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A REPRESENTATION THEOREM FOR PROBABILISTIC METRIC SPACES IN GENERAL

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Abstract. In this paper, we present a representation theorem for probabilistic metric spaces in general.

Keywords: probabilistic metric space

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K. Menger, B. Schweizer, A. Sklar [1], H. Sherwood [3] and R. Stevens [5] investigated the relationship between probabilistic metrics and numerical metrics. Using a collection of ordinary metrics, R. Stevens presented a representation theorem of a class for probabilistic metric spaces:

Theorem A (cf. [5], p. 267). *If (S,F) is a Menger space under the t-norm $T=\mathrm{Min}$ and if each distance distribution function $F_{pq}(x)$ $(p,q\in S,\ p\neq q)$ is continuous, then (S,F) is a metrically generated PM space.*

Since Min is the strongest possible t-norm, one conjectures that Theorem A admits a considerable improvement (cf. [5], p. 267). In this paper, we thoroughly improve Theorem A and give a representation theorem for probabilistic metric spaces in general $(\sup\limits_{a<1} T(a,a)=1)$.

Definition 1. Let S be a nonempty set and Ω an index set. Let $\{d_t\colon\ t\in\Omega\}$ be a collection of mappings from $S\times S$ into $[0,+\infty)$. Then $\{d_t\colon\ t\in\Omega\}$ is a collection of semi-metrics on S if it satisfies the following conditions:

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(SM-1) For any t in Ω and all p, q in S, $d_t(p,q)=0$ if and only if $p=q$;

(SM-2) For all t in Ω and all p, q in S, $d_t(p,q)=d_t(q,p)$;

(SM-3) For every t in Ω, there exists a τ in Ω such that $d_t(p,r)\leqslant d_\tau(p,q)+d_\tau(q,r)$ for all p, q and r in S.

Definition 2. A PM space (S,F) is semi-metrically generated if and only if there exist a probability space (Ω,B,P) and a collection of semi-metrics $\{d_t\colon\ t\in\Omega\}$ on S such that

(SMG-1) for every real number x and every pair p, q of points in S, the set $\{t\in\Omega\colon\ d_t(p,q)<x\}$ is a B-measurable set;

(SMG-2) for every real number x and every pair p, q of points belonging to S we have $F(p,q)=F_{pq}$, where F_{pq} is the distribution function defined by

$$F_{pq}(x)=P\{t\in\Omega\colon\ d_t(p,q)<x\}. \tag{1}$$

The correctness of Definition 2 follows immediately from the following Theorem 1.

Theorem 1. *Let (Ω,B,P) be a probability space and $\{d_t\colon\ t\in\Omega\}$ a collection of semi-metrics on S. If $\{d_t\colon\ t\in\Omega\}$ satisfies the condition (SMG-1) in Definition 2 and F is defined by (1), then (S,F) is a PM space.*

P r o o f. Theorem 1 can be proved by using the properties of probability measures. □

Theorem 2. *If (S,F,T) is a Menger space with $\sup_{a<1}T(a,a)=1$, then each distance distribution function $F_{pq}(x)$ $(p,q\in S,\ p\neq q)$ is right-continuous at zero if and only if (S,F) is a semi-metrically generated PM space.*

P r o o f. Necessity: Suppose that B denotes the family of all Borel sets in the open interval $(0,1)$. Let L be the Lebesgue measure on $(0,1)$. Then $((0,1),B,L)$ is a probability space. For any t in $(0,1)$ and any pair p, q of points in S, we define

$$d_t(p,q)=L\{x\geqslant 0\colon\ F_{pq}(x)<t\}. \tag{2}$$

Then $\{d_t\colon\ t\in(0,1)\}$ is a collection of mappings from $S\times S$ into $[0,+\infty)$.

For any pair $p,q\in S$ of points with $p\neq q$, by the hypothesis, the distance distribution function $F_{pq}(x)$ is right-continuous at zero. Consequently, it is not hard to show that

$$d_t(p,q)=L\{x\geqslant 0\colon\ F_{pq}(x)<t\}>0$$

for all t in $(0,1)$. Therefore it is easily seen that $\{d_t\colon\ t\in(0,1)\}$ satisfies the condition (SM-1) in Definition 1.

It is clear that $\{d_t\colon\ t \in (0,1)\}$ satisfies the condition (SM-2) in Definition 1. We now prove that $\{d_t\colon\ t \in (0,1)\}$ satisfies the condition (SM-3) in Definition 1. In fact, for every t in $(0,1)$, by $\sup\limits_{a<1} T(a,a) = 1$, it follows that there exists a τ in $(0,1)$ such that $T(\tau,\tau) > t$. Hence for any positive integer n and all p, q, r in S, by (2), we have $F_{pq}(d_\tau(p,q) + 1/n) \geqslant \tau$ and $F_{qr}(d_\tau(q,r) + 1/n) \geqslant \tau$. Therefore

$$\begin{aligned}&F_{pr}(d_\tau(p,q) + d_\tau(q,r) + 2/n)\\ &\geqslant T(F_{pq}(d_\tau(p,q) + 1/n), F_{qr}(d_\tau(q,r) + 1/n))\\ &\geqslant T(\tau,\tau) > t.\end{aligned}$$

Consequently, it follows from (2) that $d_t(p,r) \leqslant d_\tau(p,q) + d_\tau(q,r) + 2/n$. Letting $n \to \infty$, we obtain $d_t(p,r) \leqslant d_\tau(p,q) + d_\tau(q,r)$.

From (2), it is easy to see that for every pair p, q of points in $S, d_t(p,q)$ is a nondecreasing function of t on $(0,1)$. Therefore it can be readily seen that for any pair p, q of points in S and any real number x, the set $\{t \in (0,1)\colon\ d_t(p,q) < x\}$ is Borel-measurable, that is, $\{d_t\colon\ t \in (0,1)\}$ satisfies the condition (SMG-1) in Definition 2. Now we show that the condition (SMG-2) in Definition 2 is satisfied. Indeed, for each pair p, q of points in S, it follows from the definition of the PM space that $F_{pq}(x)$ is a nondecreasing, left-continuous function of x. Therefore, by (2) and Proposition 2 in [4], we have

$$F_{pq}(x) = L\{t \in (0,1)\colon\ d_t(p,q) < x\}$$

for all real numbers x. From the above argument it follows that (S,F) is a semi-metrically generated PM space.

Sufficiency: The proof proceeds in the same way as that of Theorem 2 from [5], and is therefore omitted. □

Remark. Obviously the condition $\sup\limits_{a<1} T(a,a) = 1$ is much weaker than $T =$ Min. Moreover, B. Morrel and J. Nagata [2] showed that no condition weaker than $\sup\limits_{a<1} T(a,a) = 1$ can guarantee that the ε, λ neighbourhoods induce a bona fide topology.

References

[1] *K. Menger, B. Schweizer, A. Sklar*: On probabilistic metrics and numerical metrics with probability 1. Czechoslovak Math. J. *9(84)* (1959), 459–466.

[2] *B. Morrel, J. Nagata*: Statistical metric spaces as related to topological spaces. Gen. Topology Appl. *9* (1978), 233–237.

[3] *H. Sherwood*: On E spaces and their relation to other classes of probabilistic metric spaces. J. London Math. Soc. *44* (1969), 441–448.

[4] *H. Sherwood, M. D. Taylor*: Some PM structures on the set of distribution functions. Rev. Roumaine Math. Pures Appl. *10* (1974), 1251–1260.
[5] *R.Stevens*: Metrically generated probabilistic metric spaces. Fund. Math. 61 (1968), 259–269.

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