Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 50 (2000), No. 2, 331–340

Persistent URL: `http://dml.cz/dmlcz/127572`

Czechoslovak Mathematical Journal, 50 (125) (2000), 331–340

ON UNIFORM DISTRIBUTION OF SEQUENCES $(a_n x)_1^\infty$

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(Received July 13, 1997)

Keywords: uniform distribution, Baire category, Lebesgue measure, dyadic number of set, continued fraction

MSC 2000: 11K06, 11B57

Introduction

There are two possible approaches to the study of uniform distribution (mod 1) of sequences

$$(a_n x)_1^\infty \tag{1}$$

where $a_n \in \mathbb{R}$ $(n = 1, 2, \ldots)$ and $x \in \mathbb{R}$. The first such approach is the study of (1) with a fixed sequence $(a_n)_1^\infty$, x running over real numbers, the second is the study of (1) with a fixed $x \in \mathbb{R}$ and $(a_n)_1^\infty$ running over a class of sequences of real numbers. The second approach leads to the concept od α-good sequences (cf. [1]).

In the first part of the paper we will apply the first and in the second part the second approach to the investigation of sequences (1).

The work on this paper was supported by grant No. 5123 of Slovak Academy of Sciences.

1. Uniform distribution (mod 1) of sequences $(a_nx)_1^\infty$ with fixed $(a_n)_1^\infty$

In this part we restrict ourselves to the study of (1) with a fixed sequence $(a_n)_1^\infty$ of real numbers. Denote by $H\ (a_1, a_2, \ldots)$ the set of all $x \in \mathbb{R}$ for which the sequence (1) is uniformly distributed (mod 1) (shortly: u.d. mod 1). It is wellknown that if $a_n \in \mathbb{N}$ $(n = 1, 2, \ldots)$, $a_i \neq a_j$ for $i \neq j$, then the set $H(a_1, a_2, \ldots)$ has full measure (i.e. the set $\mathbb{R} \setminus H(a_1, a_2, \ldots)$ is a null set—cf. [2], [4] pp. 32–33). This results evokes the question what is the Baire category of the set $H(a_1, a_2, \ldots)$. We will show that the "topological magnitude" of $H(a_1, a_2, \ldots)$ depends on the sequence $(a_n)_1^\infty$. Indeed, if we choose $a_n = n$ or more generally $a_n = a + nd$ $(n = 1, 2, \ldots)$, $d \geqslant 1$, a, d integers, then by Weyl's criterion (cf. [4] pp. 7–8) the sequence $(a_nx)_1^\infty$ is u.d. mod 1 for each irrational x. Hence $H(a_1, a_2, \ldots)$ contains in this case all irrational numbers and so it is a residual set. In what follows we will give a class of sequences $(a_n)_1^\infty$ of positive integers such that $H(a_1, a_2, \ldots)$ is a set of the first category.

Theorem 1.1. *Let $(q_k)_1^\infty$ be a sequence of positive integers greater than* 1. *Put*

$$a_n = q_1 q_2 \ldots q_n \quad (n = 1, 2, \ldots).$$

Then $H(a_1, a_2, \ldots)$ is a set of the first Baire category in $\mathbb{R}$.

Proof. For $x \in \mathbb{R}$ we put

$$S(m, x) = \frac{1}{m} \sum_{n=1}^{m} \mathrm{e}^{2\pi \mathrm{i} a_n x} \quad (m = 1, 2, \ldots).$$

Then by Weyl's criterion we have

$$H(a_1, a_2, \ldots) \subseteq H_0(a_1, a_2, \ldots),$$

where $H_0(a_1, a_2, \ldots) = \left\{x \in \mathbb{R}\colon \lim\limits_{m\to\infty} S(m, x) = 0\right\}$. Denote by $C(a_1, a_2, \ldots)$ the set of all $x \in \mathbb{R}$ for which there exists $\lim\limits_{m\to\infty} S(m, x) = S(x)$. Then evidently

$$H(a_1, a_2, \ldots) \subseteq H_0(a_1, a_2, \ldots) \subseteq C(a_1, a_2, \ldots). \tag{2}$$

Each of these sets has the full measure. By (2) it suffices to prove that $C = C(a_1, a_2, \ldots)$ is a set of the first category in $\mathbb{R}$. We prove it in the following.

Obviously each of the functions $S(m, x)$ $(m = 1, 2, \ldots)$ is continuous on C (i.e. the restrictions $S(m, x)\big|C$ are continuous on C). Hence the function $S(x) = \lim\limits_{m\to\infty} S(m, x)$ defined on C is in the first Baire class on C. But then the set of all discontinuity points of S is a set of the first category in C (cf. [8] p. 185), and so it is a set of the first category in $\mathbb{R}$, as well.

To complete the proof it suffices to show that the function S is discontinuous at every $x \in C$. For this it suffices to show that each of the sets

$$M_0 = \{x \in C\colon\ S(x) = 0\}, \quad M_1 = \{x \in C\colon\ S(x) = 1\}$$

is dense in C.

The density of M_0 follows from the fact that $M_0 \subseteq C$ and M_0 has the full measure.

We prove that M_1 is dense in C. It is wellknown that every $x \in \mathbb{R}$ has the Cantor series expansion

$$x = c_0 + \sum_{j=1}^{\infty} \frac{c_j}{q_1 q_2 \dots q_j} = c_0 + \sum_{j=1}^{\infty} \frac{c_j}{a_j},$$

where c_j are integers, $0 \leqslant c_j < q_j$, $a_j = q_1 q_2 \dots q_j$ $(j = 1, 2, \dots)$.

Denote by A_k the set of all $x \in \mathbb{R}$ of the form

$$x = c_0 + \sum_{j=1}^{k} \frac{c_j}{a_j}, \tag{3}$$

where $k \in \mathbb{N}$, c_0 is an integer and $0 \leqslant c_j < q_j$ $(j = 1, 2, \dots, k)$. If $x \in A_k$, then $a_n x$ is an integer for $n > k$. Thus for $m > k$ we have

$$S(m, x) = \frac{1}{m} \sum_{n=1}^{k} + \frac{1}{m} \sum_{n=k+1}^{m} 1 = O(1) + \frac{m-k}{m} \to 1 \quad \text{if } m \to \infty.$$

Put $A = \bigcup\limits_{k=1}^{\infty} A_k$. Then $A \subseteq M_1 \subseteq C$ and A is obviously dense in C. The density of M_1 in C follows. This completes the proof. □

We give the following simple observation.

Proposition 1.1. *Let $(a_j)_1^{\infty}$ be an arbitrary sequence of real numbers. Then $H(a_1, a_2, \dots)$ is an $F_{\sigma\delta}$-set in $\mathbb{R}$.*

P r o o f. Using Weyl's criterion we can easily check that

$$H(a_1, a_2, \dots) = \bigcap_{h \neq 0} \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} A(n, h, k),$$

where

$$A(n, h, k) = \left\{ x \in \mathbb{R}\colon\ \left| \frac{1}{n} \sum_{j=1}^{n} \mathrm{e}^{2\pi \mathrm{i} h a_j x} \right| \leqslant \frac{1}{k} \right\}.$$

Since $\frac{1}{n}\sum_{j=1}^{n} \mathrm{e}^{2\pi i h a_j x}$ $(n = 1, 2, \ldots)$ are continuous functions, we see that $A(n, h, k)$ is a closed set (if n, h, k are fixed) and so $H(a_1, a_2, \ldots)$ is an $F_{\sigma\delta}$-set in $\mathbb{R}$. □

Remark 1.1. a) If $(a_n)_1^\infty$ is a sequence of distinct integers then by [2] and Proposition 1.1 the set $H(a_1, a_2, \ldots)$ is an $F_{\sigma\delta}$-set of the full measure.

b) For some particular choices of $(a_n)_1^\infty$ the set $H(a_1, a_2, \ldots)$ can belong to lower Borel classes. For instance if $a_n = a \in \mathbb{R}$, $(n = 1, 2, \ldots)$, then the set $H(a_1, a_2, \ldots)$ is empty while it coincides with the set $\mathbb{Q}' = \mathbb{R} \setminus \mathbb{Q}$ of all irrational numbers if $a_n = n$ $(n = 1, 2, \ldots)$.

2. Uniform distribution (mod 1) of sequences $(a_n x)_1^\infty$ with fixed x

Let α be an irrational number. A sequence $a_1 < a_2 < \ldots$ of positive integers is said to be α-good provided the sequence $(a_n\alpha)_1^\infty$ is uniformly distributed (mod 1) (cf. [1]). The sequence $1 < 2 < \ldots < n < \ldots$ and the sequence $p_1 < p_2 < \ldots < p_n < \ldots$ of all prime numbers are α-good for each irrational α (cd. [1], [4] p. 22).

For $\alpha \in \mathbb{Q}'$ $(\mathbb{Q}' = \mathbb{R} \setminus \mathbb{Q})$ denote by $D(\alpha)$ the set of all α-good sequences. Note that every infinite sequence $a_1 < a_2 < \ldots < a_n < \ldots$ of positive integers belongs to $D(\alpha)$ for almost all $\alpha \in \mathbb{Q}'$ (cf. [4] p. 32, Theorem 4.1).

We will investigate the properties of the classes $D(\alpha)$ for $\alpha \in \mathbb{Q}'$. We will show that these classes have several common properties (for all $\alpha \in \mathbb{Q}'$).

It seems to be interesting to deal with the question about magnitude of classes $D(\alpha)$ $(\alpha \in \mathbb{Q}'$). This "magnitude" will be studied from the point of view of dyadic numbers of sets $A \subseteq \mathbb{N}$.

Denote by U the class of all infinite sets

$$A = \{a_1 < a_2 < \ldots < a_n < \ldots\} \subseteq \mathbb{N}.$$

In what follows we identify the set A with the sequence $a_1 < a_2 < \ldots < a_n < \ldots$ Put

$$\varrho(A) = \sum_{k=1}^{\infty} 2^{-a_k} \in (0, 1]$$

for each $A \subseteq U$. Then ϱ is a one-to-one mapping of U onto $(0, 1]$. If S is a class of infinite subsets of $\mathbb{N}$, then we put $\varrho(S) = \{\varrho(A)\colon\ A \in S\}$. The set $\varrho(S)$ "measures" the magnitude of the class S (cf. [5] p. 17).

We will investigate metric and topological properties of the sets $\varrho(D(\alpha))$.

Recall that a measurable set $M \subseteq (0,1]$ is called homogeneous if there is a real number $d \in [0,1]$ such that for every interval $I \subseteq (0,1]$ we have

$$\frac{\lambda(I \cap M)}{\lambda(I)} = d,$$

λ being the Lebesgue measure (cf. [9], [10]).

Theorem 2.1. *For each $\alpha \in \mathbb{Q}'$ the set $\varrho(D(\alpha))$ is a homogeneous $F_{\sigma\delta}$-set in $(0,1]$.*

Proof. According to Weyl's criterion a sequence $a_1 < a_2 < \ldots$ of positive integers belongs to $D(\alpha)$ if and only if

$$(\forall h \in Z, h \neq 0)\colon \lim_{m\to\infty} \frac{1}{m} \sum_{n=1}^{m} \mathrm{e}^{2\pi i h a_n \alpha} = 0.$$

This condition is equivalent to the condition

$$(4) \qquad (\forall |h| \geqslant 1)(\forall k \geqslant 1)(\exists v \in \mathbb{N})(\forall m \geqslant v)\colon \left| \frac{1}{m} \sum_{n=1}^{m} \mathrm{e}^{2\pi i h \alpha a_n} \right| \leqslant \frac{1}{k}.$$

From (4) we get

$$\varrho(D(\alpha)) = \bigcap_{|h|\geqslant 1} \bigcap_{k=1}^{\infty} \bigcup_{v=1}^{\infty} \bigcap_{m=v}^{\infty} M(m,h,k),$$

where

$$(6) \qquad M(m,h,k) = \left\{ x = \sum_{j=1}^{\infty} 2^{-a_j} \in (0,1]\colon \left| \frac{1}{m} \sum_{n=1}^{m} \mathrm{e}^{2\pi i h \alpha a_n} \right| \leqslant \frac{1}{k} \right\}.$$

Construct the functions

$$f_{m,h}(x) = \frac{1}{m} \sum_{n=1}^{m} \mathrm{e}^{2\pi i h \alpha a_n} \quad (m = 1, 2, \ldots;\ h \in Z, h \neq 0),$$

where $x = \sum\limits_{j=1}^{\infty} 2^{-a_j} \in (0,1]$. These functions are defined for each $x \in (0,1]$. We will verify that their restrictions to $\mathbb{Q}' \cap (0,1]$ are continuous on $\mathbb{Q}' \cap (0,1]$.

Let $x_0 \in \mathbb{Q}' \cap (0,1]$, $x_0 = \sum\limits_{j=1}^{\infty} 2^{-b_j}$ $(b_1 < b_2 < \ldots)$ be the dyadic expansion of x_0. Fix the number m. Notice that the set of all numbers of the form $x = \sum\limits_{j=1}^{\infty} 2^{-a_j}$,

$a_j = b_j$ $(j = 1, 2, \ldots, m)$ fils up an interval I_m containing x_0, the left-hand endpoint of which is the rational number $\sum\limits_{j=1}^{m} 2^{-b_j}$ and the right-hand endpoint is

$$\sum_{j=1}^{m} 2^{-b_j} + \sum_{j=b_m+1}^{\infty} 2^{-j} = \sum_{j=1}^{m} 2^{-b_j} + 2^{-b_m}.$$

Obviously the function $f_{m,h}|\mathbb{Q}' \cap (0,1]$ is constant on I_m and so it is continuous at x_0.

From the continuity of functions $f_{m,h}|\mathbb{Q}' \cap (0,1]$ the closedness of the sets $M(m,h,k)$ in $\mathbb{Q}' \cap (0,1]$ follows (see (6)). But then by (5) the set $\mathbb{Q}' \cap \varrho(D(\alpha))$ is an $F_{\sigma\delta}$-set in $(0,1]$. Notice that

$$\varrho(D(\alpha)) = [\mathbb{Q}' \cap \varrho(D(\alpha))] \cup [\mathbb{Q} \cap \varrho(D(\alpha))],$$

the second "summand" on the right-hand side being countable. From this we see that $\varrho(D(\alpha))$ is an $F_{\sigma\delta}$-set in $(0,1]$.

The homogeneity of the set $\varrho(D(\alpha))$ can be proved by using a result form [7] (cf. [7], Lemma 1, pp. 255–256). We will use the following special case of Lemma 1 from [7]:

(T) Let $B \subseteq (0,1]$ be a measurable set. Suppose that for each $n = 1, 2, \ldots$ and $k, k' \in \{0, 1, \ldots, 2^n - 1\}$ we have

$$\lambda(B \cap i_n^{(k)}) = \lambda(B \cap i_n^{(k')}),$$

where

$$i_n^{(v)} = \left(\frac{v}{2^n}, \frac{v+1}{2^n}\right], \quad v \in \{0, 1, \ldots, 2^n - 1\}.$$

Then B is a homogeneous set in $(0,1]$.

If now $a_1 < a_2 < \ldots < a_n < \ldots$ is an α-good sequence and a sequence $d_1 < d_2 < \ldots < \ldots$ differs from $a_1 < a_2 < \ldots < a_n < \ldots$ only in a finite number of terms, then evidently also $d_1 < d_2 < \ldots < \ldots$ is an α-good sequence. Hence the assumptions in (T) are satisfied and so by (T) the set $\varrho(D(\alpha))$ is homogeneous in $(0,1]$.

It is wellknown that the Lebesgue measure of a homogeneous set $A \subseteq (0,1]$ is 0 or 1 (cf. [9], [10]). Hence by Theorem 2.1 we have $\lambda(\varrho(D(\alpha))) = 0$ or $\lambda(\varrho(D(\alpha))) = 1$ for each $\alpha \in \mathbb{Q}'$. We will show that this measure is equal to 1 for each $\alpha \in \mathbb{Q}'$. □

Theorem 2.2. *For each $\alpha \in \mathbb{Q}'$ we have $\lambda(\varrho(D(\alpha))) = 1$.*

Proof. Let $\alpha \in \mathbb{Q}'$. Then the sequence $(n\alpha)_{n=1}^{\infty}$ is u.d. mod 1 (cf. [4] pp. 7–8). By a theorem of Peterson (cf. [6]), if $(v_n)_1^{\infty}$ is a u.d. mod 1 sequence then for almost all

$x = \sum_{j=1}^{\infty} 2^{-a_j} \in (0,1]$ the sequence $(v_{a_j})_{j=1}^{\infty}$ (subsequence of $(v_n)_1^{\infty}$) is u.d. mod 1 as well. Hence for almost all $x = \sum_{j=1}^{\infty} 2^{-a_j} \in (0,1]$ the sequence $(a_j\alpha)_{j=1}^{\infty}$ is u.d. mod 1. But this means that almost all $x \in (0,1]$ belong to the set $\varrho(D(\alpha))$. □

We now will investigate the magnitude of sets $\varrho(D(\alpha))$ from the topological point of view. We prove the following universal theorem.

Theorem 2.3. *For every $\alpha \in \mathbb{Q}'$ the set $\varrho(D(\alpha))$ is a dense $F_{\sigma\delta}$-set of the first Baire category in* $(0,1]$.

Proof. Let $\alpha \in \mathbb{Q}'$. By Theorem 2.1 the set $\varrho(D(\alpha))$ is an $F_{\sigma\delta}$-set in $(0,1]$.

Further, the set $D(\alpha)$ is non-empty (and such is also the set $\varrho(D(\alpha))$ since the sequence $1 < 2 < \ldots < n \ldots$ belongs to $D(\alpha)$. The density of $\varrho(D(\alpha))$ follows from the above mentioned fact that together with $1 < 2 < \ldots < n \ldots$ the class $D(\alpha)$ contains every sequence $a_1 < a_2 < \ldots < a_n < \ldots$ which differs from $1 < 2 < \ldots < n \ldots$ only in a finite number of terms.

We prove that $\varrho(D(\alpha))$ is a set of the first Baire category. For $t = \varrho(A)$, $A = a_1 < a_2 < \ldots < a_n < \ldots$ we put

$$g_m(t) = \frac{1}{m} \sum_{n=1}^{m} s^{2\pi i \alpha a_n} \quad (m = 1, 2, \ldots).$$

Denote by M the set of all $t \in (0,1]$ ($t = \varrho(A)$, $A = a_1 < a_2 < \ldots$) for which there exists $\lim_{m\to\infty} g_m(t) = g(t)$. By Weyl's criterion we get

$$\varrho(D(\alpha)) \subseteq M. \tag{7}$$

It is easy to verify that the functions $g_m|\mathbb{Q}' \cap (0,1]$ are continuous on $\mathbb{Q}' \cap (0,1]$ (and so they are continuous on $M \cap \mathbb{Q}' \subseteq \mathbb{Q}' \cap (0,1]$ as well). This can be proved in an analogous way as the continuity of the functions $f_{m,h}|\mathbb{Q}' \cap (0,1]$ in the proof of Theorem 2.1. Thus the function $g|M \cap \mathbb{Q}'$ is in the first Baire class on $M \cap \mathbb{Q}'$. This implies that the set of discontinuity points of $g|M \cap \mathbb{Q}'$ is a set of the first category in $M \cap \mathbb{Q}'$ (cf. [8] p. 185).

We will show that the function g is discontinuous at every point of $M \cap \mathbb{Q}'$. To show this it suffices to prove that each of the sets

$$M_0 = \{x \in M \cap \mathbb{Q}' : \ g(x) = 0\}, M_1 = \{x \in M \cap \mathbb{Q}' : \ g(x) = 1\}$$

is dense in $M \cap \mathbb{Q}'$.

In the first place we prove the density of M_0 in $M \cap \mathbb{Q}'$. If $p_1 < p_2 < \ldots < p_n < \ldots$ is the sequence of all primes then $x_0 = \sum_{k=1}^{\infty} 2^{-p_k}$ belongs to M_0 (cf. [1], [4] p. 22). Together with x_0 each $\varrho(A)$ belongs to M_0, where A is an infinite set of positive integers which differs from $\{p_1 < p_2 < \ldots < p_n < \ldots\}$ only in a finite number of elements. From this the density of M_0 in $M \cap \mathbb{Q}'$ follows.

For the proof of density of M_1 it suffices to construct a sequence $A_0 = a_1 < a_2 < \ldots < a_n < \ldots$ such that $y_0 = \varrho(A_0)$ is an irrational number with $g(y_0) = 1$. Such a sequence can be obtained by the following procedure:

Take into account the continued fraction of α. It is wellknown that if $\frac{p_n}{q_n}$ $(n = 1, 2, \ldots)$ are convergents of this continued fraction, then

$$|q_n\alpha - p_n| < \frac{1}{q_n} \quad (n = 1, 2, \ldots)$$

(cf. [3] p. 27). Further, if n is even then $\frac{p_n}{q_n} < \alpha$ (cf. [3] p. 22). But then for such even n we have $0 < q_n\alpha - p_n < \frac{1}{q_n}$, thus $\{q_n\alpha\} = q_n\alpha - [q_n\alpha] = q_n\alpha - p_n < \frac{1}{q_n}$. So we get

$$\{q_n\alpha\} = \frac{\vartheta_n}{q_n}, \quad 0 < \vartheta_n < 1.$$

Choose a set $N_2 = \{k_1 < k_2 < \ldots < k_n < \ldots\}$ of even numbers such that

$$\lim_{n\to\infty} (q_{k_{n+1}} - q_{k_n}) = +\infty \tag{8}$$

and put $q'_n = q_{k_n}$ $(n = 1, 2, \ldots)$. Then $y_0 = \sum_{n \in N_2} 2^{-q'_n}$ belongs to $\mathbb{Q}'$ since the condition (8) guarantees that the dyadic expansion of y_0 is not periodic. Further,

$$\mathrm{e}^{2\pi\mathrm{i}\alpha q'_n} = \mathrm{e}^{2\pi\mathrm{i}([\alpha q'_n] + \{\alpha q'_n\})} = \mathrm{e}^{2\pi\mathrm{i}\{\alpha q'_n\}} = \mathrm{e}^{2\pi\mathrm{i}\frac{\vartheta'_n}{q'_n}} \quad (0 < \vartheta'_n < 1,\ n \in N_2).$$

For all sufficiently large n's (e.g. for $n > n_0$) we have

$$0 < \frac{2\pi\vartheta'_n}{q'_n} < \frac{1}{2}.$$

So we get

$$g_m(y_0) = \frac{1}{m}\sum_{n=1}^{n_0} + \frac{1}{m}\sum_{n=n_0+1}^{m} \mathrm{e}^{2\pi\mathrm{i}\{\alpha q'_n\}} = O(1)$$
$$+ \left(\frac{1}{m}\sum_{n=n_0+1}^{m} \cos\frac{2\pi\vartheta'_n}{q'_n} + i\frac{1}{m}\sum_{n=n_0+1}^{m} \sin\frac{2\pi\vartheta'_n}{q'_n}\right).$$

Note that

$$\left|\sin\frac{2\pi\vartheta'_n}{q'_n}\right| \leqslant \frac{2\pi\vartheta'_n}{q'_n}.$$

Since $q'_n \to \infty$ $(n \to \infty)$, we have

$$\left|\frac{1}{m}\sum_{n=n_0+1}^{m}\sin\frac{2\pi\vartheta'_n}{q'_n}\right| \leqslant \frac{1}{m}\sum_{n=1}^{m}\frac{2\pi}{q'_n} \to 0$$

(for $m \to \infty$) (Cesàro means).

Further, by the inequality

$$\cos x > 1 - \frac{x^2}{2} \quad (x \in (0,1))$$

we get (for $n > n_0$)

$$\cos\frac{2\pi\vartheta'_n}{q'_n} > 1 - \frac{1}{2}\left(\frac{2\pi\vartheta'_n}{q'_n}\right)^2 > 1 - \frac{2\pi^2}{{q'_n}^2}.$$

Therefore we have

$$\begin{aligned}\frac{1}{m}\sum_{n=n_0+1}^{m}\cos\frac{2\pi\vartheta'_n}{q'_n} &> \frac{1}{m}\sum_{n=n_0+1}^{m}\left(1 - \frac{2\pi^2}{{q'_n}^2}\right)\\ &= \frac{1}{m}\sum_{n=n_0+1}^{m} 1 - \frac{2\pi^2}{m}\sum_{n=n_0+1}^{m}\frac{1}{{q'_n}^2}.\end{aligned}$$

The second summand on the right-hand side has the limit 0 if $m \to \infty$ while the first tends to 1. Hence $\lim\limits_{m\to\infty} g_m(y_0) = 1$.

So we have proved that g is a function in the first Baire class on $M \cap \mathbb{Q}'$, discontinuous at every point of $M \cap \mathbb{Q}'$. Therefore $M \cap \mathbb{Q}'$ is a set of the first category in $M \cap \mathbb{Q}'$ (cf. [8] p. 185) and so of the first category in $(0,1]$ as well. Since $M \cap \mathbb{Q}$ is a countable set, we see that $M = (M \cap \mathbb{Q}) \cup (M \cap \mathbb{Q}')$ is a set of the first category in $(0,1]$. On account of (7) we get that $\varrho(D(\alpha))$ is a set of the first category in $(0,1]$. This completes the proof. □

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