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Jae Myung Park

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THE DENJOY EXTENSION OF THE RIEMANN AND MCSHANE INTEGRALS

Jae Myung Park, Taejon

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Abstract. In this paper we study the Denjoy-Riemann and Denjoy-McShane integrals of functions mapping an interval $[a,b]$ into a Banach space X. It is shown that a Denjoy-Bochner integrable function on $[a,b]$ is Denjoy-Riemann integrable on $[a,b]$, that a Denjoy-Riemann integrable function on $[a,b]$ is Denjoy-McShane integrable on $[a,b]$ and that a Denjoy-McShane integrable function on $[a,b]$ is Denjoy-Pettis integrable on $[a,b]$. In addition, it is shown that for spaces that do not contain a copy of c_0, a measurable Denjoy-McShane integrable function on $[a,b]$ is McShane integrable on some subinterval of $[a,b]$. Some examples of functions that are integrable in one sense but not another are included.

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The Denjoy-Dunford, Denjoy-Pettis, and Denjoy-Bochner integrals are the extensions of Dunford, Pettis, and Bochner integrals, respectively. These integrals were defined and studied by Gordon [4]. He showed that a Denjoy-Dunford (Denjoy-Bochner) integrable function on $[a,b]$ is Dunford (Bochner) integrable on some subinterval of $[a,b]$ and that for spaces that do not contain a copy of c_0, a Denjoy-Pettis integrable function on $[a,b]$ is Pettis integrable on some subinterval of $[a,b]$. Here c_0 represents the classical Banach space of all bounded sequences of scalars converging to 0. It follows from the Bessaga-Pelczyński Theorem that a Banach space X contains no copy of c_0 if and only if all series $\sum\limits_n x_n$ in X, with $\sum\limits_n |x^*x_n| < \infty$ for all x^* in the dual X^*, are unconditionally convergent in the norm [1, Corollary I.4.5]. This theorem is useful in proving results in the theory of integrals of vector-valued functions.

In this paper we will define and study the Denjoy extension of the Riemann and McShane integrals of functions mapping an interval $[a,b]$ into a Banach space X. We will also examine the relationship between these integrals and other vector-valued integrals.

Throughout this paper X will denote a real Banach space and X^* its dual.

Definition 1. Let $F\colon [a,b] \to X$ be a function and let E be a subset of $[a,b]$.

(1) The function F is BV on E if $\sup\Big\{\sum\limits_{i=1}^{n} \|F(d_i) - F(c_i)\|\Big\}$ is finite where the supremum is taken over all finite collections $\{[c_i, d_i]\}_{i\leqslant n}$ of non-overlapping intervals that have endpoints in E.

(2) The function F is BVG on E if E can be expressed as a countable union of sets on each of which F is BV.

(3) The function F is AC on E if for each $\varepsilon > 0$ there exists $\eta > 0$ such that $\sum\limits_{i=1}^{n} \|F(d_i) - F(c_i)\| < \varepsilon$ whenever $\{[c_i, d_i]\}_{i\leqslant n}$ is a finite collection of non-overlapping intervals that have endpoints in E and satisfy $\sum\limits_{i=1}^{n} (d_i - c_i) < \eta$.

(4) The function F is ACG on E if F is continuous on E and E can be expressed as a countable union of sets on each of which F is AC.

The following theorem was proved by Gordon [8].

Theorem 2. *Let $F\colon [a,b] \to X$, let $E \subset [a,b]$, and let $\overline{E}$ be the closure of E.*

(1) *Suppose that F is BV on E. If $t \in \overline{E}$, then each of the limits $\lim\limits_{\substack{s\to t^+\\ s\in E}} F(s)$ and $\lim\limits_{\substack{s\to t^-\\ s\in E}} F(s)$ exists.*

(2) *Suppose that F is measurable. If F is BV on E, then there exists a measurable set $H \subset [a,b]$ such that $E \subset H$ and F is BV on H.*

Proof. The proof of (1) is similar to the well-known proof that a BV function on an interval has one-sided limits at each point.

We turn now to the proof of (2). Let E_1 be the set of all points t in $\overline{E} - E$ such that t is a right-hand limit point of E but not a left-hand limit point of E and let $E_2 = \overline{E} - (E \cup E_1)$. Define $G_1\colon \overline{E} \to X$ as follows: $G_1(t) = F(t)$ for $t \in E$, $G_1(t) = \lim\limits_{\substack{s\to t^+\\ s\in E}} F(s)$ for $t \in E_1$, and $G_1(t) = \lim\limits_{\substack{s\to t^-\\ s\in E}} F(s)$ for $t \in E_2$. The function G_1 is well-defined by (1) above and it is not difficult to show that G_1 is BV on $\overline{E}$. Let c and d be the bounds of $\overline{E}$ and let $G\colon [c,d] \to X$ be the function that equals G_1 on $\overline{E}$ and is linear on the intervals contiguous to $\overline{E}$. By [4, Theorem 3] the function G is BV on $[c,d]$. Let $H = \{t \in [c,d] : F(t) = G(t)\}$. Then H is a measurable set since F and G are measurable functions, the function F is BV on H, and $E \subset H$. This completes the proof. $\square$

Definition 3. (1) A *tagged partition* of $[a,b]$ is a finite sequence $\langle [a_i,b_i], t_i\rangle_{i\leqslant n}$ such that $\langle [a_i,b_i]\rangle_{i\leqslant n}$ is a non-overlapping family of intervals covering $[a,b]$ and $t_i \in [a_i,b_i]$ for each i. A function $f\colon [a,b]\to X$ is *Riemann integrable* on $[a,b]$, with *Riemann integral* z, if for each $\varepsilon>0$ there exists $\eta>0$ such that

$$\left\|\sum_{i=1}^{n} f(t_i)(b_i-a_i)-z\right\|<\varepsilon$$

for every tagged partition $\langle [a_i,b_i], t_i\rangle_{i\leqslant n}$ of $[a,b]$ that satisfies $\max\limits_{1\leqslant i\leqslant n}\{b_i-a_i\}<\eta$.

(2) A *McShane partition* of $[a,b]$ is a finite sequence $\langle [a_i,b_i], t_i\rangle_{i\leqslant n}$ such that $\langle [a_i,b_i]\rangle_{i\leqslant n}$ is a non-overlapping family of intervals covering $[a,b]$ and $t_i\in[a,b]$ for each i. A *gauge* on $[a,b]$ is a function $\delta\colon [a,b]\to(0,\infty)$. A McShane partition $\langle [a_i,b_i], t_i\rangle_{i\leqslant n}$ is *subordinate* to a gauge δ if $[a_i,b_i]\subset(t_i-\delta(t_i), t_i+\delta(t_i))$ for each $i\leqslant n$. A function $f\colon [a,b]\to X$ is *McShane integrable*, with *McShane integral* w, if for each $\varepsilon>0$ there exists a gauge $\delta\colon [a,b]\to(0,\infty)$ such that

$$\left\|\sum_{i=1}^{n} f(t_i)(b_i-a_i)-w\right\|<\varepsilon$$

for every McShane partition $\langle [a_i,b_i], t_i\rangle_{i\leqslant n}$ of $[a,b]$ subordinate to δ.

The function f is Riemann (McShane) integrable on the set $E\subset[a,b]$ if the function $f\chi_E$ is Riemann (McShane) integrable on $[a,b]$.

Definition 4. Let $F\colon [a,b]\to X$ and let $E\subset[a,b]$.

(1) The function F is *approximately differentiable* at $t\in(a,b)$ if there exists a vector z in X with the following property: there exists a measurable set $E\subset[a,b]$ that has t as a point of density such that

$$\lim_{\substack{s\to t\\ s\in E}}\frac{F(s)-F(t)}{s-t}=z$$

for the norm topology of X. We will write $F'_{ap}(t)=z$.

(2) The function $f\colon E\to X$ is an *approximate scalar derivative* of F on E if for each x^* in X^* the function $x^*F\colon E\to\mathbb{R}$ is approximately differentiable almost everywhere on E and $(x^*F)'_{ap}=x^*f$ almost everywhere on E.

(3) The function $f\colon [a,b]\to X$ is a *scalar derivative* of F on E if for each x^* in X^* the function x^*F is differentiable almost everywhere on E and $(x^*F)'=x^*f$ almost everywhere on E.

If $F\colon [a,b]\to X$ is ACG on $[a,b]$, then for each x^* in X^*, x^*F is ACG on $[a,b]$ and hence approximately differentiable almost everywhere on $[a,b]$ [4, Theorem 9].

Now we define the Denjoy-Riemann and Denjoy-McShane integrals.

Definition 5. (1) The function $f\colon [a,b] \to X$ is *Denjoy-Riemann integrable* on $[a,b]$ if there exists an ACG function $F\colon [a,b] \to X$ such that $(x^*F)'_{ap} = x^*f$ almost everywhere on $[a,b]$ for each x^* in X^*.

(2) The function $f\colon [a,b] \to X$ is *Denjoy-McShane integrable* on $[a,b]$ if there exists a continuous function $F\colon [a,b] \to X$ such that each x^*F is ACG on $[a,b]$ and $(x^*F)'_{ap} = x^*f$ almost everywhere on $[a,b]$ for each x^* in X^*.

The function f is Denjoy-Riemann (Denjoy-McShane) integrable on the set $E \subset [a,b]$ if the function $f\chi_E$ is Denjoy-Riemann (Denjoy-McShane) integrable on $[a,b]$.

The function $f\colon [a,b] \to R$ is *Denjoy integrable* on $[a,b]$ if there exists an ACG function $F\colon [a,b] \to R$ such that $F'_{ap} = f$ almost everywhere on $[a,b]$.

Definition 5 implies that if f is Denjoy-Riemann (Denjoy-McShane) integrable on $[a,b]$, then for each x^* in X^*, x^*f is Denjoy integrable on $[a,b]$.

The following theorem shows that the Denjoy-Riemann integral is an extension of the Riemann integral.

Theorem 6. *If $f\colon [a,b] \to X$ is Riemann integrable on $[a,b]$, then f is Denjoy-Riemann integrable on $[a,b]$.*

Proof. Suppose that $f\colon [a,b] \to X$ is Riemann integrable on $[a,b]$ and let $F(t) = \int_a^t f$ for each $t \in [a,b]$. Then F is AC on $[a,b]$ [6, Theorem 8]. Since for each x^* in X^*, x^*F is AC on $[a,b], x^*F$ is differentiable almost everywhere on $[a,b]$ and $(x^*F)'_{ap} = (x^*F)' = x^*f$ almost everywhere on $[a,b]$. Hence, f is Denjoy-Riemann integrable on $[a,b]$. □

Definition 7. (1) A function $f\colon [a,b] \to X$ is *Dunford integrable* on $[a,b]$ if x^*f is Lebesgue integrable on $[a,b]$ for each x^* in X^*. The *Dunford integral* of f on the measurable set $E \subset [a,b]$ is the vector x_E^{**} in X^{**} such that $x_E^{**}(x^*) = \int_E x^*f$ for all x^* in X^*.

(2) A function $f\colon [a,b] \to X$ is *Pettis integrable* on $[a,b]$ if f is Dunford integrable on $[a,b]$ and $x_E^{**} \in X$ for every measurable set E in $[a,b]$.

(3) A function $f\colon [a,b] \to X$ is *Bochner integrable* on $[a,b]$ if there exists an AC function $F\colon [a,b] \to X$ such that F is differentiable almost everywhere on $[a,b]$ and $F' = f$ almost everywhere on $[a,b]$.

A function f is Dunford (Pettis, Bochner) integrable on the set $E \subset [a,b]$ if the function $f\chi_E$ is Dunford (Pettis, Bochner) integrable on $[a,b]$.

Definition 8. (1) A function $f\colon [a,b] \to X$ is *Denjoy-Dunford integrable* on $[a,b]$ if for each x^* in X^* the function x^*f is Denjoy integrable on $[a,b]$ and if for every interval I in $[a,b]$ there exists a vector x_I^{**} in X^{**} such that $x_I^{**}(x^*) = \int_I x^*f$ for all x^* in X^*.

(2) A function $f\colon [a,b] \to X$ is *Denjoy-Pettis integrable* on $[a,b]$ if f is Denjoy-Dunford integrable on $[a,b]$ and if $x_I^{**} \in X$ for every interval I in $[a,b]$.

(3) A function $f\colon [a,b] \to X$ is *Denjoy-Bochner integrable* on $[a,b]$ if there exists an ACG function $F\colon [a,b] \to X$ such that F is approximately differentiable almost everywhere on $[a,b]$ and $F'_{ap} = f$ almost everywhere on $[a,b]$.

Theorem 9. *If a function $f\colon [a,b] \to X$ is Denjoy-Bochner integrable on $[a,b]$, then f is Denjoy-Riemann integrable on $[a,b]$.*

Proof. Suppose that f is Denjoy-Bochner integrable on $[a,b]$ and let $F(t) = \int_a^t f$ for each $t \in [a,b]$. Then F is ACG on $[a,b]$, approximately differentiable almost everywhere on $[a,b]$ and $F'_{ap} = f$ almost everywhere on $[a,b]$. Since for each $x^* \in X^*$, $(x^*F)'_{ap} = x^*f$ almost everywhere on $[a,b]$, f is Denjoy-Riemann integrable on $[a,b]$. □

The following example shows that the converse of Theorem 9 is not true.

Example 10. A Denjoy-Riemann integrable function that is not Denjoy-Bochner integrable.

Define $f\colon [0,1] \to l_\infty[0,1]$ by $f(t) = \chi_{[0,t]}$. Since f is not essentially separably-valued, it is not measurable. By [4, Theorem 26] f is not Denjoy-Bochner integrable on $[0,1]$. But since f is Riemann integrable on $[0,1]$ [6, Example 12], f is Denjoy-Riemann integrable on $[0,1]$ by Theorem 6.

Let $F\colon [a,b] \to R$ be a function. If we define $F([c,d]) = F(d) - F(c)$ for an interval $[c,d] \subset [a,b]$ and $F\Big(\bigcup\limits_{i=1}^{n} I_i\Big) = \sum\limits_{i=1}^{n} F(I_i)$ for every finite collection $\{I_1, I_2, \ldots, I_n\}$ of non-overlapping intervals in $[a,b]$, F can be treated as a function having the unions of a finite number of intervals in $[a,b]$ for its domain and a Banach space X for its range.

The following theorem was proved by Pettis [10].

Theorem 11. *Let $F\colon [a,b] \to X$ be BV on $[a,b]$ and suppose that $f\colon [a,b] \to X$ is the scalar derivative of F on $[a,b]$. If f is separably-valued, then F is differentiable almost everywhere on $[a,b]$ and $F' = f$ almost everywhere on $[a,b]$.*

If a Denjoy-Riemann integrable function is separably-valued, then it is Denjoy-Bochner integrable. To prove this we need the following theorem, which was proved by Gordon [8].

Theorem 12. *Let $F\colon [a,b] \to X$ be measurable, let E be a measurable subset of $[a,b]$, and let $f\colon E \to X$ be an approximate scalar derivative of F on E. If F*

is BVG on E and if f is separably-valued, then F is approximately differentiable almost everywhere on E and $F_{ap}^{'} = f$ almost everywhere on E.

P r o o f. Let $E = \bigcup_n E_n$ where F is BV on each E_n. Using Theorem 2 we may assume that each E_n is measurable. It is sufficient to prove that $F_{ap}^{'} = f$ almost everywhere on each E_n. To this end, fix n and let $\varepsilon > 0$. Let $H \subset E_n$ be a closed set such that $\mu(E_n - H) < \varepsilon$ and let c and d be the bounds of H. Let $G\colon [c,d] \to X$ be the function that equals F on H and is linear on the intervals contiguous to H. Note that G is BV on $[c,d]$ by [4, Theorem 3]. Let $(c,d) - H = \bigcup_k (c_k, d_k)$ and define $g\colon [c,d] \to X$ by $g(t) = f(t)$ for $t \in H$ and $g(t) = \frac{F(d_k)-F(c_k)}{d_k-c_k}$ for $t \in (c_k, d_k)$. We will show that g is the scalar derivative of G on $[c,d]$. Fix $x^* \in X^*$. The function x^*G is BV on $[c,d]$ and hence differentiable almost everywhere on $[c,d]$. It is clear that $(x^*G)' = x^*g$ almost everywhere on $(c,d) - H$. Let H_1 be the set of all points t in H such that $(x^*G)'(t)$ exists, $(x^*F)_{ap}^{'}(t) = x^*f(t)$, and t is a point of density of H. Then $\mu(H - H_1) = 0$ and for each s in H_1 we see that

$$x^*g(s) = x^*f(s) = \lim_{\substack{t\to s\\ t\in A}} \frac{x^*F(t) - x^*F(s)}{t-s} = \lim_{\substack{t\to s\\ t\in A\cap H}} \frac{x^*G(t) - x^*G(s)}{t-s} = (x^*G)'(s)$$

where A is a measurable subset of $[c,d]$ that has s as a point of density. We conclude that $(x^*G)' = x^*g$ on H_1 and it follows that $(x^*G)' = x^*g$ almost everywhere on $[c,d]$. Since x^* was arbitrary, the function g is the scalar derivative of G on $[c,d]$.

Since G is BV on $[c,d]$ and since g is separably-valued, we find that $G' = g$ almost everywhere on $[c,d]$ by Theorem 11. Let B be the set of all points t in H such that $G'(t) = g(t)$ and t is a point of density of H. Then $\mu(H - B) = 0$ and for each s in B we have

$$f(s) = g(s) = \lim_{t\to s} \frac{G(t) - G(s)}{t-s} = \lim_{\substack{t\to s\\ t\in H}} \frac{F(t) - F(s)}{t-s}.$$

Hence, the function F is approximately differentiable on B and $F_{ap}^{'} = f$ on B. Since $\mu(E_n - H) < \varepsilon$ and $\mu(H - B) = 0$, we have $\mu(E_n - B) < \varepsilon$. For each positive integer k, choose a measurable set B_k such that $\mu(E_n - B_k) < \frac{1}{k}$, F is approximately differentiable on B_k and $F_{ap}^{'} = f$ on B_k. Let $A = \bigcup_{k=1}^{\infty} B_k$. Then the function F is approximately differentiable on A and $F_{ap}^{'} = f$ on A. Since $\mu(E_n - A) \leqslant \mu(E_n - B_k) < \frac{1}{k}$ for every positive integer k, we have $\mu(E_n - A) = 0$. Hence, $F_{ap}^{'} = f$ almost everywhere on E_n. This completes the proof. □

Corollary 13. *Let $f\colon [a,b] \to X$ be Denjoy-Riemann integrable on $[a,b]$. If f is separably-valued, then f is Denjoy-Bochner integrable on $[a,b]$.*

P r o o f. Suppose that $f\colon [a,b] \to X$ is Denjoy-Riemann integrable on $[a,b]$. Then there exists an ACG function $F\colon [a,b] \to X$ such that for each x^* in $X^*, (x^*F)'_{ap} = x^*f$ almost everywhere on $[a,b]$. Since F is continuous on $[a,b]$, x^*F is measurable for each x^* in X^* and the set $\{F(t)\colon t \in [a,b]\}$ is compact and hence separable. It follows from the Pettis Measurability Theorem that F is measurable. By Theorem 12 F is approximately differentiable almost everywhere on $[a,b]$ and $F'_{ap} = f$ almost everywhere on $[a,b]$. Hence, f is Denjoy-Bochner integrable on $[a,b]$. □

The next theorem follows immediately from Definition 5.

Theorem 14. *If $f\colon [a,b] \to X$ is Denjoy-Riemann integrable on $[a,b]$, then f is Denjoy-McShane integrable on $[a,b]$.*

A Denjoy-Bochner integrable function is measurable [4, Theorem 26]. But since there exists a Riemann integrable function that is not measurable (Example 10), it follows that a Denjoy-Riemann (Denjoy-McShane) integrable function is not measurable in general.

Theorem 15. *Let $f\colon [a,b] \to X$ be a Denjoy-Riemann (Denjoy-McShane) integrable function on $[a,b]$ and let $F(t) = \int_a^t f$ for each $t \in [a,b]$. If F is approximately differentiable almost everywhere on $[a,b]$, then f is measurable.*

P r o o f. Let $f\colon [a,b] \to X$ be a Denjoy-Riemann (Denjoy-McShane) integrable function on $[a,b]$ and let $F(t) = \int_a^t f$ for each $t \in [a,b]$. Then for each x^* in X^*, x^*f is Denjoy integrable on $[a,b]$ and hence measurable. Since F is continuous on $[a,b]$ and approximately differentiable almost everywhere on $[a,b]$. The rest of the proof is quite similar to the proof of [4, Theorem 26]. □

Theorem 16. *Suppose that $f\colon [a,b] \to X$ is separably-valued. If f is Denjoy-Riemann integrable on $[a,b]$, then there exists a subinterval of $[a,b]$ on which f is Bochner integrable.*

P r o o f. Suppose that $f\colon [a,b] \to X$ is separably-valued and Denjoy-Riemann integrable on $[a,b]$. Let $F(t) = \int_a^t f$ for each $t \in [a,b]$. Since F is ACG on $[a,b]$, there exists a subinterval $[c,d]$ of the perfect set $[a,b]$ on which F is AC [4, Theorems 2 and 4]. Hence, for each x^* in X^*, x^*F is differentiable almost everywhere on $[c,d]$ and $(x^*F)' = (x^*F)'_{ap} = x^*f$ almost everywhere on $[c,d]$. Since f is separably-valued, F is differentiable almost everywhere on $[c,d]$ and $F' = f$ almost everywhere on $[c,d]$ by Theorem 11. Hence, f is Bochner integrable on $[c,d]$. □

The following theorem shows that the Denjoy-McShane integral is an extension of the McShane integral.

Theorem 17. *If $f\colon\ [a,b]\to X$ is McShane integrable on $[a,b]$, then f is Denjoy-McShane integrable on $[a,b]$.*

P r o o f. Suppose that $f\colon\ [a,b]\to X$ is McShane integrable on $[a,b]$ and let $F(t)=\int_a^t f$ for each $t\in[a,b]$. Then F is continuous on $[a,b]$.

Since for each x^* in X^*, x^*f is McShane integrable on $[a,b]$ and

$$x^*F(t)=(M)\int_a^t x^*f,$$

each x^*F is AC on $[a,b]$ and $(x^*F)'_{ap}=(x^*F)'=x^*f$ almost everywhere on $[a,b]$. Hence, f is Denjoy-McShane integrable on $[a,b]$. □

Theorem 18. *If $f\colon\ [a,b]\to X$ is Denjoy-McShane integrable on $[a,b]$, then f is Denjoy-Pettis integrable on $[a,b]$.*

P r o o f. Suppose that $f\colon\ [a,b]\to X$ is Denjoy-McShane integrable on $[a,b]$ and let $F(t)=\int_a^t f$ for each $t\in[a,b]$. Since for each x^* in X^*, x^*f is Denjoy integrable on $[a,b]$, for every interval $[c,d]$ in $[a,b]$ we have

$$\begin{aligned}x^*\left(F(d)-F(c)\right)&=x^*F(d)-x^*F(c)\\&=\int_a^d x^*f-\int_a^c x^*f=\int_c^d x^*f.\end{aligned}$$

Since this is valid for all x^* in X^* and since $F(d)-F(c)\in X$, f is Denjoy-Pettis integrable on $[a,b]$. □

A *portion* of a set $E\subset R$ is a nonempty set P of the form $P=E\cap I$ where I is an open interval.

Corollary 19. *Suppose that X contains no copy of c_0 and let $f\colon\ [a,b]\to X$ be measurable. If f is Denjoy-McShane integrable on $[a,b]$, then every perfect set in $[a,b]$ contains a portion on which f is McShane integrable.*

P r o o f. Suppose that $f\colon\ [a,b]\to X$ is measurable and Denjoy-McShane integrable on $[a,b]$. Let E be a perfect set in $[a,b]$. Since f is Denjoy-Pettis integrable on $[a,b]$ by Theorem 18, there exists an interval $[c,d]$ in $[a,b]$ such that f is Pettis integrable on $E\cap[c,d]$ [4, Theorem 38]. Hence, $f\chi_E$ is Pettis integrable on $[c,d]$. Since f is measurable on $[c,d]$, $f\chi_E$ is McShane integrable on $[c,d]$ by [5, Theorem 17]. Hence, f is McShane integrable on $E\cap(c,d)$. □

The above theorem shows that for spaces that do not contain a copy of c_0, a measurable and Denjoy-McShane integrable function on $[a,b]$ is McShane integrable on some subinterval of $[a,b]$.

The next two examples are the corrected versions of examples of Gordon [4].

Example 20. A Denjoy-McShane integrable function that is not Denjoy-Riemann integrable.

Let $\{\gamma_k\}$ be a listing of the rational numbers in $[0,1)$ and for each pair of positive integers n and k let

$$I_n^k = \Big(\gamma_k + \frac{1}{n+1}, \gamma_k + \frac{1}{n}\Big).$$

For each k define $f_k\colon\ [0,1] \to l_2$ by

$$f_k\,(t) = \big\{(n+1)\,\chi_{I_n^k}\,(t)\big\}\,.$$

Then the series $\sum\limits_k 4^{-k} f_k$ is l_2-valued almost everywhere on $[0,1]$. To show this, let $A_i = \bigcup\limits_k \{t \in [0,1]\colon\ |t - r_k| < 2^{-i-k}\}$ for each positive integer i and let $A = \bigcap\limits_i A_i$. Then $\{r_k\} \subset A$, and $\mu(A) = 0$ since $\mu(A) \leqslant \mu(A_i) < 2^{1-i}$ for each i. If $t \notin A$, then $t \notin A_{i_0}$ for some i_0 and $|t - r_k| \geqslant 2^{-i_0-k}$ for all k. Hence $\|f_k(t)\| \leqslant 2^{i_0+k}$ for all k and $\sum\limits_k \|4^{-k} f_k(t)\| \leqslant 2^{i_0}$. It follows that $\sum\limits_k 4^{-k} f_k(t)$ converges in l_2. This shows that $\sum\limits_k 4^{-k} f_k(t)$ is l_2-valued almost everywhere on $[0,1]$.

Let A be a set of measure zero such that $\sum\limits_k 4^{-k} f_k$ is l_2-valued for all t in $[0,1]-A$. Define $f\colon\ [0,1] \to l_2$ by $f\,(t) = \sum\limits_k 4^{-k} f_k\,(t)$ for $t \in [0,1] - A$ and $f\,(t) = 0$ for t in A. Then f is separably-valued, measurable and Pettis integrable on $[0,1]$, but f is not Bochner integrable on any subinterval of $[0,1]$ [4, Proof of Example 42]. By Theorem 16, f is not Denjoy-Riemann integrable on $[0,1]$. But by [5, Theorem 17] f is McShane integrable on $[0,1]$ and hence f is Denjoy-McShane integrable on $[0,1]$.

Example 21. A Denjoy-Pettis integrable function that is not Denjoy-McShane integrable.

For each positive integer n let

$$I_n' = \Big(\frac{1}{n+1}, \frac{n+\frac{1}{2}}{n\,(n+1)}\Big), \quad I_n'' = \Big(\frac{n+\frac{1}{2}}{n\,(n+1)}, \frac{1}{n}\Big)$$

and define $f_n\colon\ [0,1] \to R$ by $f_n\,(t) = 2n\,(n+1)\,\big(\chi_{I_n'}\,(t) - \chi_{I_n''}\,(t)\big)$. Then the sequence $\{f_n\}$ converges to 0 pointwise and $\big\{\int_I f_n\big\}$ converges to 0 for each interval $I \subset [0,1]$. Define $f\colon\ [0,1] \to c_0$ by $f\,(t) = \{f_n\,(t)\}$. Then f is Dunford integrable on $[0,1]$, $\int_E f = \left\{\int_E f_n\right\}$ for every measurable set $E \subset [0,1]$ and f is Denjoy-Pettis integrable on $[0,1]$ [4, Example 44].

Now we will show that f is not Denjoy-McShane integrable on $[0,1]$. Let $G(t) = \int_0^t f$ be an indefinite Dunford integral of f and let $t_n = \frac{n+\frac{1}{2}}{n(n+1)}$ for each n. Then $t_n \to 0$ as $n \to \infty$, but $\int_0^{t_n} f_n = 1$ for each n. Hence $\|G(t_n)\|_{c_0} \geqslant 1$ and G is not continuous at 0.

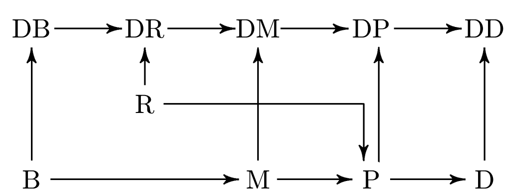
Suppose that f is Denjoy-McShane integrable on $[0,1]$. Then there exists a continuous function F on $[0,1]$ such that for each x^* in c_0^*, x^*F is ACG on $[0,1]$ and $(x^*F)'_{ap} = x^*f$ almost everywhere on $[0,1]$. For each x^* in c_0^*, x^*f is Denjoy integrable on $[0,1]$ and $x^*F(t) = (D)\int_0^t x^*f$ for each $t \in [0,1]$. Since x^*f is Lebesgue integrable on $[0,1]$, for each $t \in [0,1]$ we have

$$x^*F(t) = (D)\int_0^t x^*f = (L)\int_0^t x^*f = x^*G(t).$$

Since this valid for all x^* in c_0^*, we have $F = G$ on $[0,1]$, a contradiction. This shows that f is not Denjoy-McShane integrable on $[0,1]$.

Now we have a table indicating the relations between the various types of integrals we have been discussing.

We present a diagram relating the following integrals: Bochner integral (B), Riemann integral (R), McShane integral (M), Pettis integral (P), Dunford integral (D), Denjoy-Bochner integral (DB), Denjoy-Riemann integral (DR), Denjoy-McShane integral (DM), Denjoy-Pettis integral (DP), and Denjoy-Dunford integral (DD).



In the above diagram, an arrow stands for implication. For example, the implication $\lceil$DB $\longrightarrow$ DR$\rfloor$ represents that if a function f is Denjoy-Bochner integrable, then it is Denjoy-Riemann integrable.

References

[1] *J. Diestel, J. J. Uhl*: Vector Measures. Amer. Math. Soc., Providence, R. I., 1977.

[2] *D. H. Fremlin*: The Henstock and McShane integrals of vector-valued functions. Illinois J. Math. *38* (1994), 471–479.

[3] *D. H. Fremlin, J. Mendoza*: On the integration of vector-valued functions. Illinois J. Math. *38* (1994), 127–147.

[4] *R. A. Gordon*: The Denjoy extension of the Bochner, Pettis, and Dunford integrals. Studia Math. *92* (1989), 73–91.

[5] *R. A. Gordon*: The McShane integral of Banach-valued functions. Illinois J. Math. *34* (1990), 557–567.

[6] *R. A. Gordon*: Riemann integration in Banach spaces. Rocky Mountain J. Math. *21* (1991), 923–949.

[7] *R. A. Gordon*: The Integrals of Lebesgue, Denjoy, Perron, and Henstock. Amer. Math. Soc., 1994.

[8] *R. A. Gordon*: Differentiation in Banach spaces. preprint.

[9] *J. M. Park*: Bounded convergence theorem and integral operator for operator valued measures. Czechoslovak Math. J. *47(122)* (1997), 425–430.

[10] *B. J. Pettis*: Differentiation in Banach spaces. Duke Math. J. *5* (1939), 254–269.

Author's address: Department of Mathematics, Chungnam National University, Taejon 305-764, Republic of Korea.