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Czechoslovak Mathematical Journal, Vol. 50 (2000), No. 2, 359–366

Persistent URL: http://dml.cz/dmlcz/127575

Czechoslovak Mathematical Journal, 50 (125) (2000), 359–366

ORTHOMODULAR LATTICES WITH STATE-SEPARATED NONCOMPATIBLE PAIRS

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(Received October 17, 1997)

Abstract. In the logico-algebraic foundation of quantum mechanics one often deals with the orthomodular lattices (OML) which enjoy state-separating properties of noncompatible pairs (see e.g. [18], [9] and [15]). These properties usually guarantee reasonable "richness" of the state space—an assumption needed in developing the theory of quantum logics. In this note we consider these classes of OMLs from the universal algebra standpoint, showing, as the main result, that these classes form quasivarieties. We also illustrate by examples that these classes may (and need not) be varieties. The results supplement the research carried on in [1], [3], [4], [5], [6], [11], [12], [13] and [16].

Keywords: orthomodular lattice, state, noncompatible pairs, (quasi)variety

MSC 2000: 06C15, 81P10

1. Basic notions

Let us formally introduce the class of OMLs we shall deal with in the sequel. We allow ourselves to assume that the reader is acquainted with the basics of the orthomodular lattice theory as presented in the introductory chapters of the books [10] or [2].

Let us first recall some standard notions of OMLs. Let $L = (L, 0, 1, \wedge, \vee, ')$ be an OML (i.e., let L be a lattice with 0, 1 and with the complementation operation so that the orthomodular law $a \vee b = a \vee (a' \wedge (a \vee b))$ holds). Two elements $a, b \in L$ are said to be *compatible* in L if $a = (a \wedge b) \vee (a \wedge b')$ or $b = (a \wedge b) \vee (b \wedge a')$, which is equivalent, and they are said to be *noncompatible* if they are not compatible. We

[1] This research was partially supported by the grant GAČR 201/96/0117 of the Czech Grant Agency and the grant VS 96049 of the Czech Ministry of Education.

denote a noncompatible pair $a, b \in L$ by $a \not\leftrightarrow b$. Obviously, L is a Boolean algebra exactly when all its elements are compatible.

We need to recall the notion of the state on L. By a state on L we mean a probability measure on L. Thus, a mapping $s\colon L \to \langle 0,1\rangle$, where $\langle 0,1\rangle$ is the unit interval of reals, is said to be *a state on* L if (i) $s(1) = 1$, and (ii) $s(a \vee b) = s(a) + s(b)$ provided $a \leqslant b'$. Let us denote the set of all states on L by $\mathscr{S}(L)$. (It should be noted that $\mathscr{S}(L)$ may be empty, may be a singleton [17] or may be an infinite convex set.)

We are in the position to introduce the classes of OMLs, denoted by $\mathscr{L}_D^C$, which we will be interested in.

Definition 1.1. Let C and D be nonempty closed subsets of the interval $\langle 0,1\rangle$. Let the symbol $\mathscr{L}_D^C$ stand for the class of OMLs which are determined as follows:

$$\begin{aligned}\mathscr{L}_D^C = \{L \in \text{ OML} \mid \text{if } a, b \in L \text{ and } a \not\leftrightarrow b, \text{ then there is a state}\\ s \in \mathscr{S}(L) \text{ such that } s(a) \in C \text{ and } s(b) \in D\}.\end{aligned}$$

In what follows we will show that each $\mathscr{L}_D^C$ is a quasivariety in the class of all OMLs. (Observe that each $\mathscr{L}_D^C$ contains the class of all Boolean algebras.)

2. Results

Theorem 2.1. *Each $\mathscr{L}_D^C$ is a quasivariety of OML.*

Proof. Let C and D be nonempty closed subsets of $\langle 0,1\rangle$. It is sufficient to show that the class $\mathscr{L}_D^C$ is closed under the formation of sub-OMLs, products and ultraproducts [7]. Let us check these properties in the given order.

1. $\mathscr{L}_D^C$ is closed under sub-OMLs. Indeed, let $L \in \mathscr{L}_D^C$ and let K be a sub-OML of L. Let $a \not\leftrightarrow b$ in K. Then $a \not\leftrightarrow b$ in L since the relation of noncompatibility is obviously hereditary. Thus, if $s \in \mathscr{L}_D^C$ is such that $s(a) \in C$ and $s(b) \in D$, then the restriction of s to K, $\tilde{s}$, is a state in $\mathscr{S}(K)$ so that $\tilde{s}(a) \in C$ and $\tilde{s}(b) \in D$.
2. $\mathscr{L}_D^C$ is closed under products. Indeed, let L_α ($\alpha \in I$) be a collection of OMLs and let $L_\alpha \in \mathscr{L}_D^C$ for any $\alpha \in I$. Take the (direct) product $\prod\limits_{\alpha\in I} L_\alpha$ of the collection L_α ($\alpha \in I$). Suppose that $a, b \in \prod\limits_{\alpha\in I} L_\alpha$ and that $a \not\leftrightarrow b$ in $\prod\limits_{\alpha\in I} L_\alpha$. Write $a = (a_\alpha), b = (b_\alpha)$, where $a_\alpha, b_\alpha \in L_\alpha$ ($\alpha \in I$) are the respective coordinates. Then there is an index α_0 such that $a_{\alpha_0} \not\leftrightarrow b_{\alpha_0}$ in L_{α_0} (obviously, $a \not\leftrightarrow b$ in $\prod\limits_{\alpha\in I} L_\alpha$ if and only if $a_\alpha \not\leftrightarrow b_\alpha$ in L_α for some $\alpha \in I$). Take a state $s_{\alpha_0} \in \mathscr{S}(L_{\alpha_0})$ such that $s_{\alpha_0}(a_{\alpha_0}) \in C$ and $s_{\alpha_0}(b_{\alpha_0}) \in D$. Let us define a state $s \in \mathscr{S}\Big(\prod\limits_{\alpha\in I} L_\alpha\Big)$

by putting $s(k) = s_{\alpha_0}(k_{\alpha_0})$ for any $k \in \prod\limits_{\alpha\in I} L_\alpha$. Then $s(a) \in C$ and $s(b) \in D$, which we wanted to show.

3. Let $\prod\limits_{\alpha\in I} L_\alpha$ be the direct product of L_α $(\alpha \in I)$ and let $\mathscr{F}$ be a free ultrafilter on I. If we let, for any two elements $a, b \in \prod\limits_{\alpha\in I} L_\alpha$, $a \underset{\mathscr{F}}{\sim} b$ if and only if the set $W = \{\alpha \in I \mid a_\alpha = b_\alpha\}$ belongs to $\mathscr{F}$, then $\underset{\mathscr{F}}{\sim}$ is a congruence relation on $\prod\limits_{\alpha\in I} L_\alpha$. The corresponding factor OML with respect to $\underset{\mathscr{F}}{\sim}$ is called the ultraproduct of L_α $(\alpha \in I)$ with respect to $\mathscr{F}$. Let us denote it by L.

We must show that if any L_α $(\alpha \in I)$ belongs to $\mathscr{L}_D^C$, then so does L. Before doing so, let us observe that L can be viewed as an epimorphic image of $\prod\limits_{\alpha\in I} L_\alpha$ under the natural epimorphism $e_{\mathscr{F}}\colon \prod\limits_{\alpha\in I} L_\alpha \to L$. Moreover, if $a \not\leftrightarrow b$ in L and if $\tilde a, \tilde b$ are arbitrary preimages in $\prod\limits_{\alpha\in I} L_\alpha$ of a, b under the mapping $e_{\mathscr{F}}$, then the set $\mathscr{I} = \{\alpha \in I \mid \tilde a_\alpha \not\leftrightarrow \tilde b_\alpha\}$ belongs to $\mathscr{F}$. In what follows, we will frequently refer to this fact and to the set $\mathscr{I}$.

Let $a \not\leftrightarrow b$ in L and let $\tilde a, \tilde b$ be the preimages. For any pair $\tilde a_\alpha, \tilde b_\alpha$ such that $\alpha \in \mathscr{I}$, let us take states $s_\alpha \in \mathscr{S}(L_\alpha)$ such that $s_\alpha(\tilde a_\alpha) \in C$ and $s_\alpha(\tilde b_\alpha) \in D$, otherwise take an arbitrary state $s_\alpha \in \mathscr{S}(L_\alpha)$ $(\alpha \in I - \mathscr{I})$. We claim that the corresponding ultraproduct state of states s_α $(\alpha \in I)$, denoted by s, $s \in \mathscr{S}(L)$, enjoys the property of $s(a) \in C$ and $s(b) \in D$. Let us only sketch the argument since it follows the standard pattern. Let us summarize the properties which are needed in the next proposition. □

Proposition 2.2. *Let I be an infinite set and let L_α $(\alpha \in I)$ be a collection of OML. Let $s_\alpha \in \mathscr{S}(L_\alpha)$ for any $\alpha \in I$. Let $\mathscr{F}$ be a free ultrafilter on I and let L be the corresponding ultraproduct. Then the following statement holds true:*

If $a \in \prod\limits_{\alpha\in I} L_\alpha$, then there exists exactly one real number $t_a \in \langle 0,1\rangle$ such that, for any $\varepsilon > 0$, the set $F_\varepsilon = \{\alpha \in I \mid |s_\alpha(a_\alpha) - t_a| < \varepsilon\}$ belongs to $\mathscr{F}$. If we define a mapping $s_{\mathscr{F}}\colon \prod\limits_{\alpha\in I} L_\alpha \to \langle 0,1\rangle$ by setting $s_{\mathscr{F}}(a) = t_a$ for any $a \in L$, we obtain a state on $\prod\limits_{\alpha\in I} L_\alpha$. Moreover, if $a \underset{\mathscr{F}}{\sim} b$, then $s_{\mathscr{F}}(a) = s_{\mathscr{F}}(b)$, and if $s_\alpha(a_\alpha) \in C$ and $s_\alpha(b_\alpha) \in D$ $(\alpha \in I)$, then $s_{\mathscr{F}}(a) \in C$ and $s_{\mathscr{F}}(b) \in D$. Thus, if $e_{\mathscr{F}}$ denotes the natural factor epimorphism, $e_{\mathscr{F}}\colon \prod\limits_{\alpha\in I} L_\alpha \to L$, then there exists exactly one state $s \in \mathscr{S}(L)$—the ultraproduct state of s_α $(\alpha \in I)$—such that $s \circ e_{\mathscr{F}} = s_{\mathscr{F}}$.

Proof. We will only indicate the proof of the first part of the proposition, the rest being routine. Consider first the sets $G_{\langle 0,\frac12\rangle} = \{\alpha \in I \mid s_{\alpha(a_\alpha)} \in \langle 0, \frac12\rangle\}$ and $G_{\langle \frac12,1\rangle} = \{\alpha \in I \mid s_\alpha(a_\alpha) \in \langle \frac12, 1\rangle\}$. If the set $G_{\frac12} = \{\alpha \in I \mid s_\alpha(a_\alpha) = \frac12\}$ belongs to $\mathscr{F}$, we set $t_a = \frac{1}{2}$. If not, then there is exactly one set of the two sets $G_{\langle 0,\frac{1}{2}\rangle}$ and $G_{\langle\frac{1}{2},1\rangle}$ which belongs to $\mathscr{F}$. Suppose it is the set $G_{\langle 0,\frac{1}{2}\rangle}$. Consider the sets $G_{\langle 0,\frac{1}{4}\rangle}$ and $G_{\langle\frac{1}{4},\frac{1}{2}\rangle}$. If the set $G_{\frac{1}{4}} = \{\alpha \in I \mid s_\alpha(a_\alpha) = \frac{1}{4}\}$ belongs to $\mathscr{F}$, we set $t_a = \frac{1}{4}$. If not, then there is exactly one set of the two sets $G_{\langle 0,\frac{1}{4}\rangle}$ and $G_{\langle\frac{1}{4},\frac{1}{2}\rangle}$ which belongs to $\mathscr{F}$, etc. The process either stops, in which case we set the value t_a to be the corresponding middle point, or it goes on, in which case we obtain a decreasing sequence of closed intervals $\langle u_n, v_n\rangle$ in $\langle 0,1\rangle$, and then it suffices to set $t_a = \bigcap\limits_{n\in\mathbb{N}} \langle u_n, v_n\rangle$. It can be easily shown that this mapping $s_{\mathscr{F}}(a) = t_a$ defines a state on $\prod\limits_{\alpha\in I} L_\alpha$ and that s satisfies all the other required properties as well.

The verification of the last part of Proposition 2.2 is now easy. Since we have chosen states $s_\alpha \in \mathscr{S}(L_\alpha)$ such that $s_\alpha(a_\alpha) \in C$ and $s_\alpha(b_\alpha) \in D$ for any $\alpha \in I$ we see that the corresponding ultraproduct state $s \in \mathscr{S}(L)$ fulfils $s(\tilde{a}) \in C$ and $s(\tilde{b}) \in D$ (the sets C and D are closed). The proof of Theorem 2.1 is complete. □

Remark. We could in principle define the classes $\mathscr{L}_D^C$ for arbitrary nonempty sets C, D of $\langle 0,1\rangle$. However, it can be shown that then $\mathscr{L}_D^C$ will rarely be closed under the formation of ultraproducts, see also [5].

It would be desirable to characterize which of the quasivarieties $\mathscr{L}_D^C$ are varieties (recall that a quasivariety is called a variety if it is closed under the formation of all epimorphisms). We have not been able to find this characterization. The fact is however that $\mathscr{L}_D^C$ is a variety for some C, D and is not a variety for some C, D. We will now demonstrate it. (Recall that L is called *unital* if for any b, $b \neq 0$, there is a state $s \in \mathscr{S}(L)$ such that $s(b) = 1$, see [9], [15]. Let us denote the class of unital logics by $\mathscr{L}(1)$.)

Theorem 2.3. *For every nonempty closed set D the class $\mathscr{L}_D^{\{1\}}$ is a variety. In particular, $\mathscr{L}_{\langle 0,1\rangle}^1$ is the variety $\mathscr{L}(1)$ of unital OMLs.*

Proof. The proof can be extracted from the general result of Th. 1, §5 in [12]. Let us indicate a direct proof. Let us first observe that

(i) for any $L \in$ OML, the set $\mathscr{S}(L)$ is a compact convex set when equipped with the pointwise topology,

(ii) the following simple proposition holds true [4]: If $f\colon L \to M$ is an OML epimorphism and if $s \in \mathscr{S}(L)$ is a state such that $s(h) = 1$ for any element $t \in f^{-1}(1)$, then there exists a state $t \in \mathscr{S}(M)$ such that $s = t \circ f$.

We only have to show that our class $\mathscr{L}_D^{\{1\}}$ is closed under the formation of epimorphic images. Suppose that $f\colon L \to M$ is an OML epimorphism onto M and suppose that $L \in \mathscr{L}_D^{\{1\}}$. We have to show that $M \in \mathscr{L}_D^{\{1\}}$. Suppose that

$m_1 \not\leftrightarrow m_2$ in M. Suppose that $k_2 \in L$ is such that $f(k_2) = m_2$. Consider the set $P = \{k \in L \mid f(k) \geqslant m_1\}$. Let us observe that if $p \in P$ and $k_1 \in L$ is such an element that $f(k_1) = m_1$, then $f(p \wedge k_1) = m_1$. Since OML morphisms must send pairs of compatible elements into pairs of compatible elements, we see that $(p \wedge k_1) \not\leftrightarrow k_2$. It follows that there is a state $s \in \mathscr{S}(L)$ such that $s(p \wedge k_1) = 1$ and $s(k_2) \in D$. Obviously, $s(k_1) = 1$ and $s(k_2) \in D$.

For any $k \in P$ put $S_k = \{s \in \mathscr{S}(L) \mid s(k) = 1 \text{ and } s(m_2) \in D\}$. Then any S_k is nonempty. Since every S_k is obviously closed in $\mathscr{S}(L)$ and since P is a filter in L, we infer that the collection $\mathscr{S} = \{S_k \mid k \in P\}$ is a centred collection of closed sets in a compact space. It follows that $\bigcap\limits_{k \in P} S_k \neq \emptyset$. Take a state $s \in \mathscr{S}(L)$ such that $s \in \bigcap\limits_{k \in P} S_k$. Then we easily see that $s(k) = 1$ for any $k \in f^{-1}(1)$ and therefore, by the above observation (ii), there exists a state $t \in \mathscr{S}(M)$ such that $s = t \circ f$. Obviously, $t(m_1) = 1$ and $t(m_2) \in D$. This completes the proof of the statement that $\mathscr{L}_D^{\{1\}}$ is a variety.

We will now show that $\mathscr{L}_{\langle 0,1\rangle}^{\{1\}}$ consists of unital OMLs. Denote by $\mathscr{L}(1)$ the variety of all unital OMLs. Let $L \in \mathscr{L}(1)$ and $a \not\leftrightarrow b$. Then $b \neq 0$ and therefore there is a state $s \in \mathscr{S}(L)$ such that $s(b) = 1$. Thus, $L \in \mathscr{L}_{\langle 0,1\rangle}^{\{1\}}$. Conversely, suppose that $L \in \mathscr{L}_{\langle 0,1\rangle}^{\{1\}}$ and take an arbitrary $b \in L$, $b \neq 0$. If there is $a \in L$ such that $a \not\leftrightarrow c$ for some $c \leqslant b$ we have a state $s \in \mathscr{S}(L)$ such that $s(c) = 1$. Since $c \leqslant b$, we see that $s(b) = 1$. If there is no element $a \in L$ such that $a \not\leftrightarrow c$ for some c, $c \leqslant b$, it follows that the set $L_b = \{x \in L \mid x \leqslant b\}$ is a subset of the centre of L. It follows that L_b (considered with operations inherited from L) is a Boolean algebra. Take the natural epimorphism $e\colon L \to L_b$, $e(y) = y \wedge b$, and a state $\tilde{s} \in \mathscr{S}(L_b)$. Then $s = e \circ \tilde{s}$ is a state on L such that $s(b) = 1$. Thus, $\mathscr{L}_{\langle 0,1\rangle}^{\{1\}} \subset \mathscr{L}(1)$. It follows that $\mathscr{L}_{\langle 0,1\rangle}^{\{1\}} = \mathscr{L}(1)$.

It may be observed in connection with the last proposition and with the foundations of quantum mechanics that the variety $\mathscr{L}_{\langle 0,1\rangle}^{\{1\}}$ contains the lattice $L(H)$ of projectors in the Hilbert space H. This is known, of course. What does not seem to be explicitly known is that the equality $\mathscr{L}_{\langle 0,1\rangle}^{1} = \mathscr{L}(1)$ verified above remains true, and the proof remains fully analogous, when we replace 1 by an arbitrary interval $\langle a, 1\rangle$ $(a \geqslant 0)$. Thus, $\mathscr{L}_{\langle 0,1\rangle}^{\langle a,1\rangle} = \mathscr{L}(\langle a, 1\rangle)$ and therefore $\mathscr{L}(\langle a, 1\rangle)$ is a quasivariety (by analogy, $\mathscr{L}(\langle a, 1\rangle)$ stands for the class of the OMLs where each nonzero element can be sent into $\langle a, 1\rangle$ by a state). The equality $\mathscr{L}_{\langle 0,1\rangle}^{\langle a,1\rangle} = \mathscr{L}(\langle a, 1\rangle)$ may be of its own interest for quantum logics. □

Proposition 2.4. *The class $\mathscr{L}_{\langle 0,1\rangle}^{\langle 0,1\rangle}$ is not a variety.*

Proof. The class $\mathscr{L}_{\langle 0,1\rangle}^{\langle 0,1\rangle}$ coincides with the class of all OMLs which possess a state. We want to show that the latter class is not a variety. In other words, we must show that it is not closed under the formation of OML epimorphisms. Let K be a stateless OML (see [8]) and let $\{0,1\}$ be a two-point Boolean algebra. Let $L = K \times \{0,1\}$ in the category of OMLs, and let $p\colon\ L \to K$ be the natural projection. It remains to be shown that L possesses a state. Define $\hat{s}(k,1) = 1$ for any $k \in K$ and $\hat{s}(k,0) = 0$ for any $k \in K$. It can be easily seen that $\hat{s}$ admits a unique extension to a state on K (see also [14]). This completes the proof. $\square$

An important quasivariety among the classes $\mathscr{L}_D^C$ seems to be the quasivariety $\mathscr{L}_{\{\frac{1}{2}\}}^{\{\frac{1}{2}\}}$. This quasivariety contains the variety of unital OMLs as a proper subclass (see Prop. 2.5). The clarification of whether $\mathscr{L}_{\{\frac{1}{2}\}}^{\{\frac{1}{2}\}}$ is a variety would be helpful in deciding the same question for general quasivarieties $\mathscr{L}_D^C$. So, the following concrete open question seems to be the first step in solving the general question on when $\mathscr{L}_D^C$ is a variety.

Open question. Suppose that $f\colon\ L \to K$ is an OML morphism onto K. Suppose that for each noncompatible pair $a \not\leftrightarrow b$ in L there is a state in L such that $s(a) = \frac{1}{2} = s(b)$. Does this property remain valid for K as well?

Proposition 2.5. *Suppose that L is unital (i.e., suppose that L is an OML such that for any $a \in L$ there is a state $s \in \mathscr{S}(L)$ such that $s(a) = 1$). Then $L \in \mathscr{L}_{\{\frac{1}{2}\}}^{\{\frac{1}{2}\}}$. On the other hand, there exists an $L \in \mathscr{L}_{\{\frac{1}{2}\}}^{\{\frac{1}{2}\}}$ which is not unital.*

Proof. Suppose that $a \not\leftrightarrow b$. Let us first show that there is a state $t \in \mathscr{S}(L)$ such that $t(a) = t(b) \geqslant \frac{1}{2}$: If $t(a) = t(b) = 1$ for some state, then we are done. Otherwise there are states $s_1, s_2 \in \mathscr{S}(L)$ such that $s_1(a) = 1$, $s_2(b) = 1$, $s_1(b) < 1$ and $s_2(a) < 1$. Let

$$t = \frac{(1 - s_2(a))s_1 + (1 - s_1(b))s_2}{2 - s_1(b) - s_2(a)}.$$

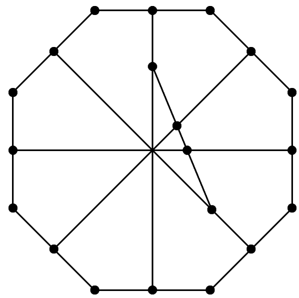
Then $t(a) \geqslant \frac{1}{2}, t(b) \geqslant \frac{1}{2}$ and $t(a) = t(b)$.

Let us now take such a state, t_1, for a and b, and such a state, t_2, for a' and b'. If $t_1(a) = \frac{1}{2}$, we are done. If $t_1(a) > \frac{1}{2}$, then we put

$$s = \frac{(\frac{1}{2} - t_2(a))t_1 + (t_1(a) - \frac{1}{2})t_2}{t_1(a) - t_2(a)},$$

and we see that s is a state and $s(a) = s(b) = \frac{1}{2}$.

For the remaining part of the proposition, we assume the reader to be acquainted with the Greechie diagram technique for OMLs (e.g. [15]). We claim, leaving the details to be checked by the reader, that the diagram below presents an orthomodular lattice L such that $L \in \mathscr{L}_{\{\frac{1}{2}\}}^{\{\frac{1}{2}\}}$ and L is not unital (see also [12], §VIII). $\square$



Let us remark in the conclusion of this note that in analogy with [13] there is a general way of describing all quasivarieties $\mathscr{L}_D^C$ in terms of implicative equalities. The methods to be used are essentially model theoretic and cover also cases of many other (more general and less quantum physic motivated) quasivarieties. We intend to investigate this description elsewhere.

Acknowledgement. The second author acknowledges the grant support and hospitality of the host institution "Institut Girard Desargues", Université Claude Bernard—Lyon 1 while he worked with the first author towards the results of this research.

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