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STRONG TOPOLOGIES ON VECTOR-VALUED FUNCTION SPACES

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Abstract. Let $(X, \|\cdot\|_X)$ be a real Banach space and let E be an ideal of L^0 over a σ-finite measure space (Ω, Σ, μ). Let (X) be the space of all strongly Σ-measurable functions $f\colon \Omega \to X$ such that the scalar function $\widetilde{f}$, defined by $\widetilde{f}(\omega) = \|f(\omega)\|_X$ for $\omega \in \Omega$, belongs to E. The paper deals with strong topologies on $E(X)$. In particular, the strong topology $\beta(E(X), E(X)^{\sim}_n)$ ($E(X)^{\sim}_n =$ the order continuous dual of $E(X)$) is examined. We generalize earlier results of [PC] and [FPS] concerning the strong topologies.

Keywords: vector valued function spaces, locally solid topologies, strong topologies, Mackey topologies, absolute weak topologies

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Introduction and preliminaries

Vector-valued function spaces $E(X)$ endowed with some natural topologies have been examined by many authors (cf. [FPS], [FN], [G], [M], [PC], [R]). In the case when E is provided with a locally convex-solid topology ξ one can topologize the space $E(X)$ as follows. Let $\{p_\alpha\colon \alpha \in \mathscr{A}\}$ be a family of Riesz seminorms on E that generates ξ. By putting $\overline{p}_\alpha(f) = p_\alpha(\widetilde{f})$ for $f \in E(X)$ $(\alpha \in \mathscr{A})$ we obtain a family $\{\overline{p}_\alpha\colon \alpha \in \mathscr{A}\}$ of solid seminorms on $E(X)$ that defines a locally convex-solid topology $\overline{\xi}$ on $E(X)$ (called the topology associated with ξ). In particular, one can consider the topologies $\overline{\beta(E, E')}$, $\overline{\tau(E, E')}$, $\overline{|\sigma|(E, E')}$ associated with the strong topology $\beta(E, E')$, the Mackey topology $\tau(E, E')$ and the absolute weak topology $|\sigma|(E, E')$ ($E' =$ the Köthe dual of E). These topologies have been examined by N. Phuang-Các [PC] and M. Florencio, P. J. Paul, C. Sáez [FPC]. The topology $\overline{\beta(E, E')}$ is called *the natural topology* on $E(X)$ (see [FPS]). In particular, in [FPS] it is shown that if $\beta(E, E') = \tau(E, E')$ then the topological dual of $(E(X), \overline{\beta(E, E')})$

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is identifiable with $E'(X^*)$ iff the topological dual X^* of X has the Radon-Nikodym Property (briefly RNP) with respect to μ.

Following the definition of the order dual in the theory of Riesz spaces one can define the order dual $E(X)^\sim$ of $E(X)$ as the space of all those linear functionals F on $E(X)$ for which $\sup\{|F(h)|: h \in E(X), \widetilde{h} \leqslant \widetilde{f}\} < \infty$ for each $f \in E(X)$ (see Section 1). In this paper we consider strong topologies $\beta(E(X), I)$, where I is an ideal of $E(X)^\sim$. We show that the topologies $\beta(E(X), I)$ are locally solid. In particular, we obtain that $\beta(E(X), E(X)^\sim)$ coincides with the Mackey topology $\tau(E(X), E(X)^\sim)$ and $\overline{\beta(E, E^\sim)} = \beta(E(X), E(X)^\sim)$ (see Theorem 3.3).

First of all we are interested in the topology $\beta(E(X), E(X)_n^\sim)$, where $E(X)_n^\sim$ stands for the order continuous dual of $E(X)$ (see Section 1). Due to A.V. Bukhvalov ([B_3], [B_4]) we know that $E(X)_n^\sim$ is identifiable with the space $E'(X^*, X)$ of X-weak measurable functions and $E'(X^*, X) = E'(X^*)$ iff X^* has the RNP with respect to μ. It turns out that the formal similarity between the dual systems $\langle E, E'\rangle$ and $\langle E(X), E'(X^*, X)\rangle$ is complete. In fact, we prove that the strong topology $\beta(E(X), E'(X^*, X))$ coincides with the natural topology $\overline{\beta(E, E')}$ (see Theorem 3.4). Due to this identity we can examine the topology $\beta(E(X), E'(X^*, X))$ by making use of the properties of the topology $\beta(E, E')$ (see Corollary 3.5). We generalize earlier results of [PC], [FPS] concerning the strong topologies on $E(X)$, where the dual pair $\langle E(X), E'(X^*)\rangle$ with X^* satisfying the RNP is considered. In particular, we easily obtain that if $\beta(E, E') = \tau(E, E')$ then the topological dual of $E(X)$ endowed with $\beta(E(X), E'(X^*, X))$ is identifiable with $E'(X^*, X)$ (see Theorem 3.6.).

Finally we show that if $(E, \|\cdot\|_E)$ is a Banach function space with the norm $\|\cdot\|_E$ satisfying the σ-Fatou property, then the strong topology $\beta(E(X), E'(X^*, X))$ coincides with the topology of the norm $\|\cdot\|_{E(X)}$ on $E(X)$ (see Theorem 3.8).

For the terminology concerning Riesz spaces we refer to [AB_1], [AB_2]. Given a topological vector space (L, τ), by $(L, \tau)^*$ and $\mathrm{Bd}(L, \tau)$ we will denote its topological dual and the collection of all τ-bounded subsets of L respectively.

Throughout the paper let (Ω, Σ, μ) be a complete σ-finite measure space and let L^0 denote the corresponding space of equivalence classes of all Σ-measurable real valued functions.

Let E be an ideal of L^0 with $\operatorname{supp} E = \Omega$. As usual, let $E^\sim$ stand for the order dual of E. The Köthe dual E' of E is defined by

$$E' = \left\{ v \in L^0 \colon \int_\Omega |u(\omega)v(\omega)| \,\mathrm{d}\mu < \infty \quad \text{for all} \ \ u \in E \right\}.$$

Since the measure space (Ω, Σ, μ) is assumed to be σ-finite, the order continuous dual $E_n^\sim$ coincides with the σ-order continuous dual $E_c^\sim$ (see [KA, Chap. 10, §2]), and by [KA, Theorem 6.1.1] we have $E_n^\sim = \{\varphi_v \colon v \in E'\}$, where $\varphi_v(u) = \int_\Omega u(\omega)v(\omega)\,\mathrm{d}\mu$ for all $u \in E$. It is known that $E_n^\sim$ separates points of E iff $\operatorname{supp} E' = \Omega$.

Let $(X, \|\cdot\|_X)$ be a real Banach space, and let S_X and B_X denote the unit sphere and the unit ball in X respectively. Let X^* stand for the topological dual of $(X, \|\cdot\|_X)$. By $L^0(X)$ we will denote the linear space of equivalence classes of all strongly Σ-measurable functions $f\colon \Omega \to X$. For $f \in L^0(X)$ let $\widetilde{f}(\omega) = \|f(\omega)\|_X$ for $\omega \in \Omega$. Let

$$E(X) = \{f \in L^0(X)\colon \ \widetilde{f} \in E\}$$

(see [B$_1$], [CHM], [FN]).

Now we recall terminology concerning the solid structure of $E(X)$ (see [FN]).

A subset H of $E(X)$ is said to be *solid* whenever $\widetilde{f_1} \leqslant \widetilde{f_2}$ with $f_1 \in E(X)$, $f_2 \in H$ implies $f_1 \in H$. A linear subspace B of $E(X)$ is called *an ideal* of $E(X)$ whenever B is a solid subset of $E(X)$.

A linear topology τ on $E(X)$ is said to be *locally solid* if it has a local base at zero consisting of solid sets. A linear topology τ on $E(X)$ that is at the same time locally solid and locally convex will be called a *locally convex-solid topology* on $E(X)$.

A seminorm ϱ on $E(X)$ is said to be *solid* if $\varrho(f_1) \leqslant \varrho(f_2)$ whenever $\widetilde{f_1} \leqslant \widetilde{f_2}$.

1. Order dual and order continuous dual of vector valued function spaces

We begin by recalling the terminology concerning the duality theory of vector valued function spaces as set out in [N$_1$]. For a linear functional F on $E(X)$ let us put

$$|F|(f) = \sup\{|F(h)|\colon \ h \in E(X), \widetilde{h} \leqslant \widetilde{f}\}.$$

The set

$$E(X)^\sim = \{F \in E(X)^\#\colon \ |F|(f) < \infty \quad \text{for all} \quad f \in E(X)\}$$

will be called *the order dual* of $E(X)$ (here $E(X)^\#$ denotes the algebraic dual of $E(X)$). For $F_1, F_2 \in E(X)^\sim$ we will write $|F_1| \leqslant |F_2|$ whenever $|F_1|(f) \leqslant |F_2|(f)$ for all $f \in E(X)$.

A subset M of $E(X)^\sim$ is said to be *solid* whenever $|F_1| \leqslant |F_2|$ with $F_1 \in E(X)^\sim, F_2 \in M$ implies $F_1 \in M$. A linear subspace I of $E(X)^\sim$ is called *an ideal* of $E(X)^\sim$ if I is a solid subset of $E(X)^\sim$.

Theorem 1.1 (cf. [N$_1$, Theorem 3.2]). *Let τ be a locally solid topology on $E(X)$. Then $(E(X), \tau)^*$ is an ideal of $E(X)^\sim$.*

For a subset M of $E(X)^\sim$ we will denote by $S(M)$ its solid hull, i.e., the smallest solid set in $E(X)^\sim$ containing M. Note that

$$S(M) = \{F \in E(X)^\sim\colon \ |F| \leqslant |G| \quad \text{for some} \quad G \in M\}.$$

We shall need the following lemma.

Lemma 1.2 (cf. [N_1, Lemma 2.1]). *Let M be a subset of $E(X)^\sim$. Then for $f \in E(X)$ we have*

$$\begin{aligned}\sup\{|F|(f)\colon\ F \in M\} &= \sup\{|G(f)|\colon\ G \in S(M)\}\\ &= \sup\{|G(f)|\colon\ G \in \operatorname{conv}(S(M))\}.\end{aligned}$$

A linear functional F on $E(X)$ is said to be *order continuous*, whenever for a net (f_σ) in $E(X)$, $\widetilde{f}_\sigma \xrightarrow{(o)} 0$ in E implies $F(f_\sigma) \to 0$ (see [B_3], [B_4]). The set consisting of all order continuous linear functionals on $E(X)$ will be denoted by $E(X)_n^\sim$ and called *the order continuous dual* of $E(X)$ (see [N_1, Definition 2.3]).

It is known that $E(X)_n^\sim$ is an ideal of $E(X)^\sim$ (see [N_1]).

To describe the space $E(X)_n^\sim$ we now recall the terminology concerning spaces of X-weak measurable functions (see [B_2], [B_3], [B_4]).

For a given function $g\colon\ \Omega \to X^*$ and $x \in X$ we denote by g_x the real function on Ω defined by $g_x(\omega) = g(\omega)(x)$ for $\omega \in \Omega$. A function g is said to be X-*weak measurable* if the functions g_x are measurable for each $x \in X$. We shall say that two X-weak measurable functions g_1, g_2 are equivalent whenever $g_1(\omega)(x) = g_2(\omega)(x)$ μ-a.e. for each $x \in X$.

By $L^0(X^*, X)$ we will denote the linear space consisting of the equivalence classes of all X-weak measurable functions $g\colon\ \Omega \to X^*$. In view of the super Dedekind completeness of L^0 the set $\{|g_x|\colon\ x \in B_X\}$ is order bounded in L^0 for each $g \in L^0(X^*, X)$. Thus we can define the so-called *abstract norm* $\vartheta\colon\ L^0(X^*, X) \to L^0$ by

$$\vartheta(g) = \sup\{|g_x|\colon\ x \in B_X\} \quad \text{for} \quad g \in L^0(X^*, X).$$

Then $L^0(X^*) \subset L^0(X^*, X)$ and $\vartheta(g) = \widetilde{g}$ for $g \in L^0(X^*)$. For an ideal K of L^0 let

$$K(X^*, X) = \{g \in L^0(X^*, X)\colon\ \vartheta(g) \in K\}.$$

A subset C of $K(X^*, X)$ is said to be *solid* if $\vartheta(g_1) \leqslant \vartheta(g_2)$ with $g_1 \in K(X^*, X)$ and $g_2 \in C$ implies $g_1 \in C$. A solid linear subspace of $K(X^*, X)$ is called an *ideal* of $K(X^*, X)$ (see [N_1, Definition 1.2]).

In particular, the space $E'(X^*, X)$ is of importance. Due to A. V. Bukhvalov [B_4, Theorem 3.5], $E'(X^*, X) = E'(X^*)$ iff X^* has the RNP with respect to μ. It is known that reflexive Banach spaces and separable dual Banach spaces have the RNP (see [DU]).

The following important theorem describes order continuous linear functionals on $E(X)$ in terms of the space $E'(X^*, X)$ (see [B_3, Theorem 4.1]).

Theorem 1.3. *Assume that* $\operatorname{supp} E' = \Omega$. *Then for a linear functional* F *on* $E(X)$ *the following statements are equivalent:*

(i) F *is order continuous.*

(ii) *There exists a unique* $g \in E'(X^*, X)$ *such that*

$$F(f) = F_g(f) = \int_\Omega \langle f(\omega), g(\omega)\rangle \,\mathrm{d}\mu \quad \text{for all} \;\; f \in E(X).$$

Moreover, for each $g \in E'(X^*, X)$ *we have*

$$|F_g|(f) = \int_\Omega \widetilde{f}(\omega)\vartheta(g)(\omega)\,\mathrm{d}\mu \quad \text{for all} \;\; f \in E(X).$$

Since $E(X)^\sim_n$ is an ideal of $E(X)^\sim$, it is clear that a subset I of $E(X)^\sim_n$ is an ideal of $E(X)^\sim$ iff I is an ideal of $E(X)^\sim_n$, i.e., $|F_1| \leqslant |F_2|$ with $F_1 \in E(X)^\sim_n$, $F_2 \in I$ implies $F_1 \in I$.

The following theorem generalizes [PC, Proposition 6] and will be needed later.

Theorem 1.4. *Let* K *be an ideal of* E' *with* $\operatorname{supp} K = \Omega$ *and assume that* C *is a solid subset of* $K'(X^*, X)$. *Then for each* $f \in E(X)$ *the following identities hold:*

$$\begin{aligned}\sup\Big\{\Big|\int_\Omega \langle f(\omega), g(\omega)\rangle \,\mathrm{d}\mu\Big| \colon\; g \in C\Big\} &= \sup\Big\{\int_\Omega |\langle f(\omega, g(\omega)\rangle| \,\mathrm{d}\mu \colon\; g \in C\Big\}\\ &= \sup\Big\{\int_\Omega \widetilde{f}(\omega)\vartheta(g)(\omega)\,\mathrm{d}\mu \colon\; g \in C\Big\}.\end{aligned}$$

P r o o f. Observe that the set $\{F_g \colon\; g \in C\}$ is a solid subset of $E(X)^\sim$. In fact, let $|F| \leqslant |F_g|$, where $F \in E(X)^\sim$ and $g \in C$. Since $F_g \in E(X)^\sim_n$ and $E(X)^\sim_n$ is an ideal of $E(X)^\sim$ we conclude that $F \in E(X)^\sim_n$. Hence by Theorem 1.3, $F = F_{g'}$ for some $g' \in E'(X^*, X)$, and $|F_{g'}| \leqslant |F_g|$. By [N_1, Corollary 2.4] we see that $\vartheta(g') \leqslant \vartheta(g)$, so $g' \in C$, because C is a solid subset of $K(X^*, X)$. Thus $S(\{F_g \colon\; g \in C\}) = \{F_g \colon\; g \in C\}$. Combining Lemma 1.2 and Theorem 1.3 we obtain our identities. □

Throughout this section let I be an ideal of $E(X)^\sim$ that separates points of $E(X)$. We have the dual system $\langle E(X), I\rangle$ with the duality $\langle f, F\rangle = F(f)$ for $f \in E(X)$, $F \in I$ (see [N$_1$]). For each $f \in E(X)$ let us put

$$\varrho_f(F) = |F|(f) \quad \text{for all} \quad F \in I.$$

Then ϱ_f is a solid seminorm on I, that is, $\varrho_f(F_1) \leqslant \varrho_f(F_2)$ whenever $|F_1| \leqslant |F_2|$. We define the *absolute weak topology* $|\sigma|(I, E(X))$ on I as the locally convex-solid topology generated by the family $\{\varrho_f\colon f \in E(X)\}$.

Theorem 2.1. *For a subset M of I the following statements are equivalent:*
(i) *M is $|\sigma|(I, E(X))$-bounded.*
(ii) *M is $\sigma(I, E(X))$-bounded.*

P r o o f. (i) $\Rightarrow$ (ii) Obvious.

(ii) $\Rightarrow$ (i) For $0 \leqslant e \in E$ let $E_e = \{u \in E\colon |u| \leqslant \lambda e$ for some $\lambda > 0\}$. Let $p_e(u) = \inf\{\lambda > 0\colon |u| \leqslant \lambda e\}$ for $u \in E$. Then (E_e, p_e) is a Banach space (see [V, Theorem 7.4.2]) and $B_{p_e}(1) = \{u \in E\colon p_e(u) \leqslant 1\} = [-e, e]$. Let $E_e(X) = \{h \in L^0(X)\colon \widetilde{h} \in E_e\}$ and let $\overline{p}_e(h) = p_e(\widetilde{h})$. Then the space $(E_e(X), \overline{p}_e)$ is a Banach space (see [B$_1$, Theorem 2]). It is easy to observe that $B_{\overline{p}_e}(1) = \{h \in E_e(X)\colon \overline{p}_e(h) \leqslant 1\} = \{h \in E_e(X)\colon \widetilde{h} \leqslant e\}$.

Let $F \in M$ and let $\overline{e} = ex_0$, where $x_0 \in S_X$. Then $\sup\{|F(h)|\colon h \in E(X), \widetilde{h} \leqslant e\} < \infty$, because $|F(h)| \leqslant |F|(h) \leqslant |F|(\overline{e}) < \infty$ for each $h \in E(X)$ with $\widetilde{h} \leqslant e = \widetilde{\overline{e}}$. This shows that the functional $F|_{E_e(X)}$ restricted to $E_e(X)$ is bounded on $B_{\overline{\varrho}_e}(1)$. Thus $F|_{E_e(X)}$ is $\overline{p}_e$-continuous on $E_e(X)$, that is, $F|_{E_e(X)} \in (E_e(X), \overline{p}_e)^* = E_e(X)^*$. Since M is $\sigma(I, E(X))$-bounded, $\sup\{|F(h)|\colon F \in M\} < \infty$ for each $h \in E(X)$. It follows that the set $\{F|_{E_e(X)}\colon F \in M\}$ is $\sigma(E_e(X)^*, E_e(X))$-bounded. Hence by the uniform boundedness theorem (see [Wi, Theorem 3.3.6]) the set $\{F|_{E_e(X)}\colon F \in M\}$ is bounded in $E_e(X)^*$, so there exists $c > 0$ such that $\sup\{|F(h)|\colon F \in M,\ h \in B_{\overline{p}_e}(1)\} \leqslant c$, i.e.,

$$\begin{aligned}&\sup\{|F(h)|\colon F \in M, h \in E_e(X), \widetilde{h} \leqslant e\}\\ &\quad = \sup\{|F(h)|\colon F \in M, h \in E(X), \widetilde{h} \leqslant \widetilde{\overline{e}}\} \leqslant c.\end{aligned}$$

It follows that $\sup\{|F|(\overline{e})\colon F \in M\} \leqslant c$.

For $f \in E(X)$ let us put $e = \widetilde{f}$. Then $\widetilde{\overline{e}} = e = \widetilde{f}$, so $|F|(\overline{e}) = |F|(f)$ and $\sup\{|F|(f)\colon F \in M\} \leqslant c$. This shows that M is $|\sigma|(I, E(X))$-bounded. □

Corollary 2.2. *The solid hull $S(M)$ of a $\sigma(I, E(X))$-bounded subset of I is also $\sigma(I, E(X))$-bounded.*

P r o o f. Assume that M is a $\sigma(I, E(X))$-bounded subset of I. By Theorem 2.1, M is $|\sigma|(I, E(X))$-bounded. Hence also its solid hull $S(M)$ is $|\sigma|(I, E(X))$-bounded. Hence $S(M)$ is $\sigma(I, E(X))$-bounded, as desired. □

3. Strong topologies

Let I be an ideal of $E(X)^\sim$ that separates points of $E(X)$. For each $M \in \mathrm{Bd}(I, \sigma(I, E(X))$ (= the collection of all $\sigma(I, E(X))$-bounded subsets of I) let

$$\varrho_M(f) = \sup\{|F(f)|\colon\ F \in M\}.$$

The strong topology $\beta(E(X), I)$ is the Hausdorff locally convex topology on $E(X)$ generated by the family $\{\varrho_M\colon\ M \in \mathrm{Bd}(I, \sigma(I, E(X)))\}$.

Theorem 3.1. *The strong topology* $\beta(E(X), I)$ *is locally solid and is generated by the family of solid seminorms*

$$\varrho_M(f) = \sup\{|F|(f)\colon\ F \in M\}$$

where M *runs over the family* $\mathrm{Bd}_S(I, \sigma(I, E(X)))$ *of all* $\sigma(I, E(X))$*-bounded solid subsets of* I.

P r o o f. Assume that $M \in \mathrm{Bd}(I, \sigma(I, E(X)))$. Then by Corollary 2.2 its solid hull $S(M)$ is $\sigma(I, E(X))$-bounded and $\varrho_M(f) \leqslant \varrho_{S(M)}(f)$ for all $f \in E(X)$. Moreover, in view of Lemma 1.2, $\varrho_{S(M)} = \sup\{|G(f)|\colon\ G \in S(M)\} = \sup\{|F|(f)\colon\ F \in M\}$, so $\varrho_{S(M)}$ is a solid seminorm. This shows that to generate $\beta(E(X), I)$ it is enough to restrict ourselves to the family $\{\varrho_M\colon\ M \in \mathrm{Bd}_S(I, \sigma(I, E(X)))\}$, where $\varrho_M(f) = \sup\{|F|(f)\colon\ F \in M\}$. □

To describe the mutual connection between strong topologies on E and $E(X)$ we briefly explain the general relationship between topological structures of E and $E(X)$ (see [FN]).

Let $x \in S_X$. Given $u \in E$ let us put $u(\omega) = u(\omega)x$ for $\omega \in \Omega$. Then $u \in L^0(X)$ and $\|u(\omega)\|_X = |u(\omega)|$ for $\omega \in \Omega$, so $u \in E(X)$. For a solid seminorm ϱ on $E(X)$ let us set

$$\widetilde{\varrho}(u) = \varrho(u) \quad \text{for all} \quad u \in E.$$

Clearly $\widetilde{\varrho}$ is well defined, because $\varrho(u)$ does not depend on $x \in S_X$ in virtue of the solidness of ϱ. It is easy to check that $\widetilde{\varrho}$ is a Riesz seminorm on E.

Assume that τ is a locally convex-solid topology on $E(X)$. Then τ is generated by a family $\{\varrho_\alpha\colon\ \alpha \in \mathscr{A}\}$ of solid seminorms defined on $E(X)$ (see [FN, Theorem 2.2]).

By $\widetilde{\tau}$ we will denote the locally convex-solid topology on E generated by the family $\{\widetilde{\varrho}_\alpha\colon\ \alpha\in\mathscr{A}\}$ of Riesz seminorms on E. Clearly $\widetilde{\tau}$ is a Hausdorff topology, whenever τ is a Hausdorff topology.

We will need the following result.

Theorem 3.2 (cf. [FN]). *Let* ξ,ξ_1,ξ_2 *be locally convex-solid topologies on* E *and let* τ, τ_1, τ_2 *be locally convex-solid topologies on* $E(X)$. *Then:*

(i) $\widetilde{\overline{\xi}}=\xi$ *and* $\overline{\widetilde{\tau}}=\tau$.
(ii) *If* $\xi_1\subset\xi_2$, *then* $\overline{\xi}_1\subset\overline{\xi}_2$.
(iii) *If* $\tau_1\subset\tau_2$, *then* $\widetilde{\tau}_1\subset\widetilde{\tau}_2$.

Now we are in position to describe the relationship between the strong topologies $\beta(E,E^{\sim})$ and $\beta(E(X),E(X)^{\sim})$.

Theorem 3.3. *The strong topology* $\beta(E(X),E(X)^{\sim})$ *coincides with the Mackey topology* $\tau(E(X),E(X)^{\sim})$. *Hence* $\tau(E(X),E(X)^{\sim})$ *is locally solid. Moreover, the following identities hold:*

$$\overline{\beta(E,E^{\sim})}=\beta(E(X),E(X)^{\sim})\quad\text{and}\quad\widetilde{\beta(E(X),E(X)^{\sim})}=\beta(E,E^{\sim}).$$

P r o o f. Since $\beta(E(X),E(X)^{\sim})$ is a locally solid topology (see Theorem 3.1), in view of Theorem 1.1 we have $(E(X),\beta(E(X),E(X)^{\sim})^*)\subset E(X)^{\sim}$. It follows that $\beta(E(X),E(X)^{\sim})\subset\tau(E(X),E(X)^{\sim})$, so $\beta(E(X),E(X)^{\sim})=\tau(E(X),E(X)^{\sim})$, as desired.

In view of Theorem 1.1, $I_\tau=\big(E(X),\overline{\tau(E,E^{\sim})}\big)^*\subset E(X)^{\sim}$, so by the Mackey-Arens theorem $\overline{\tau(E,E^{\sim})}\subset\tau(E(X),I_\tau)$. Moreover, $\sigma(E(X),I_\tau)\subset\sigma(E(X),E(X)^{\sim})$, so $\tau(E(X),I_\tau)\subset\tau(E(X),E(X)^{\sim})$ (see [Ro]). Thus $\overline{\tau(E,E^{\sim})}\subset\tau(E(X),E(X)^{\sim})$. Hence by Theorem 3.2 we get

$$\tau(E,E^{\sim})=\widetilde{\overline{\tau(E,E^{\sim})}}\subset\widetilde{\tau(E(X),E(X)^{\sim})}.$$

Moreover, since $(E,\widetilde{\tau(E(X),E(X)^{\sim})})^*\subset E^{\sim}$ (see [AB$_1$, Theorem 5.7]), we get $\widetilde{\tau(E(X),E(X)^{\sim})}\subset\tau(E,E^{\sim})$.

Hence, by applying Theorem 3.2 we conclude that $\overline{\tau(E,E^{\sim})}\subset\tau(E(X),E(X)^{\sim})$ and $\tau(E(X),E(X)^{\sim})\subset\overline{\tau(E,E^{\sim})}$, so $\overline{\tau(E,E^{\sim})}=\tau(E(X),E(X)^{\sim})$. In view of Theorem 3.2 it follows that $\tau(E,E^{\sim})=\widetilde{\tau(E(X),E(X)^{\sim})}$. Since $\beta(E(X),E(X)^{\sim})=\tau(E(X),E(X)^{\sim})$ and $\beta(E,E^{\sim})=\tau(E,E^{\sim})$ (see [F, 81 I(g)]) the proof is complete. □

Now we examine the strong topology $\beta(E(X), I)$, where I is an ideal of $E(X)^{\sim}_n$. Recall that $E(X)^{\sim}_n = \{F_g\colon g \in E'(X^*, X)\}$, where for each $g \in E'(X^*, X)$

$$F_g(f) = \int_\Omega \langle f(\omega), g(\omega)\rangle \, \mathrm{d}\mu \quad \text{and} \quad |F_g|(f) = \int_\Omega \widetilde{f}(\omega)\vartheta(g)(\omega) \, \mathrm{d}\mu$$

for all $f \in E(X)$ (see Theorem 1.3).

Given an ideal of $E(X)^{\sim}_n$ let $A_I = \{g \in E'(X^*, X)\colon F_g \in I\}$. Then A_I is an ideal of $E'(X^*, X)$ and $A_I = \widetilde{A}_I(X^*, X)$, where

$$\widetilde{A}_I = \{v \in E'\colon |v| \leqslant \vartheta(g) \quad \text{for some} \quad g \in A_I\}$$

is an ideal of E' (see [N_1, Theorem 2.6, Theorem 1.2]).

Conversely, if K is an ideal of E' then $K(X^*, X)$ is an ideal of $E'(X^*, X)$ and the set $I_K = \{F_g\colon g \in K(X^*, X)\}$ is an ideal of $E(X)^{\sim}_n$.

Thus instead of the topologies $\beta(E(X), I)$ we can consider topologies $\beta(E(X), K(X^*, X))$, where K is an ideal of E'.

For each $C \in \mathrm{Bd}_S(K(X^*, X), \sigma(K(X^*, X), E(X)))$ (= the collection of all $\sigma(K(X^*, X), E(X))$-bounded solid subsets of $K(X^*, X)$) let us put

$$\varrho_C(f) = \sup\{|F_g(f)|\colon g \in C\}.$$

Note that $M_C = \{F_g\colon g \in C\} \in \mathrm{Bd}_S(I_K, \sigma(I_K, E(X)))$ and by Lemma 1.2 we get

$$\begin{aligned}\varrho_C(f) &= \sup\{|F_g(f)|\colon F_g \in M_C\} \\ &= \sup\{|F_g|(f)\colon F_g \in M_C\} = \sup\left\{\int_\Omega \widetilde{f}(\omega)\vartheta(g)(\omega)\, \mathrm{d}\mu\colon g \in C\right\}.\end{aligned}$$

Thus the strong topology $\beta(E(X), K(X^*, X))$ $(= \beta(E(X), I_K))$ is generated by the family $\{\varrho_C\colon C \in \mathrm{Bd}_S(K(X^*, X), \sigma(K(X^*, X), E(X)))\}$, where

$$\varrho_C(f) = \sup\left\{\int_\Omega \widetilde{f}(\omega)\vartheta(g)(\omega)\, \mathrm{d}\mu\colon g \in C\right\} \quad \text{for all} \quad f \in E(X).$$

Now let K be an ideal of E' with $\operatorname{supp} K = \Omega$. Let $\beta(E, K)$ and $|\sigma|(E, K)$ stand for the strong topology and the absolute weak topology on E with respect to the dual system $\langle E, K\rangle$. Since $\mathrm{Bd}(K, \sigma(K, E)) = \mathrm{Bd}(K, |\sigma|(K, E))$ (see [AB_1, Theorem 19.15]), arguing as in the proof of Theorem 3.1 we obtain that the strong topology $\beta(E, K)$ is generated by the family $\{p_D\colon D \in \mathrm{Bd}_S(K, \sigma(K, E))\}$ of Riesz seminorms, where $\mathrm{Bd}_S(K, \sigma(K, E))$ denotes the collection of all $\sigma(K, E)$-bounded solid subsets of K and

$$p_D(u) = \sup\left\{\int_\Omega |u(\omega)v(\omega)|\, \mathrm{d}\mu\colon v \in D\right\} \quad \text{for all} \quad u \in E.$$

Now we are ready to state our main result that shows that the formal similarity between the dual systems $\langle E, E'\rangle$ and $\langle E(X), E'(X^*, X)\rangle$ is complete.

Theorem 3.4. *Let K be an ideal of E' with* $\operatorname{supp} K = \Omega$. *Then the following identities hold:*

$$\overline{\beta(E,K)} = \beta(E(X), K(X^*, X)) \quad \text{and} \quad \widetilde{\beta(E(X), K(X^*, X))} = \beta(E, K).$$

In particular, we get

$$\overline{\beta(E,E')} = \beta(E(X), E'(X^*, X)) \quad \text{and} \quad \widetilde{\beta(E(X), E'(X^*, X))} = \beta(E, E').$$

Proof. To show that $\overline{\beta(E,K)} \subset \beta(E(X), K(X^*, X))$ assume that $D \in \mathrm{Bd}_S(K, \sigma(K, E))$. One can easily check that the set $C_D = \{g \in K(X^*, X)\colon \vartheta(g) \in D\}$ is a solid subset of $K(X^*, X)$. Moreover, by Theorem 1.4, for each $f \in E(X)$ we have

$$\begin{aligned}
\sup\Big\{\Big|\int_\Omega \langle f(\omega), g(\omega)\rangle \,\mathrm{d}\mu\Big|\colon\ g \in C_D\Big\} &= \sup\Big\{\int_\Omega \widetilde{f}(\omega)\vartheta(g)(\omega)\,\mathrm{d}\mu\colon\ g \in C_D\Big\}\\
&= \sup\Big\{\int_\Omega \widetilde{f}(\omega)|v(\omega)|\,\mathrm{d}\mu\colon\ v \in D\Big\} = p_D(\widetilde{f}) = \overline{p}_D(f).
\end{aligned}$$

It follows that $C_D \in \mathrm{Bd}_S(K(X^*, X), \sigma(K(X^*, X), E(X)))$ and $\varrho_{C_D}(f) = \overline{p}_D(f)$ for each $f \in E(X)$. Hence $\overline{\beta(E,K)} \subset \beta(E(X), K(X^*, X))$.

In turn, to see that $\widetilde{\beta(E(X), K(X^*, X))} \subset \beta(E, K)$, assume that

$$C \in \mathrm{Bd}_S(K(X^*, X), \sigma(K(X^*, X), E(X))).$$

Let $D_C = \{v \in K\colon\ |v| \leqslant \vartheta(g)$ for some $g \in C\}$. To prove that D_C is a solid subset of K, assume that $|v_1| \leqslant |v_2|$, where $v_1 \in K$ and $v_2 \in D_C$. Then $|v_1| \leqslant |v_2| \leqslant \vartheta(g)$ for some $g \in C$. Hence $v_1 \in D_C$. By Theorem 1.4, for each $u \in E$ we have

$$\begin{aligned}
\sup\Big\{\Big|\int_\Omega u(\omega)v(\omega)\,\mathrm{d}\mu\Big|\colon\ v \in D_C\Big\} &= \sup\Big\{\int_\Omega |u(\omega)v(\omega)|\,\mathrm{d}\mu\colon\ v \in D_C\Big\}\\
&= \sup\Big\{\int_\Omega |u(\omega)|\vartheta(g)(\omega)\,\mathrm{d}\mu\colon\ g \in C\Big\} = \sup\Big\{\Big|\int_\Omega \langle u(\omega), g(\omega)\rangle\,\mathrm{d}\mu\Big|\colon\ g \in C\Big\}\\
&= \varrho_C(\overline{\mathrm{u}}) = \widetilde{\varrho}_C(u).
\end{aligned}$$

It follows that $D_C \in \mathrm{Bd}_S(K, \sigma(K, E))$ and $p_{D_C}(u) = \widetilde{\varrho}_C(u)$ for each $u \in E$. Hence $\beta(E, K) \supset \widetilde{\beta(E(X), K(X^*, X))}$, as desired. Since $\overline{\beta(E,K)} \subset \beta(E(X), K(X^*, X))$ and $\widetilde{\beta(E(X), K(X^*, X))} \subset \beta(E, K)$, by Theorem 3.2 we get

$$\beta(E, K) = \widetilde{\overline{\beta(E,K)}} \subset \widetilde{\beta(E(X), K(X^*, X))} \subset \beta(E, K)$$

and

$$\beta(E(X),K(X^*,X)) = \overline{\widetilde{\beta(E(X),K(X^*,X))}} \subset \overline{\beta(E,K)} \subset \beta(E(X),K(X^*,X)).$$

Thus the proof is complete. □

As a consequence of Theorem 3.4 we obtain the following result.

Corollary 3.5. *Let E be a perfect function space (i.e., $E'' = E$). Then the space $(E(X), \beta(E(X), E(X)^\sim_n))$ is complete.*

P r o o f. In view of [F, 81 I(d)] the space $(E, \beta(E, E'))$ is complete and satisfies the Fatou property, so by [AB$_1$, Theorem 11.4] for $D \in \mathrm{Bd}_S(E', \sigma(E', E))$ the seminorms p_D have the Fatou property (i.e., $0 \leqslant u_\alpha \uparrow u$ in E implies $p_D(u_\alpha) \uparrow p_D(u)$). Hence by [B$_1$, Theorem 3] the space $(E(X), \overline{\beta(E, E')})$ is complete. In view of Theorem 3.4 the space $(E(X), \beta(E(X), E(X)^\sim_n))$ is complete as well. □

Remark. The above result extends [PC, Corollary of Proposition 10] where X^* is assumed to be separable (so X^* satisfies the RNP).

Now we examine the properties of $\beta(E(X), E(X)^\sim_n))$ in the case when $\beta(E, E')$ coincides with the Mackey topology $\tau(E, E')$. Since the space $(E^\sim_n, \sigma(E^\sim_n, E))$ is sequentially complete (see [KA, Corollary 10.3.1]), in view of [W, Proposition 4.15] the identity $\tau(E, E') = \beta(E, E')$ holds whenever the space $(E', \beta(E', E))$ is separable (cf. [We], [K, 30.7(1)]).

Theorem 3.6. *Assume that $\tau(E, E') = \beta(E, E')$. Then the following statements hold:*

(i) *$\beta(E(X), E(X)^\sim_n)$ is a Lebesgue topology (i.e., $\widetilde{f_n} \xrightarrow{(o)} 0$ in E imply $f_n \to 0$ for $\beta(E(X), E(X)^\sim_n)$).*

(ii) *$(E(X), \beta(E(X), E(X)^\sim_n))^* = E(X)^\sim_n$.*

(iii) *$\beta(E(X), E(X)^\sim_n)$ coincides with the Mackey topology $\tau(E(X), E(X)^\sim_n)$, so the space $(E(X), \tau(E(X), E(X)^\sim_n))$ is barreled and $\tau(E(X), E(X)^\sim_n)$ is locally solid.*

(iv) *Every $\sigma(E(X)^\sim_n, E(X))$-compact absolutely convex subset of $E(X)^\sim_n$ is contained in a solid $\sigma(E(X)^\sim_n, E(X))$-compact absolutely convex subset of $E(X)^\sim_n$.*

P r o o f. (i) Assume that (f_n) is a sequence in $E(X)$ with $\widetilde{f_n} \xrightarrow{(o)} 0$ in E. Then $\widetilde{f_n} \to 0$ for $\beta(E, E')$ because $\beta(E, E') = \tau(E, E') = \tau(E, E^\sim_n)$ and $\tau(E, E^\sim_n)$ is a Lebesgue topology (see [MR, Corollary 2.4], [AB$_1$, Theorem 9.1]). Hence $p_D(\widetilde{f_n}) \to 0$ for each $D \in \mathrm{Bd}_S(E', \sigma(E', E))$. Since $p_D(\widetilde{f}) = \overline{p}_D(f_n)$ for $n \in \mathbb{N}$ and $\overline{\beta(E, E')} = \beta(E(X), E(X)^\sim_n)$ (see Theorem 3.4) we conclude that $f_n \to 0$ for $\beta(E(X), E(X)^\sim_n)$, as desired.

(ii) From (i) it easily follows that $(E(X), \beta(E(X), E(X)_n^\sim))^* \subset E(X)_n^\sim$. Since $\tau(E(X), E(X)_n^\sim) \subset \beta(E(X), E(X)_n^\sim)$, we obtain that $(E(X), \beta(E(X), E(X)_n^\sim)^* \supset E(X)_n^\sim$.

(iii) In view of the Mackey-Arens theorem (ii) implies that $\beta(E(X), E(X)_n^\sim) \subset \tau(E(X), E(X)_n^\sim)$.

(iv) Let M be a $\sigma(E(X)_n^\sim, E(X))$-compact absolutely convex subset of $E(X)_n^\sim$. Since the Mackey topology $\tau(E(X), E(X)_n^\sim)$ is solid there exists a solid neighbourhood of 0 for $\tau(E(X), E(X)_n^\sim)$, say U, such that $U \subset C^0$. Hence $C = C^{00} \subset U^0$, where U^0 is a $\sigma(E(X)_n^\sim, E(X))$-compact absolutely convex and solid subset of $E(X)_n^\sim$, because polars of solid sets are solid (see [N_1, Theorem 3.3]). □

Hence as a consequence of Theorem 3.6 we get the following result.

Corollary 3.7. *Assume that* $\tau(E, E') = \beta(E, E')$. *Then*

$$(E(X), \beta(E(X), E(X)_n^\sim))^* = \{F_g \colon g \in E'(X^*)\}$$

iff X^* *has the RNP with respect to* μ.

Remark. In the case when Ω is a locally compact Hausdorff topological space and μ is a positive Radon measure on Ω the result of Corollary 3.7 was obtained by M. Florencio, P. J. Paúl, C. Sáez [FPS, Theorem 1].

Now we will deal with strong topologies on Köthe-Bochner spaces. Let $(E, \|\cdot\|_E)$ be a Banach function space. The space $E(X)$ provided with the solid norm $\|\cdot\|_{E(X)}$ defined by $\|f\|_{E(X)} = \|\widetilde{f}\|_E$ is usually called a Köthe-Bochner space (see [CHM]). The most important examples of Köthe-Bochner spaces are the Lebesgue-Bochner space $L^p(X)$ $(1 \leqslant p \leqslant \infty)$ and their generalization, the Orlicz-Bochner spaces $L^\varphi(X)$. We will denote by $\mathscr{T}_E$ and $\mathscr{T}_{E(X)}$ the topologies of the norms $\|\cdot\|_E$ and $\|\cdot\|_{E(X)}$ respectively. It is known that (see [N_1, Theorem 3.5]):

$$E(X)^* = (E(X), \mathscr{T}_{E(X)})^* = E(X)^\sim.$$

Assume that $\|\cdot\|_E$ satisfies the σ-Fatou property (i.e., $0 \leqslant u_n \uparrow u$ in E implies $\|u_n\|_E \uparrow \|u\|_E$). Then

$$\|u\|_E = \sup\left\{\left|\int_\Omega u(\omega)v(\omega)\,\mathrm{d}\mu\right| \colon\ v \in E', \|v\|_{E'} \leqslant 1\right\} \tag{3.1}$$

where $\|\cdot\|_{E'}$ is the associated norm on the Köthe dual E' of E, i.e.,

$$\|v\|_{E'} = \sup\left\{\left|\int_\Omega u(\omega)v(\omega)\,\mathrm{d}\mu\right| \colon\ u \in E, \|u\|_E \leqslant 1\right\}$$

(see [KA, Theorem 6.1.6]). Since $\mathscr{T}_E$ is the finest locally solid topology on E (see [AB$_1$, Theorem 16.7]), we obtain that $\beta(E, E') \subset \mathscr{T}_E$. Moreover, making use of the identity (3.1) we can easily obtain that $\mathscr{T}_E \subset \beta(E, E')$. Thus (cf. [F, 81 I(e)])

$$\beta(E, E') = \mathscr{T}_E. \tag{3.2}$$

As an application of (3.2) and Theorem 3.4 we have

Theorem 3.8. *Assume that $(E, \|\cdot\|_E)$ is a Banach function space with $\|\cdot\|_E$ satisfying the σ-Fatou property. Then $\beta(E(X), E(X)^\sim_n) = \mathscr{T}_E(X)$.*

Corollary 3.9. *Assume that $(E, \|\cdot\|_E)$ is a Banach function space with $\|\cdot\|_E$ satisfying the σ-Fatou property. Then the following statements are equivalent:*
(i) *The space $(E(X), \tau(E(X), E(X)^\sim_n))$ is barreled.*
(ii) $\tau(E(X), E(X)^\sim_n) = \mathscr{T}_E(X)$.
(iii) $E(X)^\sim_n = E(X)^*$.
(iv) $\|\cdot\|_E$ *is order continuous.*
(v) $\tau(E, E') = \mathscr{T}_E$.
(vi) $\tau(E, E') = \beta(E, E')$.

Proof. (i) $\Rightarrow$ (ii) Assume that the space $(E(X), \tau(E(X), E(X)^\sim_n))$ is barreled, i.e., $\tau(E(X), E(X)^\sim_n) = \beta(E(X), E(X)^\sim_n)$. By Theorem 3.8 we conclude that $\tau(E(X), E(X)^\sim_n) = \mathscr{T}_E(X)$.

(ii) $\Rightarrow$ (iii) Obvious.

(iii) $\Rightarrow$ (iv) See ([N$_2$, Corollary 2.5]).

(iv) $\Rightarrow$ (v) Assume that $\|\cdot\|_E$ is order continuous. Then $E^\sim_n = (E, \|\cdot\|_E)^* = E^*$ (see [KA, Corollary 6.1.1]), so $\tau(E, E') = \tau(E, E^\sim_n) = \tau(E, E^*) = \mathscr{T}_E$.

(v) $\Rightarrow$ (vi) It follows from (3.2).

(vi) $\Rightarrow$ (i) See Theorem 3.6. □

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