain a copy of c_0. In fact, if $\{e_n\}$ denotes the unit vector basis of ℓ_2 and $T_n \in \mathscr{L}(\ell_2,\ell_2)$ is defined by $T_n\xi = \xi_n e_n$ for each $n \in \mathbb{N}$, then each T_n is a compact norm-one linear operator. Since $\left\|\sum\limits_{i=1}^{n} c_i T_i \xi\right\|_2 \leqslant \|\xi\|_2 \sup\limits_{1\leqslant i\leqslant n} |c_i|$ for each $\xi \in \ell_2$ and each finite set $c_1,\ldots,c_n$ of scalars, we have $\left\|\sum\limits_{i=1}^{n} c_i T_i\right\| \leqslant \sup\limits_{1\leqslant i\leqslant n} |c_i|$ and consequently $\{T_n\}$ is a basic sequence in $\mathscr{L}(\ell_2,\ell_2)$ equivalent to the unit vector basis of c_0.

Theorem 2.1. *Assume $\mathscr{L}(X,Y)$ has a basic sequence $\{T_n\colon n \in \mathbb{N}\}$ equivalent to the unit vector basis of c_0 and there exists a bounded linear operator P from $\mathscr{L}(X,Y)$ onto $[T_n]$ such that there is an $M \in \mathscr{P}_\infty(\mathbb{N})$ with $PT_m = T_m$ for each $m \in M$. Then Y contains a copy of c_0.*

P r o o f. Set $Z = [T_n]$, let $J\colon c_0 \to Z$ be a topological isomorphism from c_0 onto Z such that $T_n := Je_n$ for each $n \in \mathbb{N}$ and denote by $\{e_n\colon n \in \mathbb{N}\}$ the unit vector basis of c_0. We assume by way of contradiction that Y does not contain any copy of c_0. Since J is a bounded linear operator, the series $\sum\limits_{n=1}^{\infty} T_n$ is wuC in $\mathscr{L}(X,Y)$ and hence there exists a $C > 0$ such that

$$\sup_{n\in\mathbb{N}}\left\|\sum_{i=1}^{n} \xi_i T_i\right\| < C\,\|\xi\|_\infty$$

for each $\xi \in \ell_\infty$. Then, given $x \in X$, the series $\sum\limits_{n=1}^{\infty} T_n x$ is wuC in Y since for each $y^* \in Y^*$ the map $u\colon \mathscr{L}(X,Y) \to \mathbb{K}$ defined by $u(T) = y^*Tx$ is a continuous linear form on $\mathscr{L}(X,Y)$, and consequently $\sum\limits_{n=1}^{\infty} |y^*T_n x| = \sum\limits_{n=1}^{\infty} |uT_n| < \infty$. As we are assuming that Y contains no copy of c_0, according to a well known result of Bessaga and Pełczyński [2] the series $\sum\limits_{n=1}^{\infty} T_n x$ is (BM)-convergent in Y. So we may consider the bounded linear operator $\varphi\colon \ell_\infty \to \mathscr{L}(X,Y)$ of a norm $\leqslant C$ defined by $(\varphi\xi)x = \sum\limits_{n=1}^{\infty} \xi_n T_n x$.

According to the hypotheses there exists some $M \in \mathscr{P}_\infty(\mathbb{N})$ such that $PT_m = T_m$ for each $m \in M$. So, considering $\ell_\infty(M)$ as a subspace of ℓ_∞ and noting that $e_m \in \ell_\infty(M)$ for each $m \in M$, if ψ denotes the restriction of φ to $\ell_\infty(M) \subseteq \ell_\infty$ and S stands for the canonical projection from c_0 onto $c_0(M)$, then the bounded linear operator $Q\colon \ell_\infty(M) \to c_0(M)$ defined by $Q = S \circ J^{-1} \circ P \circ \psi$ satisfies $Qe_m = e_m$ for each $m \in M$. In fact,

$$Qe_m = SJ^{-1}P\psi e_m = SJ^{-1}PT_m = SJ^{-1}PJe_m = SJ^{-1}Je_m = e_m$$

since $PJe_m = Je_m$, which implies that $Q\zeta = \zeta$ for each $\zeta \in c_0(M)$. Hence if $\xi \in \ell_\infty(M)$, as $Q\xi \in c_0(M)$ one has that

$$Q^2\xi = Q(Q\xi) = Q\xi.$$

However, this means that Q must be a bounded projection operator from $\ell_\infty(M)$ onto $c_0(M)$, a contradiction. □

Corollary 2.2. *If $\mathscr{L}(X,Y)$ contains a complemented copy of c_0, then Y contains a copy of c_0.*

Proof. Assume Z is a complemented copy of c_0 in $\mathscr{L}(X,Y)$ and let $J\colon c_0 \to Z$ be a topological isomorphism from c_0 onto Z. Then $\{Je_n\colon n \in \mathbb{N}\}$ is a basic sequence equivalent to the unit vector basis of c_0. If P denotes a bounded projection operator from $\mathscr{L}(X,Y)$ onto $Z = [Je_n]$, then obviously $PJe_n = Je_n$ for each $n \in \mathbb{N}$, and in particular for each $n \in M$ with $M \in \mathscr{P}_\infty(\mathbb{N})$. Consequently, Theorem 2.1 applies. □

Remark 2.1. It is shown in [6] that, assuming X is a Banach space and c_0 embeds complementably into $\mathcal{L}(X,Y)$, then c_0 embeds into either X^* or Y. On the other hand, if X is a dual Banach space, according to the previous corollary we obtain the familiar fact that X contains no complemented copy of c_0.

It is also well known that Y is linearly isometric to a norm one complemented subspace of $\mathscr{L}(X,Y)$. In fact, given $z \in X$ with $\|z\| = 1$ and $z^* \in X^*$ such that $\|z^*\| = 1$ and $z^*z = 1$, the map $H\colon Y \to \mathscr{L}(X,Y)$ defined by $(Hy)x = z^*x \cdot y$ for each $x \in X$ is a linear isometry from Y into $\mathscr{L}(X,Y)$. So the linear operator $P\colon \mathscr{L}(X,Y) \to H(Y)$ defined by $PT = H(Tz)$ is a norm one projection from $\mathscr{L}(X,Y)$ onto $H(Y)$. Consequently, if Y contains a complemented copy of c_0, then $\mathscr{L}(X,Y)$ embeds c_0 complementably. On the other hand, by noting that the map $T \to T^*$ is a linear isometry from $\mathscr{L}(X,Y)$ into $\mathscr{L}(Y^*,X^*)$ and assuming c_0 embedded into $\mathscr{L}(X,Y)$, one has that $\mathscr{L}(Y^*,X^*)$ contains a copy of ℓ_∞ since it is a dual Banach space. If $\mathscr{L}(X,Y)$ contains a copy of c_0 but X^* does not, the previous statement may be sharpened.

Theorem 2.3. *Let G be a norming set in Y^* and assume $\mathscr{L}(X,Y)$ contains a copy of c_0. If X^* does not contain a copy of ℓ_∞, then $\mathscr{L}(\langle G\rangle, X^*)$ contains a copy of ℓ_∞.*

P r o o f. Let Z be a copy of c_0 in $\mathscr{L}(X,Y)$, let $J\colon c_0 \to Z$ be a topological isomorphism from c_0 onto Z and let $\{e_n\colon n\in\mathbb{N}\}$ denote the unit vector basis of c_0. As in the proof of Theorem 2.1 set $T_n := Je_n$ for each $n\in\mathbb{N}$. Since the series $\sum\limits_{n=1}^{\infty} T_n$ is wuC in $\mathscr{L}(X,Y)$, there is $C>0$ such that $\Bigl\|\sum\limits_{i=1}^{n}\xi_i T_i\Bigr\| < C\|\xi\|_\infty$ for each $\xi\in\ell_\infty$ and $n\in\mathbb{N}$, and $\sum\limits_{n=1}^{\infty}|y^*T_nx|<\infty$ for $x\in X$ and $y^*\in Y^*$.

If X^* does not contain a copy of ℓ_∞, then according to [3, Chapter 5, Corollary 11] each series $\sum\limits_{n=1}^{\infty} y^*T_n$ is (BM)-convergent in X^*. Thus we may define a linear operator $\varphi\colon \ell_\infty \to \mathscr{L}(\langle G\rangle, X^*)$ by

$$(\varphi\xi)y^* = \sum_{n=1}^{\infty}\xi_n y^* T_n$$

for each $y^*\in\langle G\rangle$. Given $y^*\in\langle G\rangle$, $\xi\in\ell_\infty$ and $\varepsilon>0$, let $n\in\mathbb{N}$ be such that $\Bigl\|\sum\limits_{j>n}\xi_j y^*T_j\Bigr\|<\varepsilon$. Note that

$$\|(\varphi\xi)y^*\| \leqslant \Bigl\|\sum_{j=1}^{n}\xi_j y^*T_j\Bigr\| + \Bigl\|\sum_{j=n+1}^{\infty}\xi_j y^*T_j\Bigr\| \leqslant C\|y^*\|_{\langle G\rangle}\|\xi\|_\infty + \varepsilon.$$

This implies that $\|(\varphi\xi)y^*\|\leqslant C\|y^*\|_{\langle G\rangle}\|\xi\|_\infty$ for each $\xi\in\ell_\infty$ and $y^*\in\langle G\rangle$, which shows that $\varphi\xi\in\mathscr{L}(\langle G\rangle, X^*)$ for each $\xi\in\ell_\infty$ and that φ is bounded. On the other hand, since

$$\begin{aligned}\|Je_n\|_{\mathscr{L}(X,Y)} &= \sup\{|y^*Je_nx|\colon x\in X,\ \|x\|\leqslant 1 \text{ and } y^*\in G,\ \|y^*\|\leqslant 1\}\\ &= \sup\{|\langle(\varphi e_n)y^*, x\rangle|\colon x\in X,\ \|x\|\leqslant 1 \text{ and } y^*\in G,\ \|y^*\|\leqslant 1\}\\ &= \|\varphi e_n\|_{\mathscr{L}(\langle G\rangle, X^*)},\end{aligned}$$

we have $\|\varphi e_n\|_{\mathscr{L}(\langle G\rangle,X^*)} = \|Je_n\|_{\mathscr{L}(X,Y)} \geqslant \dfrac{1}{\|J^{-1}\|}$ for each $n\in\mathbb{N}$, so Rosenthal's ℓ_∞ theorem [11] yields the conclusion. □

Example 2.1. The Banach space of all bounded vector measures.

If (Ω, Σ) is a measurable space, Y a Banach space and $ba(\Sigma, Y)$ [$ba(\Sigma)$ if $Y = \mathbb{K}$] the Banach space of all bounded Y-valued measures on Σ, equipped with the semivariation norm, then the linear operator S from $\mathscr{L}(\ell_0^\infty(\Sigma), Y)$ onto $ba(\Sigma, Y)$ defined by $ST(E) = T(\chi_E)$ for each $E \in \Sigma$, where $\ell_0^\infty(\Sigma)$ denotes the Σ-simple function space equipped with the supremum norm, is a linear isometry. Hence, according to Corollary 2.2, if $ba(\Sigma, Y)$ contains a complemented copy of c_0, then Y contains a copy of c_0. On the other hand, since $ba(\Sigma)$ does not contain any copy of ℓ_∞ [because ℓ_∞ has no complemented copy of ℓ_1], if G is a norming set in Y^* it follows from Theorem 2.3 that $\mathscr{L}(\langle G \rangle, ba(\Sigma))$ contains a copy of ℓ_∞ whenever $ba(\Sigma, Y)$ contains a copy of c_0.

Theorem 2.4. *Let X and Y be two Banach spaces over the [same] field of real or complex numbers. Assume $\mathscr{L}(X, Y)$ contains a basic sequence $\{T_n\}$ equivalent to the unit vector basis of c_0 such that each map T_n is a linear isometry from X into Y. If X contains a copy of c_0, then $\mathscr{L}(Y^*, \ell_1)$ contains a copy of ℓ_∞.*

Proof. Since $\sum_{n=1}^{\infty} T_n$ is a series weak unconditionally Cauchy, on the one hand there exists a constant $C > 0$ such that $\left\| \sum_{i=1}^{n} \xi_i T_i \right\| \leqslant C \|\xi\|_\infty$ for each $\xi \in \ell_\infty$ and $n \in \mathbb{N}$, and on the other hand $\sum_{n=1}^{\infty} |y^* T_n x| < \infty$, i.e. $(y^* T_n x) \in \ell_1$, for each $x \in X$ and $y^* \in Y^*$. Consider the linear operator $S\colon X \to \mathscr{L}(Y^*, \ell_1)$ defined by $(Sx)\, y^* = (y^* T_n x)$ for each $y^* \in Y^*$. Given $x \in X$, $x \neq 0$, and $y^* \in Y^*$, then setting $\varepsilon_n = \dfrac{y^* T_n x}{|y^* T_n x|}$ whenever $y^* T_n x \neq 0$ and $\varepsilon_n = 0$ otherwise, one has

$$\sum_{n=1}^{\infty} |y^* T_n x| = \sum_{n=1}^{\infty} \varepsilon_n y^* T_n x \leqslant \sup_{n \in \mathbb{N}} \left\| \sum_{i=1}^{n} \varepsilon_i T_i \right\| \|x\| \, \|y^*\| \leqslant C \|x\| \, \|y^*\| .$$

This shows that at the same time $Sx \in \mathscr{L}(Y^*, \ell_1)$ for each $x \in X$ and S is bounded.

Let $\{x_n\}$ be a basic sequence equivalent to the unit vector basis of c_0. Since each T_i is one to one, we have $T_n x_n \neq 0$ for each $n \in \mathbb{N}$. So, according to the Hahn-Banach theorem, for each positive integer n there exists a $y_n^* \in Y^*$ with $\|y_n^*\| = 1$ such that $y_n^* T_n x_n = \|T_n x_n\|$. Hence, considering the sequence $\{S x_n\}$ in $\mathscr{L}(Y^*, \ell_1)$, one has

$$\|S x_n\| = \sup \left\{ \sum_{i=1}^{\infty} |y^* T_i x_n| : \ \|y^*\| \leqslant 1 \right\} \geqslant \|T_n x_n\| = \|x_n\|$$

for each $n \in \mathbb{N}$. If J is a topological isomorphism from c_0 onto $[x_n]$ such that $J e_n = x_n$ for each $n \in \mathbb{N}$, then $\varphi = S \circ J$ is a bounded linear operator from c_0

into $\mathscr{L}(Y^*, \ell_1)$ such that $\|\varphi e_n\| \geqslant \|Je_n\| \geqslant \dfrac{1}{\|J^{-1}\|}$ for each $n \in \mathbb{N}$. According to Rosenthal's c_0 theorem [11], this implies that $\mathscr{L}(Y^*, \ell_1)$ contains a copy of c_0. So, $\mathscr{L}(Y^*, \ell_1)$ contains a copy of ℓ_∞. $\square$

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