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ON VECTORIAL INNER PRODUCT SPACES

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Abstract. Let E be a real linear space. A vectorial inner product is a mapping from $E\times E$ into a real ordered vector space Y with the properties of a usual inner product. Here we consider Y to be a $\mathcal{B}$-regular Yosida space, that is a Dedekind complete Yosida space such that $\bigcap\limits_{J\in\mathcal{B}} J=\{0\}$, where $\mathcal{B}$ is the set of all hypermaximal bands in Y. In Theorem 2.1.1 we assert that any $\mathcal{B}$-regular Yosida space is Riesz isomorphic to the space $B(A)$ of all bounded real-valued mappings on a certain set A. Next we prove Bessel Inequality and Parseval Identity for a vectorial inner product with range in the $\mathcal{B}$-regular and norm complete Yosida algebra $(B(A), \sup\limits_{\alpha\in A}|x(\alpha)|)$.

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1. Introduction

1.1. Riesz Spaces and Yosida Spaces. The easiest and most intuitive way of defining a Riesz space (vector lattice) is through the concept of an ordered vector space. A vector space X over $\mathbb{R}$ endowed with an order relation "$\preceq$" (i.e., a binary relation satisfying $x\preceq x$, $x\preceq y\wedge y\preceq x\Rightarrow x=y$, and $x\preceq y\wedge y\preceq z\Rightarrow x\preceq z$ for all $x,y,z\in X$) is called an ordered vector space if the ordering "$\preceq$" is invariant under translations and under multiplication by positive scalars. An ordered vector space X over $\mathbb{R}$ is called a Riesz space (or a vector lattice) if any doubleton $\{x,y\}\subset X$ has a least upper bound $\sup(x,y)$ and a greatest lower bound $\inf(x,y)$. The ordering of X is called Archimedean if $nx\preceq y$ for some pair (x,y) and all $n\in\mathbb{N}$ implies $x\preceq 0$.

Let X denote a Riesz space. In addition to the lattice operations $(x,y)\to\sup(x,y)$ and $(x,y)\to\inf(x,y)$ from $X\times X$ into X, it has proved useful to define mappings from X into X: $x\to x^+:=\sup(x,0)$, $x\to x^-:=\sup(-x,0)$, and $x\to|x|:=\sup(x,-x)$. The elements x, y are called disjoint if $\inf(|x|,|y|)=0$; this is often denoted by $x \perp y$. If A is a non-empty subset of X we define the disjoint complement of A as the set $A^{\perp} = \{x \in X\colon \inf(|x|, |y|) = 0 \text{ for all } y \in A\}$.

A Riesz space X is called Dedekind complete (or order complete) if each non-empty subset bounded from above possesses a least upper bound. If there exists an element $\boldsymbol{e}$ $(0 \preceq \boldsymbol{e})$ such that for every $x \in X$ there exists a positive real number α satisfying $|x| \preceq \alpha\boldsymbol{e}$, the Riesz space is called unitary. The element $\boldsymbol{e}$ is called a unit (or a strong unit). A unitary Archimedean Riesz space is called a Yosida space.

Ideals J of a Riesz space are linear subspaces characterized by the property $(x \in J$, $y \in X \wedge |y| \preceq |x|) \Rightarrow y \in J$. Bands are ideals with the additional property that they contain the suprema of arbitrary subsets, whenever these suprema exist in X.

Let X_1 and X_2 be Riesz spaces. A linear mapping h from X_1 into X_2 is called a Riesz homomorphism provided $\inf(h(x), h(y)) = 0$ holds for every pair of elements $x, y \in X_1$ satisfying $\inf(x, y) = 0$. If, in addition, h is bijective, then it is called a Riesz isomorphis.

Let X be a unitary Riesz space ($\boldsymbol{e}$ a fixed unit), J an arbitrary fixed maximal ideal and X/J the quotient space of all equivalence classes modulo J.

The quotient space X/J consists of all real multiples of $\boldsymbol{e}$, that is, given $x \in X$, there exists a real number λ such that $[x] = \lambda[\boldsymbol{e}]$. We will denote this number by $x(J)$. For x fixed and J running through the set $\mathcal{M}$ of all maximal ideals in X, we thus obtain a bounded real-valued mapping defined on $\mathcal{M}$.

2. Regular Yosida Spaces

2.1. $\mathcal{B}$-Regular Yosida Spaces. In [6] F. Robert defined a regular Yosida space as a space where every maximal ideal is a band. In [5] we showed that this notion of regular Yosida space implies that the space has necessarily a finite number of maximal ideals and consequently is finitedimensional. In this sense in [4] we have introduced the notion of a $\mathcal{B}$-regular Yosida space that generalizes the above notion to Yosida spaces with infinitely many maximal ideals. In a Riesz space a maximal band is called a hypermaximal band if it is also a maximal ideal.

Definition 2.1.1. In a Riesz space X we will denote by $\mathcal{B}$ the set of all hypermaximal bands in X.

We will say that a Dedekind complete Riesz space is $\mathcal{B}$-regular if $\bigcap_{J \in \mathcal{B}} J = \{0\}$.

In [5] we also established the following important theorem

Theorem 2.1.1 (Yosida Theorem). *Let Y be a $\mathcal{B}$-regular Yosida space. The mapping $(x \in Y) \longrightarrow (J \in \mathcal{B} \longrightarrow x(J) \in \mathbb{R})$ defines a one-one Riesz homomorphism from Y onto the Riesz space $B(\mathcal{B})$ of all bounded real-valued mappings defined on $\mathcal{B}$.*

Moreover, we have

1. $x(J) = 0 \Leftrightarrow x \in J$, $x = 0 \Leftrightarrow (\forall J \in \mathcal{B}$, $x(J) = 0)$,
2. $((x+y)(J) = x(J) + y(J)$, $(\alpha x)(J) = \alpha x(J))$, $\forall J \in \mathcal{B}$,
3. $x \preceq y \Leftrightarrow (\forall J \in \mathcal{B}$, $x(J) \leqslant y(J))$,
4. $(|x|)(J) = |x(J)|$, $\forall J \in \mathcal{B}$,
5. $\boldsymbol{e}(J) = 1$, $\forall J \in \mathcal{B}$,
6. $((\sup(x,y))(J) = \max(x(J), y(J))$, $(\inf(x,y))(J) = \min(x(J), y(J)))$, $\forall J \in \mathcal{B}$,
7. *If* $J_1 \neq J_2$ *there exists* $x \in Y$ *such that* $x(J_1) \neq x(J_2)$.

In view of Theorem 2.1.1, we have that any $\mathcal{B}$-regular Yosida space is Riesz isomorphic to the space of all bounded real-valued mappings on a certain set. So it seems desirable to assume, without loss of generality, that Y is such a space; we will denote it by $B(A)$.

In the $\mathcal{B}$-regular Yosida space $B(A)$ of all bounded real-valued mappings on a certain set A we have $\mathcal{B} = \{J_\alpha\colon\ \alpha \in A\}$ where $J_\alpha = \{x \in B(A)\colon\ x(\alpha) = 0\}$.

2.2. Vectorially Normed Spaces. Let E be a linear space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and $B(A)$ the space of all bounded real-valued mappings on A.

A vectorial norm p is a mapping from E into $B(A)$ with the properties of a usual norm, i.e. $p(\lambda u) = |\lambda| p(u)$, $p(u+v) \preceq p(u) + p(v)$ and if $p(u) = 0$ then $u = 0$.

The space E with a vectorial norm is named a vectorially normed space.

Let p be a vectorial norm from the linear space E into $B(A)$, and let us consider the family $(\theta_\alpha)_{\alpha \in A}$ of usual seminorms defined in the following way:

For each $\alpha \in A$, we set

$$\theta_\alpha(u) = (p(u))(\alpha) \quad \forall u \in E.$$

The kernel of θ_α will be denoted by V_α, i.e. $V_\alpha = \{u \in E\colon\ (p(u))(\alpha) = 0\}$. For each $\alpha \in A$, we shall also consider the subspace $W_\alpha = \{u \in E\colon\ (p(u))(\beta) = 0, \beta \neq \alpha\}$.

For each finite subset S of A, we define $W(S)$ as the direct sum

$$W(S) = \bigoplus_{\alpha \in S} W_\alpha.$$

Denoting by $\mathcal{PF}(A)$ the set of all finite subsets of A, we also define

$$W = \bigcup_{S \in \mathcal{PF}(A)} W(S).$$

2.3. Regular Vectorial Norms. Let E be a linear space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), p a vectorial norm defined on E with range in the Banach lattice $(B(A), \|.\| = \sup_{\alpha \in A} |.(\alpha)|)$.

Let us consider the usual norm g defined by $g(u) = \|p(u)\| = \sup_{\alpha \in A}(p(u)(\alpha))$.

Definition 2.3.1. Let p be a vectorial norm defined on a linear space E and with range in $(B(A), \|.\|)$. The vectorial norm p is said to be regular if $\overline{W} = E$ (considering in E the topology induced by the norm $g(.) = \|p(.)\|$).

The equivalence of norms in Banach lattices allows us to set forth the last definition for a vectorial norm with range in any Banach lattice.

In what follows we denote by $\mathcal{PN}(A)$ the set of all finite or countably infinite subsets of A.

Now we state the following theorem for purposes of later reference, its proof can be found in [5].

Theorem 2.3.1. *Let E be a real or a complex linear space, p a regular vectorial norm defined on E with its range in $(B(A), \|.\|)$. Let us also suppose that $(E, g(.))$ is a Banach space. Then*

1. $E = V_\alpha \oplus W_\alpha\ \forall \alpha \in A$,
2. *let P_α be the projection of $E = V_\alpha \oplus W_\alpha$ onto W_α. For each $u \neq 0$ in E there exists a set $S_u \in \mathcal{PN}(A)$ such that $P_\alpha u = u_\alpha = 0$ if and only if $\alpha \notin S_u$. The family $\{u_\alpha\}_{\alpha \in S_u}$ is summable (with respect to the topology induced by the norm $g(.)$), with the sum u, i.e. $u = \sum_{\alpha \in S_u} u_\alpha$, $u_\alpha \in W_\alpha$, $u_\alpha \neq 0$, $\forall \alpha \in S_u$. Moreover, the representation of the element u in the above form is unique.*

3. Vectorial Inner Product Spaces

3.1. Vectorial Norms and Vectorial Inner Products. We will say that a real linear space E is a vectorial inner product space associated to $B(A)$ if there is a mapping F from $E \times E$ with range in $B(A)$, subject to the following axioms:

A1. $F(u, v) = F(v, u)$,
A2. $F(u + v, w) = F(u, w) + F(v, w)$,
A3. $F(ku, v) = kF(u, v)$,
A4. $F(u, u) \succeq 0$ and $F(u, u) = 0$ if and only if $u = 0$,

where u, v, w are arbitrary elements of E and k is an arbitrary real scalar. The mapping F will be called a vectorial inner product.

Let us define, for each $\alpha \in A$, the mapping

$$\begin{array}{ccc} E & \longrightarrow & \mathbb{R}_0^+ \\ u & \longrightarrow & (F(u, u)(\alpha))^{1/2}. \end{array}$$

The mappings just defined are usual seminorms, moreover

$$(F(u, u)(\alpha))^{1/2} = 0 \quad \text{if and only if } F(u, u) \in J_\alpha$$

and, for all $\alpha \in A$,

$$|F(u,v)(\alpha)| \leqslant (F(u,u)(\alpha))^{1/2}.(F(v,v)(\alpha))^{1/2} \qquad \forall u,v \in E.$$

Let us now consider the usual product of bounded real-valued mappings defined on $B(A)$. The mapping

$$\begin{array}{rccc} p\colon & E & \longrightarrow & B(A) \\ & u & \longrightarrow & p(u) = F(u,u)^{1/2} \end{array}$$

is a vectorial norm.

This definition of p is equivalent to

$$p(u)(\alpha) = (F(u,u)(\alpha))^{1/2} \ \forall \alpha \in A.$$

The inequality

$$|F(u,v)(\alpha)| \leqslant (F(u,u)(\alpha))^{1/2} \cdot (F(v,v)(\alpha))^{1/2} \quad \forall \alpha \in A$$

can be written in the form

$$|F(u,v)| \preceq p(u)p(v).$$

We assume that in E the topology of a normed linear space is given, using the norm $g(u) = \|p(u)\| = \|(F(u,u))^{1/2}\|$.

Definition 3.1.1. Let E be a linear space, F a vectorial inner product from $E \times E$ into the space $(B(A), \|.\|)$ and suppose that $p(.) := F(.,.)^{1/2}$ is a regular vectorial norm. The space E is said to be a Vectorial Hilbert Space if it is a complete space (with respect to the topology induced by the norm $g(.) = \|F(.,.)^{1/2}\|$).

3.2. Orthogonality and Orthonormal Sets. Let E be a vectorial inner product space associated to $B(A)$.

We say that $u, v \in E$ are orthogonal, if $F(u,v) = 0$ (symbolically $u \perp v$).

Let us define the following elements of $B(A)$. For each $\alpha \in A$ the element $e_\alpha \in B(A)$ is such that

$$e_\alpha(\beta) = \begin{cases} 1 & \text{if } \beta = \alpha, \\ 0 & \text{if } \beta \neq \alpha. \end{cases}$$

Definition 3.2.1. A set S of elements of E is called an orthogonal set if $u \perp v$ for every pair u, v for which $u \in S$, $v \in S$, and $u \neq v$. If in addition for every $u \in S$ there exists $\alpha_u \in A$ such that $F(u,u) = e_{\alpha_u}$, the set S is called an orthonormal set.

Definition 3.2.2. Let S be an orthonormal set in the space E. The set S is called maximal if there exists no orthonormal set of which S is a proper subset. The set S is called complete if $F(v,u) = 0$ for all $u \in S$ implies $v = 0$.

Theorem 3.2.1. *Let E be a vectorial Hilbert space associated to $B(A)$ and S an orthonormal set in E. Then S is maximal if and only if S is complete.*

4. The Space $\overline{W_{B(A)}}$

4.1. Definition and Properties. As can be easily observed the norm complete and $\mathcal{B}$-regular Yosida space $(B(A), \|.\|)$, with the usual product of bounded real-valued mappings, is a commutative normed Yosida algebra (i.e., it is a Yosida space with a product satisfying $xy = yx$, $(xy)z = x(yz)$, $x(y+z) = xy + xz$, $(kx)(y) = k(xy)$, if $0 \preceq w$ and $x \preceq y$ then $wx \preceq wy$, and $\|xy\| \leqslant \|x\| \, \|y\|$). Ideals J of the Yosida algebra are ideals of the Riesz space $B(A)$ with the additional property: $y \in J \Rightarrow (\forall x \in B(A),\ xy \in J)$. The Yosida algebra is still $\mathcal{B}$-regular since $\bigcap\limits_{J \in \mathcal{B}} J = \{0\}$, where $\mathcal{B}$ is the set of all hypermaximal bands of the algebra.

The algebra $B(A)$ can also be regarded as a vectorially normed space with the vectorial norm $m(x) = \sup(x, -x) = |x|$ with its range in $B(A)$.

We must observe that $B(A)$ as a vectorially normed space is also a Banach space with the norm $g(x) = \sup\limits_{\alpha \in A}((m(x))(\alpha)) = \|x\|$.

In this particular case we have that, given $\alpha \in A$ and $x \in B(A)$, we have

$$\theta_\alpha(x) = (m(x))(\alpha) = |x(\alpha)|, \quad V_\alpha = J_\alpha, \quad W_\alpha = J_\alpha^\perp.$$

Let $W_{B(A)}$ denote the linear subspace of $B(A)$ of all bounded real-valued mappings with finite support. We define the space $\overline{W_{B(A)}}$ as the closure of $W_{B(A)}$ with respect to the topology induced by the norm $\|.\|$. The proof of the following theorem can be found in [4].

Theorem 4.1.1. *Let $(B(A), \|.\|)$ be the $\mathcal{B}$-regular and norm complete Yosida algebra of all bounded real-valued mappings defined on A. Consider also the vectorially normed space $(B(A), m = |.|)$. Then:*

1. *For each $x \neq 0$ in $\overline{W_{B(A)}}$ there exists a set $S_x \in \mathcal{PN}(A)$ such that $x(\alpha) = 0$ if and only if $\alpha \notin S_x$.*
   The family $\{x(\alpha)e_\alpha\}_{\alpha \in S_x}$ is summable (with respect to the topology induced by the norm $\|.\|$) with the sum x, i.e. $x = \sum\limits_{\alpha \in S_x} x(\alpha)e_\alpha$. Conversely, if an element

$x \in B(A)$ is of the form $x = \sum_{\alpha \in S} k_\alpha e_\alpha$, $S \in \mathcal{PN}(A)$ then $x \in \overline{W_{B(A)}}$ and $k_\alpha = x(\alpha)$.

2. *The linear subspace $\overline{W_{B(A)}}$ (as a subset of the norm complete Yosida algebra $B(A)$) is a norm complete Archimedean and Dedekind complete Riesz algebra.*

Now let us consider the following definition.

Definition 4.1.1. Let X be a Riesz space. Given $x, y \in X$ satisfying $x \preceq y$ the subset $\{z \in X/x \preceq z \preceq y\}$ of X is called an order interval, and it is denoted by $[x, y]$. The subset A of X is said to be order bounded if A is included in some order interval.

Our objective now is to justify that every order bounded monotone sequence in the space $(\overline{W_{B(A)}}, \|.\|)$ is convergent in norm.

If (a_n) is a nonnegative order bounded increasing sequence in $\overline{W_{B(A)}}$ then, since this space is Dedekind complete, there exists $a \in \overline{W_{B(A)}}$ such that $a = \sup\{a_n/n \in \mathbb{N}\}$. Using Theorem 4.1.1 it can be shown that the sequence (a_n) converges in norm to a.

In a similar way it can be shown that every nonnegative order bounded decreasing sequence is convergent in norm to the greatest lower bound of the set of its terms.

Supposing now that (x_n) is an arbitrary order bounded increasing sequence in $(\overline{W_{B(A)}}, \|.\|)$, from

$$x_n \preceq x_{n+1} \quad \forall n \in \mathbb{N}$$

we get

$$x_n^+ \preceq x_{n+1}^+ \wedge x_{n+1}^- \preceq x_n^- \quad \forall n \in \mathbb{N},$$

i.e., (x_n^+) is a nonnegative increasing sequence and (x_n^-) is a nonnegative decreasing sequence. They are also order bounded since

$$0 \preceq x_n^+ \preceq |x_n| \wedge 0 \preceq x_n^- \preceq |x_n| \quad \forall n \in \mathbb{N}.$$

It is now evident from the foregoing that (x_n) $(x_n = x_n^+ - x_n^-)$ is convergent.

It is obvious that this result remains true for decreasing order bounded sequences and we have

Theorem 4.1.2. *Every order bounded monotone sequence in the normed Riesz space $(\overline{W_{B(A)}}, \|.\|)$ is convergent in norm.*

In what follows $(E, g = \|p(.)\|)$ is a vectorial Hilbert space with $p(.) = (F(.,.))^{1/2}$ where F is a vectorial inner product defined on $E \times E$ and with its range in the $\mathcal{B}$-regular norm complete Yosida algebra $(B(A), \|.\|)$.

5.1. Bessel Inequality in Vectorial Hilbert Spaces. Before stating the next theorem the following result is required:

Lemma 5.1.1. *Let $\{w_1, w_2, \ldots, w_k\}$ be an orthonormal set. Then*

$$\sum_{i=1}^{k} |F(u, w_i)F(v, w_i)| \preceq p(u)p(v) \quad \forall u, v \in E. \tag{5.1.1}$$

Theorem 5.1.1 (Bessel Inequality—Countably Infinite Case). *Let $(w_n)_{n\in\mathbb{N}}$ be an orthonormal sequence of elements of E. Given $u, v \in E$ the series*

$$\sum_{i=1}^{\infty} |F(u, w_i)F(v, w_i)|$$

is convergent in $\overline{W_{B(A)}}$ and we have

$$\sum_{i=1}^{\infty} |F(u, w_i)F(v, w_i)| \preceq p(u)p(v). \tag{5.1.2}$$

Proof. To begin with we must observe that $p(u) \in \overline{W_{B(A)}}$ for all $u \in E$.

The sequence

$$0 \preceq z_n = \sum_{i=1}^{n} |F(u, w_i)F(v, w_i)|$$

is an order bounded increasing sequence of elements of $\overline{W_{B(A)}}$. According to the last lemma for any integer n we have

$$0 \preceq z_n = \sum_{i=1}^{n} |F(u, w_i)F(v, w_i)| \preceq p(u)p(v),$$

hence by Theorem 4.1.2 the sequence (z_n) is convergent in $\overline{W_{B(A)}}$; let z be its limit. We also have

$$z = \lim z_n = sup\{z_k / k \in \mathbb{N}\},$$

thus

$$z \preceq p(u)p(v)$$

or equivalently

$$\sum_{i=1}^{\infty} |F(u, w_i)F(v, w_i)| \preceq p(u)p(v).$$

□

In the last theorem we have proved the convergence of the series

$$\sum_{i=1}^{\infty} |F(u, w_i)F(v, w_i)|$$

which can also be written in the form $\sum_{i=1}^{\infty} |(F(u, w_i)F(v, w_i))(\alpha_{w_i})|e_{\alpha_{w_i}}$. Now it seems important to point out that this series remains convergent to the same limit, no matter how its terms are rearranged. More precisely, the following result holds.

Lemma 5.1.2. *The series* $\sum_{i=1}^{\infty} \xi_i e_{\alpha_i}$, $\xi_i \geqslant 0$ *for all* $i \in \mathbb{N}$, *is convergent if and only if the family* $\{\xi_i e_{\alpha_i}\}_{i\in\mathbb{N}}$ *is summable. In that case we have that the sum of the series coincides with the sum of the family.*

The proof and further details can be found in [4].

Theorem 5.1.2 (Bessel Inequality—General Case). *Let S be an arbitrary orthonormal set in E. The set of $w \in S$ such that $F(u, w) \neq 0$ (u any fixed element of E) is either finite or countably infinite. Given $u, v \in E$ we have that* $\sum_{w\in S} |F(u, w)F(v, w)|$ *defines an element in* $\overline{W_{B(A)}}$, *and*

$$\sum_{w\in S} |F(u, w)F(v, w)| \preceq p(u)p(v) \tag{5.1.3}$$

it being understood that the sum on the left includes all $w \in S$ for which $F(u, w)F(v, w) \neq 0$, and is, therefore, either a finite series or a convergent series with a countable infinity of terms.

Proof. By Theorem 2.3.1 there exists $S_u, S_v \in \mathcal{PN}(A)$ such that

$$u = \sum_{\alpha\in S_u} u_\alpha, u_\alpha \in W_\alpha, u_\alpha \neq 0, \forall\alpha \in S_u, \quad v = \sum_{\beta\in S_v} v_\beta, v_\beta \in W_\beta, v_\beta \neq 0, \forall\beta \in S_v.$$

For each $\gamma \in S_u$ let S_γ be the set of $w \in S$ such that $F(w,w) = e_\gamma$. Observe that $S_\gamma = S \cap W_\gamma$. For each $u_\gamma \in W_\gamma$, if $w \notin S_\gamma$ then $F(u_\gamma, w) = 0$. Clearly we have that

$$F(u_\gamma, w) = 0$$

is equivalent to

$$F(u_\gamma, w)(\gamma) = 0.$$

As for each $\gamma \in A$ the mapping $\Pi_\gamma(.,.) := F(.,.)(\gamma)$ restricted to W_γ is a usual inner product it follows that the set $A_\gamma \subset S_\gamma$ of $w \in S_\gamma$ such that $F(u_\gamma, w) \neq 0$ is either finite or countably infinite.

Given any fixed element $w \in S$ we have

$$F(u,w) = F\left(\sum_{\alpha \in S_u} u_\alpha, w\right) = \sum_{\alpha \in S_u} F(u_\alpha, w)$$

and consequently $F(u,w) \neq 0$ if and only if $w \in \cup_{\alpha \in S_u} A_\alpha$, which is still a finite or countably infinite set. Hence the set of those w for which $|F(u,w)F(v,w)| \neq 0$ is either finite or countably infinite.

The previous theorem along with Lemma 5.1.2 allow us to conclude the inequality (5.1.3). Observe that $\sum_{w \in S} |F(u,w)F(v,w)|$ defines in fact an element in $\overline{W_{B(A)}}$ since $p(u)p(v) \in \overline{W_{B(A)}}$ for all $u, v \in E$, and $\overline{W_{B(A)}}$ is an ideal of the algebra $B(A)$. $\square$

Corollary 5.1.1. *Under the assumptions of the last theorem, for each $u \in E$ there is an element $u_S \in E$ uniquely defined by*

$$u_S = \sum_{w \in S} F(u,w)(\alpha_w)w,$$

it being understood that the sum includes all $w \in S$ for which $F(u,w)(\alpha_w) \neq 0$ and is, therefore, either a finite series or a convergent series with a countable infinity of terms.

Besides, the element $v = u - u_S$ is such that

$$F(v,w) = 0, \quad \forall w \in S.$$

The proof of this corollary can be found in [4].

5.2. Parseval Identity in Vectorial Hilbert Spaces. We begin this subsection with the following lemma:

Lemma 5.2.1. *The space E possesses a maximal orthonormal set. Besides, an orthonormal set S is maximal in E if and only if*

$$S = \bigcup_{\alpha \in A} S_\alpha$$

where each S_α is a maximal orthonormal set of the Hilbert space $(W_\alpha, F(.,.)(\alpha))$.

Theorem 5.2.1. *The orthonormal set S is complete if and only if*

$$u = \sum_{w \in S} F(u,w)(\alpha_w)w, \quad \forall u \in E.$$

P r o o f. Let us suppose that S is complete. By Corollary 5.1.1, given $u \in E$ the element $v = u - \sum_{w \in S} F(u,w)(\alpha_w)w$ is such that $F(v,w) = 0$ for all $w \in S$. Since S is complete it follows that $v = 0$, i.e. $u = \sum_{w \in S} F(u,w)(\alpha_w)w$.

Conversely, since $u = \sum_{w \in S} F(u,w)(\alpha_w)w$ for all $u \in E$, hence if $F(u,w) = 0$ for all $w \in S$ it follows readily that $u = 0$ and S is complete. □

Theorem 5.2.1 (Parseval Identity). *An orthonormal set S is maximal if and only if*

$$p^2(u) = \sum_{w \in S} (F(u,w)(\alpha_w))^2 e_{\alpha_w}, \quad \forall u \in E.$$

P r o o f. Let us suppose that given $u \in E$ we have $p^2(u) = \sum_{w \in S} (F(u,w)(\alpha_w))^2 e_{\alpha_w}$.

If $F(u,w) = 0$ for all $w \in S$ then $p(u) = 0$ and $u = 0$, i.e. S is complete or equivalently (Theorem 3.2.1) S is maximal.

Conversely, let S be maximal. Once again by Theorem 3.2.1 S is complete and by the last theorem, given $u \in E$ we have $u = \sum_{w \in S} F(u,w)(\alpha_w)w$, i.e. there is a finite or countable set $\{w_n\}$ in S such that $u = \sum_n F(u,w_n)(\alpha_{w_n})w_n$.

Since $S_m = \sum_{i=1}^{m} F(u,w_i)(\alpha_{w_i})w_i$ we have

$$F(S_m, S_m) = \sum_{i=1}^{m} (F(u,w_i)(\alpha_{w_i}))^2 e_{\alpha_{w_i}}.$$

As $S_m \longrightarrow u$ and F is continuous, it follows that

$$F(S_m, S_m) \longrightarrow F(u,u) = p^2(u),$$

i.e.

$$p^2(u) = \sum_n (F(u, w_n)(\alpha_{w_n}))^2 e_{\alpha_{w_n}} = \sum_{w \in S} (F(u, w)(\alpha_w))^2 e_{\alpha_w}.$$

$\square$

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