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ON SOME NON-OBVIOUS CONNECTIONS BETWEEN GRAPHS AND UNARY PARTIAL ALGEBRAS

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Abstract. In the present paper we generalize a few algebraic concepts to graphs. Applying this graph language we solve some problems on subalgebra lattices of unary partial algebras. In this paper three such problems are solved, other will be solved in papers [Pió I], [Pió II], [Pió III], [Pió IV]. More precisely, in the present paper first another proof of the following algebraic result from [Bar1] is given: for two unary partial algebras $\mathbf{A}$ and $\mathbf{B}$, their weak subalgebra lattices are isomorphic if and only if their graphs $\mathbf{G}^*(\mathbf{A})$ and $\mathbf{G}^*(\mathbf{B})$ are isomorphic. Secondly, it is shown that for two unary partial algebras $\mathbf{A}$ and $\mathbf{B}$ if their digraphs $\mathbf{G}(\mathbf{A})$ and $\mathbf{G}(\mathbf{B})$ are isomorphic, then their (weak, relative, strong) subalgebra lattices are also isomorphic. Thirdly, we characterize pairs $\langle \mathbf{L}, \mathbf{A} \rangle$, where $\mathbf{A}$ is a unary partial algebra and $\mathbf{L}$ is a lattice such that the weak subalgebra lattice of $\mathbf{A}$ is isomorphic to $\mathbf{L}$.

Introduction

In Universal Algebra many papers describe connections between a (total) algebra and its lattice of (also total) subalgebras. We recall, for example, that full characterization of the subalgebra lattice of a (total) algebra is given in [BiFr]. There are also several results which characterize subalgebra lattices for algebras which belong to a given variety or a given type (see [Jón]). Some papers investigate algebras with special subalgebra lattices (e.g. distributive, modular, etc.) or varieties which contain algebras such that their subalgebra lattices satisfy some given conditions ([EvGa], [Sha1], [Sha2]). A few of these papers concern also classical algebras ([GP1], [GP2]).

Another way is to investigate connections between two algebras of the same type or from the same variety, if we have connections between their subalgebra lattices. The paper [Sach] on Boolean algebras is a very good example. Sachs shows that any two Boolean algebras are isomorphic if and only if their subalgebra lattices are isomorphic. But results so strong are rather scarce in Universal Algebra.

Unfortunately, very few results of this kind are known for partial algebras, although at least three structures may be considered in this case. More precisely, for any partial algebra $\mathbf{A} = \langle A, (f^{\mathbf{A}})_{f \in F} \rangle$ we have three different subalgebra lattices (see e.g. [Bur]): the lattice of weak subalgebras $\mathbf{S}_w(\mathbf{A})$, the lattice of relative subalgebras $\mathbf{S}_r(\mathbf{A})$ and the lattice of strong subalgebras $\mathbf{S}_s(\mathbf{A})$. The second structure is clearly of no interest, because any subset of an algebra is the carrier of exactly one relative subalgebra. Moreover, many properties of the third lattice are inherited from total cases, but not much is known otherwise. Recall also that in the paper [Bar1] a complete characterization of the weak subalgebra lattice is given.

The main aim of the present paper is to introduce a new language to investigate unary partial algebras and their subalgebra lattices. More precisely, we show connections between unary partial algebras and directed and undirected graphs. These connections turn out to be very useful in the solution of some problems on subalgebra lattices of unary partial algebras. For instance, applying results from this paper we will get in [Pió I], [Pió II], [Pió III] and [Pió IV] necessary and sufficient conditions for a unary partial algebra $\mathbf{A}$ of a unary type K to be uniquely determined (up to isomorphism) in the class of all unary partial algebras of the same unary type K by its weak subalgebra lattice.

In chapter one we generalize a few algebraic concepts to graphs. For instance, for any digraph (directed graph) we define four kinds of subdigraphs—weak, relative, strong and dually strong. Secondly, for any (undirected) graph we define two kinds of subgraphs—weak and relative. We also show that families of subdigraphs (weak, relative, strong and dually strong) form complete lattices. In this way, for any digraph $\mathbf{G}$ we obtain four structures: the lattice of weak subdigraphs $\mathbf{S}_w(\mathbf{G})$, the lattice of relative subdigraphs $\mathbf{S}_r(\mathbf{G})$, the lattice of strong subdigraphs $\mathbf{S}_s(\mathbf{G})$ and the lattice of dually strong subdigraphs $\mathbf{S}_d(\mathbf{G})$. Analogously for any graph $\mathbf{G}$ we obtain the lattice of weak subgraphs $\mathbf{S}_w(\mathbf{G})$ and the lattice of relative subgraphs $\mathbf{S}_r(\mathbf{G})$. Next we prove, for example, that for any digraph $\mathbf{G}$

$$\mathbf{S}_w(\mathbf{G}) \simeq \mathbf{S}_w(\mathbf{G}^*) \quad \text{and} \quad \mathbf{S}_r(\mathbf{G}) \simeq \mathbf{S}_r(\mathbf{G}^*),$$

i.e. the weak (relative) subdigraph lattice of $\mathbf{G}$ is isomorphic to the weak (relative) subgraph lattice of the graph $\mathbf{G}^*$ which is obtained from $\mathbf{G}$ by omitting the orientation of all edges.

We also recall the definition of digraph type. More precisely, let η be a cardinal number. Then a digraph $\mathbf{G}$ is of the type η if and only if for each vertex v of $\mathbf{G}$ at most η edges start from v.

In chapter two we recall first that with any unary partial algebra $\mathbf{A}$ we can associate the digraph (the directed graph) $\mathbf{G}(\mathbf{A})$ (by omitting the name of operations in

$\mathbf{A}$, see [Bar1]) and the (undirected) graph $\mathbf{G}^*(\mathbf{A})$ (by omitting the orientation of all edges of $\mathbf{G}(\mathbf{A})$).

Next we prove several results describing connections between graphs and unary partial algebras. For instance, we show that for any unary partial algebra $\mathbf{A}$ its lattices of weak, relative, strong subalgebras and initial segments are isomorphic to lattices of weak, relative, strong and dually strong subdigraphs of its digraph $\mathbf{G}(\mathbf{A})$, and also its lattices of weak and relative subalgebras are isomorphic to lattices of weak and relative subgraphs of its graph $\mathbf{G}^*(\mathbf{A})$. More precisely,

$$\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{S}_w(\mathbf{G}(\mathbf{A})) \simeq \mathbf{S}_w(\mathbf{G}^*(\mathbf{A})),\ \mathbf{S}_r(\mathbf{A}) \simeq \mathbf{S}_r(\mathbf{G}(\mathbf{A})) \simeq \mathbf{S}_r(\mathbf{G}^*(\mathbf{A})),$$
$$\mathbf{S}_s(\mathbf{A}) \simeq \mathbf{S}_s(\mathbf{G}(\mathbf{A})),\ \mathbf{S}_d(\mathbf{A}) \simeq \mathbf{S}_d(\mathbf{G}(\mathbf{A}))$$

(where $\mathbf{S}_d(\mathbf{A})$ is the lattice of initial segments of $\mathbf{A}$).

We show also that if $\mathbf{A}$ is a unary partial algebra of a unary type K, then the digraph $\mathbf{G}(\mathbf{A})$ is of the type $|K|$ (where $|K|$ is the cardinality of K), and also conversely, for any digraph $\mathbf{G}$ of the type $|K|$ there exists a unary partial algebra $\mathbf{A}$ of the unary type K such that the digraph $\mathbf{G}(\mathbf{A})$ is isomorphic to $\mathbf{G}$. In other words, the class of all unary partial algebras of the same unary type K is represented by the class of all digraphs of the same type $|K|$.

Applying these results we first give another proof of the following theorem from [Bar1]:

(A) Let $\mathbf{A}$ and $\mathbf{B}$ be arbitrary unary partial algebras. Then

$$\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{S}_w(\mathbf{B}) \quad \text{iff} \quad \mathbf{G}^*(\mathbf{A}) \simeq \mathbf{G}^*(\mathbf{B}),$$

i.e. their weak subalgebra lattices are isomorphic iff their graphs are isomorphic.

Secondly, we show

(B) Let $\mathbf{A}$ and $\mathbf{B}$ be arbitrary unary partial algebras such that $\mathbf{G}(\mathbf{A}) \simeq \mathbf{G}(\mathbf{B})$. Then $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{S}_w(\mathbf{B})$, $\mathbf{S}_r(\mathbf{A}) \simeq \mathbf{S}_r(\mathbf{B})$, $\mathbf{S}_s(\mathbf{A}) \simeq \mathbf{S}_s(\mathbf{B})$, $\mathbf{S}_d(\mathbf{A}) \simeq \mathbf{S}_d(\mathbf{B})$.

Thirdly, we solve the following problem:

(C) Let $\mathbf{L}$ be an arbitrary lattice and let $\mathbf{A}$ be a unary partial algebra. When does (necessary and sufficient condition) $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{L}$ hold, i.e. when is the weak subalgebra lattice of $\mathbf{A}$ isomorphic to $\mathbf{L}$?

More precisely, in the above problem we can of course assume that for $\mathbf{L}$ there exists a unary partial algebra $\mathbf{B}$ such that the weak subalgebra lattice of $\mathbf{B}$ is isomorphic to $\mathbf{L}$, i.e. we can assume that $\mathbf{L}$ satisfies conditions from [Bar1], where a full algebraic characterization of such lattices is given. Then with $\mathbf{L}$ we can associate a graph $\mathbf{G}(\mathbf{L})$, and we will prove that the weak subalgebra lattice of $\mathbf{A}$ is isomorphic to $\mathbf{L}$ if and only if the graphs $\mathbf{G}^*(\mathbf{A})$ and $\mathbf{G}(\mathbf{L})$ are isomorphic.

Applying this result (and also other results from this paper) we will characterize in a subsequent paper the pairs $\langle \mathbf{L}, K \rangle$, where $\mathbf{L}$ is a lattice and K is a unary algebraic type, such that there exists a unary partial algebra $\mathbf{A}$ of the unary type K with the weak subalgebra lattice $\mathbf{S}_w(\mathbf{A})$ isomorphic to $\mathbf{L}$.

Recall that such a characterization for arbitrary algebraic lattices and arbitrary type in the case of total algebras is an important problem of Universal Algebra (see e.g. [Jón]) which is not completely solved yet. But for weak subalgebra lattices of unary partial algebras we can give a complete solution.

1.

In this chapter we first recall a few basic definitions and notation from the theory of directed and undirected graphs (see e.g. [Ber] [Wil], [Ore] or [Tut]).

However, our main goal in this chapter is to generalize a few concepts from the theory of unary partial algebras (like the type of partial algebra, some kind of subalgebras and subalgebra lattices etc., see [BRR], [Bur] and [Gräl]) to directed and undirected graphs. We will also prove some basic properties of these notions. For instance, for a directed (undirected) graph we will define several kinds of subgraphs. Next, we will prove that on the set of all directed (undirected) subgraphs of a given kind we can define in a natural way the structure of a complete lattice.

Definitions and results of this chapter will be needed in the second chapter to describe connections between unary partial algebras and directed and undirected graphs.

Throughout the paper the cardinality of A is denoted by $|A|$. Moreover, $\mathbb{N}$ is the set of all non-negative integers, **Card** is the class of all cardinal numbers and $\aleph_0$ is the countable, infinite cardinal number, i.e. $\aleph_0 = |\mathbb{N}|$.

1.1.

In this section we recall a few basic definitions and notation from graph theory (see e.g. [Ber], [Ore], [Tut] or [Wil]). In order to avoid misunderstandings we start with the definition of directed and undirected graphs.

(An undirected) graph $\mathbf{G} = \left\langle V^{\mathbf{G}}, E^{\mathbf{G}}, I^{\mathbf{G}} \right\rangle$ is an ordered triplet such that $V^{\mathbf{G}}$ and $E^{\mathbf{G}}$ are arbitrary sets of vertices and edges respectively, and $I^{\mathbf{G}}$ is a function from $E^{\mathbf{G}}$ into the set $\{\{v, w\}\colon\ v, w \in V^{\mathbf{G}}\}$ of all undirected pairs of $V^{\mathbf{G}}$. For each $e \in E^{\mathbf{G}}$ the elements of the set $I^{\mathbf{G}}(e)$ will be called terminal vertices of e.

A digraph (directed graph) $\mathbf{G} = \left\langle V^{\mathbf{G}}, E^{\mathbf{G}}, I^{\mathbf{G}} \right\rangle$ is an ordered triplet such that $V^{\mathbf{G}}$ and $E^{\mathbf{G}}$ are arbitrary sets of vertices and edges respectively, and $I^{\mathbf{G}}$ is a function from $E^{\mathbf{G}}$ into the direct product $V^{\mathbf{G}} \times V^{\mathbf{G}}$. Let $\pi_i\colon\ V^{\mathbf{G}} \times V^{\mathbf{G}} \to V^{\mathbf{G}}$ be the projection on the i-th coordinate for $i = 1, 2$. Then for each $e \in E^{\mathbf{G}}$ the vertices $\pi_1 I^{\mathbf{G}}(e)$ and $\pi_2 I^{\mathbf{G}}(e)$ will be called initial and final vertices of e, respectively.

The class of all digraphs (graphs) will be denoted by $\mathcal{AG}_d$ ($\mathcal{AG}_n$).

Since we want to represent unary partial algebras by digraphs, we must consider in general infinite digraphs. More precisely, we do not restrict the cardinality of the vertex and edge sets, i.e. we consider digraphs and graphs such that their sets of vertices and edges are of arbitrary (not only finite) cardinality.

Observe also that the triplet $\langle \emptyset, \emptyset, \emptyset \rangle$ of empty sets is simultaneously a graph and a digraph—the empty graph, which will be also denoted by $\emptyset$.

Let $\mathbf{G}, \mathbf{H} \in \mathcal{AG}_d$ ($\mathbf{G}, \mathbf{H} \in \mathcal{AG}_n$) and let $\varphi_V \colon V^{\mathbf{G}} \to E^{\mathbf{H}}$, $\varphi_E \colon E^{\mathbf{G}} \to E^{\mathbf{H}}$ be arbitrary functions.

We say that the pair of functions $\varphi = \langle \varphi_V, \varphi_E \rangle$ is an isomorphism from $\mathbf{G}$ onto $\mathbf{H}$ iff φ_V and φ_E are bijections and for all $e \in E^{\mathbf{G}}$,

$$\pi_i I^{\mathbf{H}}(\varphi_E(e)) = \varphi_V(\pi_i I^{\mathbf{G}}(e)) \text{ for } i = 1, 2 \quad (I^{\mathbf{H}}(\varphi_E(e)) = \varphi_V(I^{\mathbf{G}}(e))).$$

We write $\mathbf{G} \simeq \mathbf{H}$ and say that $\mathbf{G}$ and $\mathbf{H}$ are isomorphic digraphs (graphs) iff there is an isomorphism φ from $\mathbf{G}$ onto $\mathbf{H}$.

Now recall that with any digraph $\mathbf{G}$ we can associate a graph $\mathbf{G}^*$ by omitting the orientation of edges. More formally,

Definition 1.1.1. Let $\mathbf{G} \in \mathcal{AG}_d$. Then $\mathbf{G}^*$ is the graph such that

$$V^{\mathbf{G}^*} := V^{\mathbf{G}}, \quad E^{\mathbf{G}^*} := E^{\mathbf{G}}$$

and for all $e \in E^{\mathbf{G}^*}$,

$$I^{\mathbf{G}^*}(e) := \{\pi_1 I^{\mathbf{G}}(e), \pi_2 I^{\mathbf{G}}(e)\}.$$

If $\mathbf{G} \in \mathcal{AG}_d$, then for any vertex $v \in V^{\mathbf{G}}$ we can define the set of edges

$$E_s^{\mathbf{G}}(v) := \{e \in E^{\mathbf{G}} \colon \ \pi_1 I^{\mathbf{G}}(e) = v\}$$

and the cardinal number

$$s^{\mathbf{G}}(v) := |E_s^{\mathbf{G}}(v)|.$$

Observe that $s^{\mathbf{G}}(v)$ may be an arbitrary cardinal number, because we consider also infinite digraphs, i.e. sets of vertices and edges may have arbitrary cardinalities.

In the rest of this section we generalize a few algebraic concepts (like the type of partial algebras, subalgebras, lattices of subalgebras, etc., see [BRR] or [Bur]) to directed and undirected graphs. Next we show that a few wellknown results from the theory of unary partial algebras (e.g. that sets of subalgebras form a complete lattice) can be generalized to graphs. Definitions and results from this section will be needed in the next chapter to formulate and prove connections between unary partial algebras and directed and undirected graphs.

The type of a digraph is a cardinal number. More precisely,

Definition 1.1.2. Let $\mathbf{G} \in \mathcal{AG}_d$ and $\eta \in \mathbf{Card}$.

(a) The digraph $\mathbf{G}$ is of type η iff for all $v \in V^{\mathbf{G}}$, $s^{\mathbf{G}}(v) \leqslant \eta$.

(b) $\mathbf{G}$ is of finite (infinite) type iff $\mathbf{G}$ is of type η and $\eta < \aleph_0$ ($\eta \geqslant \aleph_0$).

For every η the class of all digraphs of the type η will be denoted by $\mathcal{AG}_d(\eta)$.

Observe that $\mathcal{AG}(0)$ is the class of all discrete digraphs (i.e. digraphs which have no edges). Moreover, we have that for all $\eta_1, \eta_2 \in \mathbf{Card}$ if $\eta_1 \leqslant \eta_2$, then $\mathcal{AG}_d(\eta_1) \subseteq \mathcal{AG}_d(\eta_2)$.

We want to represent unary partial algebras by directed and undirected graphs, so we must define various kinds of subgraphs and subdigraphs. More precisely, we define two kinds of subgraphs and four kinds of subdigraphs.

Definition 1.1.3. Let $\mathbf{G}, \mathbf{H} \in \mathcal{AG}_n$. Then

(a) We say that $\mathbf{H}$ is a weak subgraph of $\mathbf{G}$ ($\mathbf{H} \leqslant_w \mathbf{G}$) iff
$V^{\mathbf{H}} \subseteq V^{\mathbf{G}}$, $E^{\mathbf{H}} \subseteq E^{\mathbf{G}}$ and $I^{\mathbf{H}} = I^{\mathbf{G}}|_{E^{\mathbf{H}}}$.

(b) We say that $\mathbf{H}$ is a relative subgraph of $\mathbf{G}$ ($\mathbf{H} \leqslant_r \mathbf{G}$) iff
$\mathbf{H} \leqslant_w \mathbf{G}$ and for all $e \in E^{\mathbf{G}}$, if $I^{\mathbf{G}}(e) \subseteq V^{\mathbf{H}}$, then $e \in E^{\mathbf{H}}$.

For all $\mathbf{G} \in \mathcal{AG}_n$, $S_w(\mathbf{G})$ ($S_r(\mathbf{G})$) is the family of all weak (relative) subgraphs of $\mathbf{G}$. Observe also that the empty graph is simultaneously a weak and a relative subgraph of an arbitrary digraph.

Definition 1.1.4. Let $\mathbf{G}, \mathbf{H} \in \mathcal{AG}_d$. Then:

(a) $\mathbf{H}$ is called a weak subdigraph of $\mathbf{G}$ ($\mathbf{H} \leqslant_w \mathbf{G}$) iff
$V^{\mathbf{H}} \subseteq V^{\mathbf{G}}$, $E^{\mathbf{H}} \subseteq E^{\mathbf{G}}$ and $I^{\mathbf{H}} = I^{\mathbf{G}}|_{E^{\mathbf{H}}}$.

(b) $\mathbf{H}$ is called a relative subdigraph of $\mathbf{G}$ ($\mathbf{H} \leqslant_r \mathbf{G}$) iff
$\mathbf{H} \leqslant_w \mathbf{G}$ and for all $e \in E^{\mathbf{G}}$, if $I^{\mathbf{G}}(e) \in V^{\mathbf{H}} \times V^{\mathbf{H}}$, then $e \in E^{\mathbf{H}}$.

(c) $\mathbf{H}$ is called a strong subdigraph of $\mathbf{G}$ ($\mathbf{H} \leqslant_s \mathbf{G}$) iff
$\mathbf{H} \leqslant_r \mathbf{G}$ and for all $e \in E^{\mathbf{G}}$, if $\pi_1 I^{\mathbf{G}}(e) \in V^{\mathbf{H}}$, then $\pi_2 I^{\mathbf{G}}(e) \in V^{\mathbf{H}}$.

(d) $\mathbf{H}$ is called a dually strong subdigraph of $\mathbf{G}$ ($\mathbf{H} \leqslant_d \mathbf{G}$) iff
$\mathbf{H} \leqslant_r \mathbf{G}$ and for all $e \in E^{\mathbf{G}}$, if $\pi_2 I^{\mathbf{G}}(e) \in V^{\mathbf{H}}$, then $\pi_1 I^{\mathbf{G}}(e) \in V^{\mathbf{H}}$.

For all $\mathbf{G} \in \mathcal{AG}_d$, $S_w(\mathbf{G})$, $S_r(\mathbf{G})$, $S_s(\mathbf{G})$ and $S_d(\mathbf{G})$ are the families of all weak, relative, strong and dually strong subdigraphs of the digraph $\mathbf{G}$, respectively. Observe also that the empty graph is simultaneously a weak, relative, strong and dually strong subdigraph of an arbitrary digraph.

Now we give a few simple facts (easy proofs are omitted).

Proposition 1.1.5. *Let* $\mathbf{G} \in \mathcal{AG}_d$ *(*$\mathbf{G} \in \mathcal{AG}_n$*). Then:*

(a) $S_s(\mathbf{G}) \subseteq S_r(\mathbf{G}) \subseteq S_w(\mathbf{G})$ *and* $S_d(\mathbf{G}) \subseteq S_r(\mathbf{G})$ $(S_r(\mathbf{G}) \subseteq S_w(\mathbf{G}))$.

(b) *If* $\mathbf{H} \leqslant_w \mathbf{G}$ $(\mathbf{H} \leqslant_r \mathbf{G})$, *then* $S_w(\mathbf{H}) \subseteq S_w(\mathbf{G})$ $(S_r(\mathbf{H}) \subseteq S_r(\mathbf{G}))$.

(c) *If* $\mathbf{H} \leqslant_s \mathbf{G}$ $(\mathbf{H} \leqslant_d \mathbf{G})$, *then* $S_s(\mathbf{H}) \subseteq S_s(\mathbf{G})$ $(S_d(\mathbf{H}) \subseteq S_d(\mathbf{G}))$.

Of course, for graphs only points (a) and (b) hold.

Now we want to show that for any $\mathbf{G} \in \mathcal{AG}_d$ $(\mathbf{G} \in \mathcal{AG}_n)$, the sets of all weak, relative, strong and dually strong subdigraphs (weak and relative subgraphs) form complete lattices. To this purpose we give a few facts (simple proofs are left to the reader).

Proposition 1.1.6. *Let* $\mathbf{G} \in \mathcal{AG}_d$ *(*$\mathbf{G} \in \mathcal{AG}_n$*). Then:*

(a) *For all* $\mathbf{H}_1, \mathbf{H}_2 \leqslant_w \mathbf{G}$,
$\mathbf{H}_1 \leqslant_w \mathbf{H}_2$ *iff* $V^{\mathbf{H}_1} \subseteq V^{\mathbf{H}_2}$ *and* $E^{\mathbf{H}_1} \subseteq E^{\mathbf{H}_2}$.
$\mathbf{H}_1 = \mathbf{H}_2$ *iff* $V^{\mathbf{H}_1} = V^{\mathbf{H}_2}$ *and* $E^{\mathbf{H}_1} = E^{\mathbf{H}_2}$.

(b) *For all* $\mathbf{H}_1, \mathbf{H}_2 \leqslant_r \mathbf{G}$,
$\mathbf{H}_1 \leqslant_r \mathbf{H}_2$ $(\mathbf{H}_1 = \mathbf{H}_2)$ *iff* $V^{\mathbf{H}_1} \subseteq V^{\mathbf{H}_2}$ $(V^{\mathbf{H}_1} = V^{\mathbf{H}_2})$.

(c) *For all* $\mathbf{H}_1, \mathbf{H}_2 \leqslant_s \mathbf{G}$, $\mathbf{H}_1 \leqslant_s \mathbf{H}_2$ *iff* $V^{\mathbf{H}_1} \subseteq V^{\mathbf{H}_2}$.

(d) *For all* $\mathbf{H}_1, \mathbf{H}_2 \leqslant_d \mathbf{G}$, $\mathbf{H}_1 \leqslant_d \mathbf{H}_2$ *iff* $V^{\mathbf{H}_1} \subseteq V^{\mathbf{H}_2}$.

Of course, for graphs only points (a) and (b) hold.

By the above facts we have that for any digraph (graph) $\mathbf{G}$, the relations $\leqslant_w$, $\leqslant_r$, $\leqslant_s$ and $\leqslant_d$ ($\leqslant_w$ and $\leqslant_r$) are partial orders. Secondly, it is easily shown that for any non-empty family $\{\mathbf{H}_i\}_{i\in I}$ of weak, or strong, or dually strong subdigraphs (weak subgraphs) of the digraph $\mathbf{G}$, the set-theoretical intersection $\left\langle \bigcap_{i\in I} V^{\mathbf{H}_i}, \bigcap_{i\in I} E^{\mathbf{H}_i}, \bigcap_{i\in I} I^{\mathbf{H}_i} \right\rangle$ and the set-theoretical union $\left\langle \bigcup_{i\in I} V^{\mathbf{H}_i}, \bigcup_{i\in I} E^{\mathbf{H}_i}, \bigcup_{i\in I} I^{\mathbf{H}_i} \right\rangle$ is again a weak, strong, dually strong subdigraph (a weak subgraph) of $\mathbf{G}$, respectively.

Observe also that for relative subdigraphs (subgraphs) of $\mathbf{G}$ we have the following two facts: for any set $W \subseteq V^{\mathbf{G}}$ there exists exactly one relative subdigraph (subgraph) $\mathbf{H}$ of $\mathbf{G}$ such that $V^{\mathbf{H}} = W$. Secondly, it is easily shown that for any non-empty family of relative subdigraphs (subgraphs) $\{\mathbf{H}_i\}_{i\in I}$ of $\mathbf{G}$, the set-theoretical intersection $\left\langle \bigcap_{i\in I} V^{\mathbf{H}_i}, \bigcap_{i\in I} E^{\mathbf{H}_i}, \bigcap_{i\in I} I^{\mathbf{H}_i} \right\rangle$ is a relative subdigraph (subgraph) of $\mathbf{G}$, too.

So we have proved the following results:

Proposition 1.1.7. *Let* $\mathbf{G} \in \mathcal{AG}_d$ *(*$\mathbf{G} \in \mathcal{AG}_n$*). Then:*

(a) $\mathbf{S}_x(\mathbf{G}) = \langle S_x(\mathbf{G}), \leqslant_x\rangle$, *where* $x = w, s$ *or* d ($\mathbf{S}_w(\mathbf{G}) = \langle S_w(\mathbf{G}), \leqslant_w\rangle$) *are complete lattices, where infimum* $\bigwedge$ *and supremum* $\bigvee$ *are defined in the following way: for all* $\{\mathbf{H}_i\}_{i\in I}$ ($I \neq \emptyset$),

$$\bigwedge_{i\in I} \mathbf{H}_i := \left\langle \bigcap_{i\in I} V^{\mathbf{H}_i}, \bigcap_{i\in I} E^{\mathbf{H}_i}, \bigcap_{i\in I} I^{\mathbf{H}_i} \right\rangle,$$

$$\bigvee_{i\in I} \mathbf{H}_i := \left\langle \bigcup_{i\in I} V^{\mathbf{H}_i}, \bigcup_{i\in I} E^{\mathbf{H}_i}, \bigcup_{i\in I} I^{\mathbf{H}_i} \right\rangle;$$

if $I = \{i_1, i_2\}$, *then we write* $\mathbf{H}_1 \wedge \mathbf{H}_2$ *and* $\mathbf{H}_1 \vee \mathbf{H}_2$.

(b) $\mathbf{S}_r(\mathbf{G}) = \langle S_r(\mathbf{G}), \leqslant_r\rangle$ *is a complete lattice, where the operation* $\bigwedge$ *is defined as above and for all* $\{\mathbf{H}_i\}_{i\in I}$ ($I \neq \emptyset$), $\bigvee_{i\in I} \mathbf{H}_i$ *is the relative subdigraph (subgraph) such that* $V^{\bigvee_{i\in I} H_i} = \bigcup_{i\in I} V^{\mathbf{H}_i}$.

(c) $\mathbf{S}_s(\mathbf{G})$ *and* $\mathbf{S}_d(\mathbf{G})$ *are sublattices of the lattice* $\mathbf{S}_w(\mathbf{G})$.

Let $\mathbf{G} \in \mathcal{AG}_d$ and $W \subseteq V^{\mathbf{G}}$. Then the above proposition implies that there exists the least with respect to order $\leqslant_s$ ($\leqslant_d$) strong (dually strong) subdigraph of $\mathbf{G}$ containing W. Such a strong (dually strong) subdigraph of $\mathbf{G}$ will be denoted by $\langle W\rangle^s_{\mathbf{G}}$ ($\langle W\rangle^d_{\mathbf{G}}$) and we will say that W generates this strong (dually strong) subdigraph of $\mathbf{G}$.

More formally, let $\mathbf{G} \in \mathcal{AG}_d$ and $W \subseteq V^{\mathbf{G}}$. Then we define

$$\langle W\rangle^s_{\mathbf{G}} := \bigwedge\{\mathbf{H} \leqslant_s \mathbf{G}\colon\ W \subseteq V^{\mathbf{H}}\}, \langle W\rangle^d_{\mathbf{G}} := \bigwedge\{\mathbf{H} \leqslant_d \mathbf{G}\colon\ W \subseteq V^{\mathbf{H}}\}.$$

Finally, observe that P. 1.1.5 and P. 1.1.7 easily imply the following facts (for all lattices $\mathbf{L}$ and $\mathbf{K}$, $\mathbf{K} \leqslant \mathbf{L}$ denotes that $\mathbf{K}$ is a sublattice of $\mathbf{L}$).

Proposition 1.1.8.

(a) *Let* $\mathbf{G} \in \mathcal{AG}_d$ ($\mathbf{G} \in \mathcal{AG}_n$) *and* $\mathbf{H}_1 \leqslant_w \mathbf{G}$, $\mathbf{H}_2 \leqslant_r \mathbf{G}$. *Then*

$$\mathbf{S}_w(\mathbf{H}_1) \leqslant \mathbf{S}_w(\mathbf{G}) \quad \textit{and} \quad \mathbf{S}_r(\mathbf{H}_2) \leqslant \mathbf{S}_r(\mathbf{G}).$$

(b) *Let* $\mathbf{G} \in \mathcal{AG}_d$ *and* $\mathbf{H}_1 \leqslant_s \mathbf{G}$, $\mathbf{H}_2 \leqslant_d \mathbf{G}$. *Then*

$$\mathbf{S}_s(\mathbf{H}_1) \leqslant \mathbf{S}_s(\mathbf{G}) \quad \textit{and} \quad \mathbf{S}_d(\mathbf{H}_2) \leqslant \mathbf{S}_d(\mathbf{G}).$$

1.2.

Now we prove two less trivial results which will be needed in the next chapter. First we formulate a result describing strong and dually strong subdigraphs which are generated by a given set of vertices. This is a graph-theoretical generalization of the classical result on the generation of (strong) subalgebras.

Next we prove that for any digraph $\mathbf{G}$ the lattices of weak and relative subdigraphs of $\mathbf{G}$ are isomorphic to the lattices of weak and relative subgraphs of the graph $\mathbf{G}^*$.

But first we recall the definitions of a finite chain and a path in a digraph. Let $\mathbf{G} \in \mathcal{AG}_d$ and let $\mathbf{r} = \left\langle (f_i^{\mathbf{r}})_{i=1}^{i=N_r}, (u_i^{\mathbf{r}})_{i=1}^{i=N_r+1} \right\rangle$ be a pair of finite sequences of edges and vertices respectively and $N_{\mathbf{r}} \geqslant 1$.

We say that $\mathbf{r}$ is a finite chain in $\mathbf{G}$ iff $I^{\mathbf{G}}(f_i^{\mathbf{r}}) = \langle u_i^{\mathbf{r}}, u_{i+1}^{\mathbf{r}} \rangle$ for $1 \leqslant i \leqslant N_{\mathbf{r}}$.

A finite chain $\mathbf{r}$ is a finite path in $\mathbf{G}$ iff $u_l^{\mathbf{r}} \neq u_k^{\mathbf{r}}$ for all $1 \leqslant l < k \leqslant N_{\mathbf{r}} + 1$.

$CR_{\text{fin}}(\mathbf{G})$ ($R_{\text{fin}}(\mathbf{G})$) is the family of all finite chains (paths) in $\mathbf{G}$.

Proposition 1.2.1. *Let $\mathbf{G} \in \mathcal{AG}_d$ and $W \subseteq V^{\mathbf{G}}$. Then:*

(a) *The following conditions are equivalent:*

(a.1) $v \in V^{\langle W \rangle_G^d}$,

(a.2) $v \in W$ *or there exists* $\mathbf{r} \in R_{\text{fin}}(\mathbf{G})$ *such that* $u_1^{\mathbf{r}} = v$ *and* $u_{N_r+1}^{\mathbf{r}} \in W$,

(a.3) $v \in W$ *or there exists* $\mathbf{r} \in CR_{\text{fin}}(\mathbf{G})$ *such that* $u_1^{\mathbf{r}} = v$ *and* $u_{N_r+1}^{\mathbf{r}} \in W$.

(b) *The following conditions are equivalent:*

(b.1) $v \in V^{\langle W \rangle_G^s}$,

(b.2) $v \in W$ *or there exists* $\mathbf{r} \in R_{\text{fin}}(\mathbf{G})$ *such that* $u_1^{\mathbf{r}} \in W$ *and* $u_{N_r+1}^{\mathbf{r}} = v$,

(b.3) $v \in W$ *or there exists* $\mathbf{r} \in CR_{\text{fin}}(\mathbf{G})$ *such that* $u_1^{\mathbf{r}} \in W$ *and* $u_{N_r+1}^{\mathbf{r}} = v$.

P r o o f. The implications (a.2) $\Rightarrow$ (a.3) and (b.2) $\Rightarrow$ (b.3) are trivial. Secondly, the implications (a.3) $\Rightarrow$ (a.2) and (b.3) $\Rightarrow$ (b.2) follow from the wellknown fact that for any $\mathbf{p} \in CR_{\text{fin}}(\mathbf{G})$, if $u_1^{\mathbf{p}} \neq u_{N_p+1}^{\mathbf{p}}$, then there is $\mathbf{r} \in R_{\text{fin}}(\mathbf{G})$ which connects $u_1^{\mathbf{p}}$ and $u_{N_p+1}^{\mathbf{p}}$, i.e. $u_1^{\mathbf{r}} = u_1^{\mathbf{p}}$ and $u_{N_r+1}^{\mathbf{r}} = u_{N_p+1}^{\mathbf{p}}$.

(a.1) $\Leftrightarrow$ (a.3) and (b.1) $\Leftrightarrow$ (b.3): Let V_1 (V_2) be the set of all vertices which satisfy the condition (a.3) ((b.3)) and let $\mathbf{H}_1$ ($\mathbf{H}_2$) be the relative subdigraph of $\mathbf{G}$ such that

$$V^{\mathbf{H}_1} = V_1 \qquad (V^{\mathbf{H}_2} = V_2).$$

First we prove that

$$\mathbf{H}_1 \leqslant_d \mathbf{G} \quad \text{and} \quad \mathbf{H}_2 \leqslant_s \mathbf{G}.$$

We show only that $\mathbf{H}_1 \leqslant_d \mathbf{G}$, the analogous proof of the second part is left to the reader.

Since $\mathbf{H}_1 \leqslant_r \mathbf{G}$, we must only prove that for an arbitrary edge $e \in E^{\mathbf{G}}$, if $\pi_2 I^{\mathbf{G}}(e) \in V^{\mathbf{H}_1}$, then $\pi_1 I^{\mathbf{G}}(e) \in V^{\mathbf{H}_1}$. Let us take $e \in E^{\mathbf{G}}$ such that $\pi_2 I^{\mathbf{G}}(e) \in$

$V^{\mathbf{H}_1} := V_1$. From the condition (a.3) we have that $\pi_2 I^{\mathbf{G}}(e) \in W$ or there exists $\mathbf{r} \in CR_{\text{fin}}(\mathbf{G})$ such that $u_1^{\mathbf{r}} = \pi_2 I^{\mathbf{G}}(e)$ and $u_{N_r+1}^{\mathbf{r}} \in W$. Then it is easily shown that the sequence of edges e, $f_1^{\mathbf{r}}, \ldots, f_{N_r}^{\mathbf{r}}$ (or the sequence of one element e) is a finite chain which connects $\pi_1 I^{\mathbf{G}}(e)$ and the set W. Thus from definition of V_1 we obtain that $\pi_1 I^{\mathbf{G}}(e) \in V_1$.

Since $W \subseteq V_1 = V^{\mathbf{H}_1}$, $W \subseteq V_2 = V^{\mathbf{H}_2}$ and $\mathbf{H}_1 \leqslant_d \mathbf{G}$, $\mathbf{H}_2 \leqslant_s \mathbf{G}$, the definitions of the digraphs $\langle W \rangle_{\mathbf{G}}^d$ and $\langle W \rangle_{\mathbf{G}}^s$ imply

$$V^{\langle W \rangle_G^d} \subseteq V_1 \quad \text{and} \quad V^{\langle W \rangle_G^s} \subseteq V_2.$$

Now let us take $v \in V_1$ ($v \in V_2$). Of course we can assume that $v \notin W$. Then there exists $\mathbf{r} \in CR_{\text{fin}}(\mathbf{G})$ such that $u_1^{\mathbf{r}} = v$, $u_{N_r+1}^{\mathbf{r}} \in W$ ($u_1^{\mathbf{r}} \in W$, $u_{N_r+1}^{\mathbf{r}} = v$). Now applying a simple induction (see D. 1.1.4) we get

$$u_i^{\mathbf{r}} \in V^{\langle W \rangle_G^d} \text{ for } i = N_{\mathbf{r}} + 1, \ldots, 1 \quad (u_i^{\mathbf{r}} \in V^{\langle W \rangle_G^s} \text{ for } i = 1, \ldots, N_{\mathbf{r}} + 1).$$

Hence, $v = u_1^{\mathbf{r}} \in V^{\langle W \rangle_G^d}$ ($v = u_{N_r+1}^{\mathbf{r}} \in V^{\langle W \rangle_G^s}$). Thus we have shown

$$V_1 \subseteq V^{\langle W \rangle_G^d} \quad \text{and} \quad V_2 \subseteq V^{\langle W \rangle_G^d}.$$

The above inclusions imply $V^{\langle W \rangle_G^d} = V_1$ and $V^{\langle W \rangle_G^s} = V_2$. Thus the proof of our equivalences is complete. □

Now we want to show that for any digraph $\mathbf{G}$ the function * (see D. 1.1.1) induces an isomorphism between the lattices of weak and relative subdigraphs of $\mathbf{G}$ and the lattices of weak and relative subgraphs of $\mathbf{G}^*$.

To this purpose we first formulate two simple facts.

Proposition 1.2.2. *Let* $\mathbf{G}, \mathbf{H} \in \mathcal{AG}_d$. *Then:*

(a) *If* $\mathbf{H} \leqslant_w \mathbf{G}$ ($\mathbf{H} \leqslant_r \mathbf{G}$), *then* $\mathbf{H}^* \leqslant_w \mathbf{G}^*$ ($\mathbf{H}^* \leqslant_r \mathbf{G}^*$).

(b) *If* $\mathbf{H} \leqslant_w \mathbf{G}$ *and* $\mathbf{H}^* \leqslant_r \mathbf{G}^*$, *then* $\mathbf{H} \leqslant_r \mathbf{G}$.

Proof is obtained from D. 1.1.3 and D. 1.1.4 and the definition of *.

Proposition 1.2.3. *Let* $\mathbf{G} \in \mathcal{AG}_d$. *Then for all* $\mathbf{K} \leqslant_w \mathbf{G}^*$ ($\mathbf{K} \leqslant_r \mathbf{G}^*$) *there exists exactly one* $\mathbf{H} \leqslant_w \mathbf{G}$ ($\mathbf{H} \leqslant_r \mathbf{G}$) *such that* $\mathbf{H}^* = \mathbf{K}$.

P r o o f. (a): Let us take $\mathbf{K} \leqslant_w \mathbf{G}^*$ and let $\mathbf{H} = \langle V^{\mathbf{H}}, E^{\mathbf{H}}, I^{\mathbf{H}} \rangle$ be a triplet such that $V^{\mathbf{H}} = V^{\mathbf{K}}$, $E^{\mathbf{H}} = E^{\mathbf{K}}$ and $I^{\mathbf{H}} = I^{\mathbf{G}}|_{E^H}$. Applying D. 1.1.4 and the definiton of * it is easily shown that $\mathbf{H}$ is the digraph such that $\mathbf{H}^* = \mathbf{K}$ and $\mathbf{H} \leqslant_w \mathbf{G}$.

Now let $\mathbf{H}_1$ be a digraph such that $\mathbf{H}_1 \leqslant_w \mathbf{G}$ and $\mathbf{H}_1^* = \mathbf{K}$. Then from the definiton of * we obtain in particular $V^{\mathbf{H}_1^*} = V^{\mathbf{K}} = V^{\mathbf{H}^*} = V^{\mathbf{H}}$ and $E^{\mathbf{H}_1^*} = E^{\mathbf{K}} = E^{\mathbf{H}^*} = E^{\mathbf{H}}$. Thus from P. 1.1.6 we have $\mathbf{H}_1 = \mathbf{H}$.

The case of relative subgraphs follows of course from P. 1.2.2 and from the case of weak subgraphs (which has been proved above). □

Theorem 1.2.4. *Let* $\mathbf{G} \in \mathcal{AG}_d$. *Then*

$$\mathbf{S}_w(\mathbf{G}) \simeq \mathbf{S}_w(\mathbf{G}^*) \quad \textit{and} \quad \mathbf{S}_r(\mathbf{G}) \simeq \mathbf{S}_r(\mathbf{G}^*).$$

P r o o f. (a): Let us take a function $\varphi\colon S_w(\mathbf{G}) \longrightarrow S_w(\mathbf{G}^*)$ such that

$$\varphi(\mathbf{H}) := \mathbf{H}^* \quad \text{for all} \quad \mathbf{H} \in S_w(\mathbf{G}).$$

We want to prove that φ is the required isomorphism. From P. 1.2.2 we have that φ is well defined, and from P. 1.2.3 we obtain that φ is a bijection. Since $\mathbf{S}_w(\mathbf{G})$ and $\mathbf{S}_w(\mathbf{G}^*)$ are total algebras, we must only show

$$(\mathbf{H}_1 \wedge \mathbf{H}_2)^* = \mathbf{H}_1^* \wedge \mathbf{H}_2^* \text{ and } (\mathbf{H}_1 \vee \mathbf{H}_2)^* = \mathbf{H}_1^* \vee \mathbf{H}_2^* \text{ for all } \mathbf{H}_1, \mathbf{H}_2 \in S_w(\mathbf{G}).$$

From the definition of the operations $\wedge$ and $\vee$ (see P. 1.1.7) we have

$$\begin{aligned}
V^{(\mathbf{H}_1 \wedge \mathbf{H}_2)^*} &= V^{\mathbf{H}_1 \wedge \mathbf{H}_2} = V^{\mathbf{H}_1} \cap V^{\mathbf{H}_2} = V^{\mathbf{H}_1^*} \cap V^{\mathbf{H}_2^*} = V^{\mathbf{H}_1^* \wedge \mathbf{H}_2^*},\\
E^{(\mathbf{H}_1 \wedge \mathbf{H}_2)^*} &= E^{\mathbf{H}_1 \wedge \mathbf{H}_2} = E^{\mathbf{H}_1} \cap E^{\mathbf{H}_2} = E^{\mathbf{H}_1^*} \cap E^{\mathbf{H}_2^*} = E^{\mathbf{H}_1^* \wedge \mathbf{H}_2^*},\\
V^{(\mathbf{H}_1 \vee \mathbf{H}_2)^*} &= V^{\mathbf{H}_1 \vee \mathbf{H}_2} = V^{\mathbf{H}_1} \cup V^{\mathbf{H}_2} = V^{\mathbf{H}_1^*} \cup V^{\mathbf{H}_2^*} = V^{\mathbf{H}_1^* \vee \mathbf{H}_2^*},\\
E^{(\mathbf{H}_1 \vee \mathbf{H}_2)^*} &= E^{\mathbf{H}_1 \vee \mathbf{H}_2} = E^{\mathbf{H}_1} \cup E^{\mathbf{H}_2} = E^{\mathbf{H}_1^*} \cup E^{\mathbf{H}_2^*} = E^{\mathbf{H}_1^* \vee \mathbf{H}_2^*}.
\end{aligned}$$

Since $(\mathbf{H}_1 \wedge \mathbf{H}_2)^*, (\mathbf{H}_1 \vee \mathbf{H}_2)^*, \mathbf{H}_1^* \wedge \mathbf{H}_2^*, \mathbf{H}_1^* \vee \mathbf{H}_2^* \in S_w(\mathbf{G}^*)$, so P. 1.1.6 implies the desired equalities. This is the end of the proof of (a).

(b): We want to prove that $\varphi|_{S_r(G)}$ is the required isomorphism. P. 1.2.2 and (a) imply that $\varphi|_{S_r(G)}$ goes into $\mathbf{S}_r(\mathbf{G}^*)$. Secondly, from P. 1.2.3 we have that $\varphi|_{S_r(G)}$ is a bijection. Since $\mathbf{S}_r(\mathbf{G})$ and $\mathbf{S}_r(\mathbf{G}^*)$ are total algebras, we must only show

$$(\mathbf{H}_1 \wedge \mathbf{H}_2)^* = \mathbf{H}_1^* \wedge \mathbf{H}_2^* \text{ and } (\mathbf{H}_1 \vee \mathbf{H}_2)^* = \mathbf{H}_1^* \vee \mathbf{H}_2^* \text{ for all } \mathbf{H}_1, \mathbf{H}_2 \in S_r(\mathbf{G}).$$

From the definition of the operations $\wedge$ and $\vee$ (see P1.1.7) we have

$$\begin{aligned}
V^{(\mathbf{H}_1 \wedge \mathbf{H}_2)^*} &= V^{\mathbf{H}_1 \wedge \mathbf{H}_2} = V^{\mathbf{H}_1} \cap V^{\mathbf{H}_2} = V^{\mathbf{H}_1^*} \cap V^{\mathbf{H}_2^*} = V^{\mathbf{H}_1^* \wedge \mathbf{H}_2^*},\\
V^{(\mathbf{H}_1 \vee \mathbf{H}_2)^*} &= V^{\mathbf{H}_1 \vee \mathbf{H}_2} = V^{\mathbf{H}_1} \cup V^{\mathbf{H}_2} = V^{\mathbf{H}_1^*} \cup V^{\mathbf{H}_2^*} = V^{\mathbf{H}_1^* \vee \mathbf{H}_2^*}.
\end{aligned}$$

Since $(\mathbf{H}_1 \wedge \mathbf{H}_2)^*, (\mathbf{H}_1 \vee \mathbf{H}_2)^*, \mathbf{H}_1^* \wedge \mathbf{H}_2^*, \mathbf{H}_1^* \vee \mathbf{H}_2^* \in S_w(\mathbf{G}^*)$, so P. 1.1.6 implies the desired equalities. Thus the proof of (b) is complete. □

2.

We assume knowledge of basic concepts and facts from the theory of partial and total algebras, and also from lattice theory (see e.g. [Bur], [BRR], [Grä1], [Grä2], [Jón], [MMT]).

Now we recall only some basic notation and definitions. A unary partial algebra of a unary type K (where K is a set of unary operation symbols) is an algebra $\mathbf{A} = \left\langle A, (f^{\mathbf{A}})_{f\in K}\right\rangle$ such that for all $f \in K$, $f^{\mathbf{A}}$ is a partial function from A into A, i.e. $f^{\mathbf{A}}$ is a unary partial operation on $\mathbf{A}$. The class of all unary partial algebras of a unary type K will be denoted by $\mathcal{UPA}lg(K)$, and the class of all unary partial algebras will be denoted by $\mathcal{UPA}lg$.

Recall also that for a given unary partial algebra we can define at least four kinds of subalgebras which form four different lattices of subalgebras. More precisely, let $\mathbf{A} = \left\langle A, (f^{\mathbf{A}})_{f\in K}\right\rangle, \mathbf{B} = \left\langle B, (f^{\mathbf{B}})_{f\in K}\right\rangle \in \mathcal{UPA}lg(K)$. $\mathbf{B}$ is a weak (relative) subalgebra of $\mathbf{A}$, denoted by $\mathbf{B} \leqslant_w \mathbf{A}$ ($\mathbf{B} \leqslant_r \mathbf{A}$), iff $B \subseteq A$ and $f^{\mathbf{B}} \subseteq f^{\mathbf{A}}$ ($f^{\mathbf{B}} = f^{\mathbf{A}} \cap (B \times B)$) for all $f \in K$. $\mathbf{B}$ is a strong subalgebra (initial segment) of $\mathbf{A}$, denoted by $\mathbf{B} \leqslant_s \mathbf{A}$ ($\mathbf{B} \leqslant_d \mathbf{A}$), iff $B \subseteq A$ and $f^{\mathbf{B}} = f^{\mathbf{A}} \cap (B \times A)$ ($f^{\mathbf{B}} = f^{\mathbf{A}} \cap (A \times B)$) for all $f \in K$. $S_w(\mathbf{A})$, $S_r(\mathbf{A})$, $S_s(\mathbf{A})$ and $S_d(\mathbf{A})$ are the sets of all weak, relative, strong subalgebras and initial segments of the unary partial algebra $\mathbf{A}$ respectively (of course the empty algebra is simultaneously a weak, relative, strong subalgebra and an initial segment of $\mathbf{A}$).

Recall that $\mathbf{S}_w(\mathbf{A}) = \langle S_w(\mathbf{A}), \leqslant_w\rangle$, $\mathbf{S}_r(\mathbf{A}) = \langle S_r(\mathbf{A}), \leqslant_r\rangle$, $\mathbf{S}_s(\mathbf{A}) = \langle S_s(\mathbf{A}), \leqslant_s\rangle$ and $\mathbf{S}_d(\mathbf{A}) = \langle S_d(\mathbf{A}), \leqslant_d\rangle$ are complete lattices. More precisely, the operations of infimum $\bigwedge$ and supremum $\bigvee$ in the lattices $\mathbf{S}_w(\mathbf{A})$, $\mathbf{S}_s(\mathbf{A})$ and $\mathbf{S}_d(\mathbf{A})$ are defined in the following way: for each non-empty family $\{\mathbf{B}_i\}_{i\in I}$,

$$\bigwedge_{i\in I} \mathbf{B}_i := \left\langle \bigcap_{i\in I} B_i, (\bigcap_{i\in I} f^{\mathbf{B}_i})_{f\in K} \right\rangle, \quad \bigvee_{i\in I} \mathbf{B}_i := \left\langle \bigcup_{i\in I} B_i, (\bigcup_{i\in I} f^{\mathbf{B}_i})_{f\in K} \right\rangle.$$

Secondly, for every set $X \subseteq A$ there exists exactly one $\mathbf{C} \leqslant_r \mathbf{A}$ such that $C = X$. Thus the operations of infimum $\bigwedge$ and supremum $\bigvee$ in the lattice $\mathbf{S}_r(\mathbf{A})$ are defined in the following way: $\bigwedge_{i\in I} \mathbf{B}_i := \left\langle \bigcap_{i\in I} B_i, (\bigcap_{i\in I} f^{\mathbf{B}_i})_{f\in K} \right\rangle$ and $\bigvee_{i\in I} \mathbf{B}_i$ is the unique relative subalgebra which is induced by the set $\bigcup_{i\in I} B_i$.

In this chapter we first recall that with any unary partial algebra $\mathbf{A}$ we can associate the digraph $\mathbf{G}(\mathbf{A})$ and the graph $\mathbf{G}^*(\mathbf{A})$. Next, we show the correspondence between the class of all unary partial algebras of type K—$\mathcal{UPA}lg(K)$ and the class of all digraphs of type $|K|$—$\mathcal{AG}_d(|K|)$ (where $|K|$ is the cardinality of the set K).

In the second section we prove that for any $\mathbf{A} \in \mathcal{UPA}lg$, the subalgebra lattices of $\mathbf{A}$ are isomorphic to the subdigraph lattices of the digraph $\mathbf{G}(\mathbf{A})$. From this result we will obtain in a simple way that for any two unary partial algebras (they can even be of different types) if their digraphs are isomorphic, then their subalgebra lattices are also isomorphic.

In the third section we prove that for any two graphs **G** and **H** their weak subgraph lattices are isomorphic iff the graphs **G** and **H** are isomorphic. From this result we will also obtain that for any two digraphs **G** and **H** their weak subdigraph lattices are isomorphic iff the graphs $\mathbf{G}^*$ and $\mathbf{H}^*$ are isomorphic. Secondly, we obtain another proof of the following theorem (proved first in [Bar1]): for any two unary partial algebras **A** and **B** their weak subalgebra lattices are isomorphic iff their graphs $\mathbf{G}^*(\mathbf{A})$ and $\mathbf{G}^*(\mathbf{B})$ are isomorphic.

In the end of this chapter we show that the theorem which contains a full algebraic characterization of the weak subalgebra lattice of a unary partial algebra from [Bar1] can be generalized to digraphs and graphs. Next, we will prove a result which characterizes (in graph language) unary partial algebras **A** and lattices **L** such that $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{L}$. We also give analogous results for digraphs and graphs.

2.1.

In this section we first recall that with any unary partial algebra we can associate in a natural way a digraph and a graph (see [Bar1]). Next, we will show that the class of all unary partial algebras of a unary type K—$\mathcal{UPA}lg(K)$ is represented by the class of all digraphs of type $|K|$—$\mathcal{AG}_d(|K|)$.

Definition 2.1.1. Let $\mathbf{A} = \langle A, (f^{\mathbf{A}})_{f\in K}\rangle \in \mathcal{UPA}lg(K)$. Then:

(a) $\mathbf{G}(\mathbf{A}) = \langle V^{\mathbf{G}(\mathbf{A})}, E^{\mathbf{G}(\mathbf{A})}, I^{\mathbf{G}(\mathbf{A})}\rangle$ is the digraph such that $V^{\mathbf{G}(\mathbf{A})} := A$, $E^{\mathbf{G}(\mathbf{A})} := \{\langle a, f, b\rangle \in A \times K \times A \colon \langle a, b\rangle \in f^{\mathbf{A}}\}$ and $I^{\mathbf{G}(\mathbf{A})}(\langle a, f, b\rangle) := \langle a, b\rangle$ for all $\langle a, f, b\rangle \in E^{\mathbf{G}(\mathbf{A})}$.

(b) $\mathbf{G}^*(\mathbf{A}) := (\mathbf{G}(\mathbf{A}))^*$.

Now we formulate and prove a few properties of the digraph $\mathbf{G}(\mathbf{A})$.

Proposition 2.1.2. *Let* $\mathbf{A} = \langle A, (f^{\mathbf{A}})_{f\in K}\rangle \in \mathcal{UPA}lg(K)$. *Then*

$$\mathbf{G}(\mathbf{A}) \in \mathcal{AG}_d(|K|).$$

P r o o f. Let us take an arbitrary $v \in V^{\mathbf{G}(\mathbf{A})}$. Then D. 2.1.1 implies

$$E_s^{\mathbf{G}(\mathbf{A})}(v) = \{\langle v, f, b\rangle \in \{v\} \times K \times A \colon \langle v, b\rangle \in f^{\mathbf{A}}\},$$

so we can take a function $\Phi\colon E_s^{\mathbf{G}(\mathbf{A})}(v) \to K$ such that

$$\Phi(\langle v, f, b\rangle) = f \quad \text{for all} \quad \langle v, f, b\rangle \in E_s^{\mathbf{G}(\mathbf{A})}(v).$$

Since $f^{\mathbf{A}}$ is a partial function for all $f \in K$, we easily obtain that Φ is an injection. Hence we have

$$s^{\mathbf{G}(\mathbf{A})}(v) := |E_s^{\mathbf{G}(\mathbf{A})}(v)| \leqslant |K|.$$

Thus the proof is complete. □

The inverse result is also true. More precisely,

Theorem 2.1.3. *Let $\eta \in$ **Card**, $\mathbf{G} \in \mathcal{AG}_d(\eta)$ and let K be a unary type such that*

$$(*) \qquad |K| = \eta.$$

Then there exists a unary partial algebra $\mathbf{A}$ *such that*

$$\mathbf{G} \simeq \mathbf{G}(\mathbf{A}) \quad \textit{and} \quad \mathbf{A} \in \mathcal{UPA}lg(K).$$

P r o o f. By virtue of $(*)$ there exist injections $\Phi(v)\colon E_s^{\mathbf{G}}(v) \to K$ for all $v \in V^{\mathbf{G}}$. Let us take

$$\Phi := \bigcup_{v \in V^G} \Phi(v).$$

Since $E_s^{\mathbf{G}}(w_1) \cap E_s^{\mathbf{G}}(w_2) = \emptyset$ for all $w_1 \neq w_2$ (recall that $E_s^{\mathbf{G}}(w)$ is the set of all edges which start from w), we obtain easily that Φ is a well defined function from $E^{\mathbf{G}}$ into K. Secondly, by the definition of Φ we have

1) for all $e_1, e_2 \in E^{\mathbf{G}}$, if $\pi_1 I^{\mathbf{G}}(e_1) = \pi_1 I^{\mathbf{G}}(e_2)$ and $\Phi(e_1) = \Phi(e_2)$, then $e_1 = e_2$.

Now for all $f \in K$, let $\overline{f}$ be a binary relation such that for each $a, b \in V^{\mathbf{G}}$,

$$\langle a, b\rangle \in \overline{f} \quad \text{iff} \quad \exists_{e \in E^G} \ \ \Phi(e) = f \ \text{ and } \ I^{\mathbf{G}}(e) = \langle a, b\rangle\,.$$

Applying 1) we get that for all $f \in K$, $\overline{f}$ is a partial function of $V^{\mathbf{G}}$ into $V^{\mathbf{G}}$. Thus we obtain that

$$\mathbf{A} = \left\langle A, (f^{\mathbf{A}})_{f \in K}\right\rangle \in \mathcal{UPA}lg(K),$$

where

$$A := V^{\mathbf{G}} \quad \text{and} \quad f^{\mathbf{A}} := \overline{f} \quad \text{for all } f \in K.$$

Now we want to prove that

$$\mathbf{G} \simeq \mathbf{G}(\mathbf{A}).$$

Let $\varphi_V := \mathrm{id}_{V^G}$ and let $\varphi_E\colon E^{\mathbf{G}} \to V^{\mathbf{G}} \times K \times V^{\mathbf{G}}$ be a mapping such that

$$\varphi_E(e) := \left\langle \pi_1 I^{\mathbf{G}}(e), \Phi(e), \pi_2 I^{\mathbf{G}}(e)\right\rangle \quad \text{for all } e \in E^{\mathbf{G}}.$$

From the above definitions and D. 2.1.1 we have

$$V^{\mathbf{G}} = A = V^{\mathbf{G}(\mathbf{A})}, \qquad \varphi_E(E^{\mathbf{G}}) = E^{\mathbf{G}(\mathbf{A})}$$

and for all $e \in E^{\mathbf{G}}$,

$$I^{\mathbf{G}(\mathbf{A})}(\varphi_E(e)) = I^{\mathbf{G}}(e).$$

The property 1) implies also that φ_E is an injection. From this facts we obtain that $\varphi = \langle \varphi_V, \varphi_E \rangle$ is an isomorphism of $\mathbf{G}$ onto $\mathbf{G}(\mathbf{A})$. This completes the proof of our proposition. □

The following result is an immediate consequence of Th. 2.1.3:

Corollary 2.1.4. *Let $\mathbf{G} \in \mathcal{AG}_d$. Then there exists $\mathbf{A} \in \mathcal{UPA}lg$ such that*

$$\mathbf{G}(\mathbf{A}) \simeq \mathbf{G}.$$

P r o o f. Let $\eta := |E^{\mathbf{G}}|$. Then $\mathbf{G} \in \mathcal{AG}_d(\eta)$, since $s^{\mathbf{G}}(v) \leqslant |E^{\mathbf{G}}|$ for all $v \in V^{\mathbf{G}}$. Now we apply P. 2.1.3. □

Proposition 2.1.5. *Let $\mathbf{G} \in \mathcal{AG}_n$. Then:*

(a) *There exists $\mathbf{H} \in \mathcal{AG}_d$ such that $\mathbf{H}^* \simeq \mathbf{G}$.*
(b) *There exists $\mathbf{A} \in \mathcal{UPA}lg$ such that $\mathbf{G}^*(\mathbf{A}) \simeq \mathbf{G}$.*

P r o o f. The proof of (a) is left to the reader (it is enough to apply the axiom of choice and for any $e \in E^{\mathbf{G}}$ to choose from the set $I^{\mathbf{G}}(e)$ exactly one vertex which will be the initial vertex of e). (b) is obtained from (a) and Cor. 2.1.4. □

2.2.

Now we want to prove that for any unary partial algebra $\mathbf{A}$ its subalgebra lattices are isomorphic to the subdigraph lattices of the digraph $\mathbf{G}(\mathbf{A})$. This theorem implies of course that for any two unary partial algebras (they can even be of different unary types) their lattices of subalgebras (weak, relative, strong and initial segment) are isomorphic provided their digraphs are isomorphic.

To this purpose we have to prove some facts.

Proposition 2.2.1. *Let $\mathbf{A}, \mathbf{B} \in \mathcal{UPA}lg$. Then:*

(a) $\mathbf{B} \leqslant_w \mathbf{A}$ *iff* $\mathbf{G}(\mathbf{B}) \leqslant_w \mathbf{G}(\mathbf{A})$.
(b) $\mathbf{B} \leqslant_r \mathbf{A}$ *iff* $\mathbf{G}(\mathbf{B}) \leqslant_r \mathbf{G}(\mathbf{A})$.
(c) $\mathbf{B} \leqslant_s \mathbf{A}$ *iff* $\mathbf{G}(\mathbf{B}) \leqslant_s \mathbf{G}(\mathbf{A})$.
(d) $\mathbf{B} \leqslant_d \mathbf{A}$ *iff* $\mathbf{G}(\mathbf{B}) \leqslant_d \mathbf{G}(\mathbf{A})$.

P r o o f. (a): "⇒" Let us assume that $\mathbf{B} \leqslant_w \mathbf{A}$. Then from D. 2.1.1 we obtain

$$V^{\mathbf{G}(\mathbf{B})} \subseteq V^{\mathbf{G}(\mathbf{A})}.$$

Secondly, for all $f \in K$ and $a, b \in \mathbf{B}$ we have

$$\langle a, f, b\rangle \in E^{\mathbf{G}(\mathbf{B})} \Rightarrow \langle a, b\rangle \in f^{\mathbf{B}} \Rightarrow \langle a, b\rangle \in f^{\mathbf{A}} \Rightarrow \langle a, f, b\rangle \in E^{\mathbf{G}(\mathbf{A})}.$$

Now applying the definition of $I^{\mathbf{G}(\mathbf{B})}$ and $I^{\mathbf{G}(\mathbf{A})}$ we get $\mathbf{G}(\mathbf{B}) \leqslant_w \mathbf{G}(\mathbf{A})$.

"⇐" Let us assume that $\mathbf{G}(\mathbf{B}) \leqslant_w \mathbf{G}(\mathbf{A})$. Then from D. 1.1.4 and D. 2.1.1 we obtain

$$B \subseteq A.$$

Secondly, for all $f \in K$ and $a, b \in B$ we have

$$\langle a, b\rangle \in f^{\mathbf{B}} \Rightarrow \langle a, f, b\rangle \in E^{\mathbf{G}(\mathbf{B})} \Rightarrow \langle a, f, b\rangle \in E^{\mathbf{G}(\mathbf{A})} \Rightarrow \langle a, b\rangle \in f^{\mathbf{A}}.$$

So we have shown that $\mathbf{B} \leqslant_w \mathbf{A}$.

(b): "⇒" Let $\mathbf{B} \leqslant_r \mathbf{A}$ and $\langle a, f, b\rangle \in E^{\mathbf{G}(\mathbf{A})}$. Then D. 2.1.1 implies

$$I^{\mathbf{G}(\mathbf{A})}(\langle a, f, b\rangle) \in V^{\mathbf{G}(\mathbf{B})} \times V^{\mathbf{G}(\mathbf{B})} \Rightarrow \langle a, b\rangle \in f^{\mathbf{A}}$$
$$\wedge\ a, b \in B \Rightarrow \langle a, b\rangle \in f^{\mathbf{B}} \Rightarrow \langle a, f, b\rangle \in E^{\mathbf{G}(\mathbf{B})}.$$

Now from (a) and D. 1.1.4 we obtain $\mathbf{G}(\mathbf{B}) \leqslant_r \mathbf{G}(\mathbf{A})$.

"⇐" Let $\mathbf{G}(\mathbf{B}) \leqslant_r \mathbf{G}(\mathbf{A})$ and $a, b \in A$. Then D. 1.1.4 and D. 2.1.1 imply

$$\langle a, b\rangle \in f^{\mathbf{A}} \ \text{ and } \ a, b \in B \Rightarrow$$
$$\langle a, f, b\rangle \in E^{\mathbf{G}(\mathbf{A})} \ \text{ and } \ I^{\mathbf{G}(\mathbf{A})}(\langle a, f, b\rangle) \in V^{\mathbf{G}(\mathbf{B})} \times V^{\mathbf{G}(\mathbf{B})} \Rightarrow$$
$$\langle a, f, b\rangle \in E^{\mathbf{G}(\mathbf{B})} \Rightarrow \langle a, b\rangle \in f^{\mathbf{B}}.$$

Now applying (a) we get $\mathbf{B} \leqslant_r \mathbf{A}$.

The analogous proofs of (c) and (d) are left to the reader. □

Proposition 2.2.2. *Let* $\mathbf{A}, \mathbf{B} \in \mathcal{UPA}lg$. *Then* $\mathbf{A} = \mathbf{B}$ *iff* $\mathbf{G}(\mathbf{A}) = \mathbf{G}(\mathbf{B})$.

The proof follows straightforward from D. 2.1.1 and P. 2.2.1.

Proposition 2.2.3. *Let* $\mathbf{A} = \langle A, (f^{\mathbf{A}})_{f \in K}\rangle \in \mathcal{UPA}lg(K)$ *and* $\mathbf{H} \leqslant_w \mathbf{G}(\mathbf{A})$. *Then there exists* $\mathbf{B} \leqslant_w \mathbf{A}$ *such that* $\mathbf{H} = \mathbf{G}(\mathbf{B})$.

P r o o f. Since $\mathbf{H} \leqslant_w \mathbf{G}(\mathbf{A})$, we have $V^{\mathbf{H}} \subseteq V^{\mathbf{G}(\mathbf{A})} = A$ and $E^{\mathbf{H}} \subseteq E^{\mathbf{G}(\mathbf{A})}$. Now let $\mathbf{B} = \left\langle B, (f^{\mathbf{B}})_{f \in K} \right\rangle$ be the pair such that

$$B = V^{\mathbf{H}}$$

and for all $f \in K$, $a, b \in B$,

$$\langle a, b \rangle \in f^{\mathbf{B}} \quad \text{iff} \quad \langle a, f, b \rangle \in E^{\mathbf{H}}.$$

Then we obtain (see also D. 2.1.1)

$$B \subseteq V^{\mathbf{G}}(\mathbf{A}) = A \quad \text{and} \quad f^{\mathbf{B}} \subseteq f^{\mathbf{A}} \quad \text{for all} \quad f \in K.$$

This facts clearly imply that $\mathbf{B}$ is a unary partial algebra and $\mathbf{B} \leqslant_w \mathbf{A}$. Applying once more the definitions of $\mathbf{B}$ and $\mathbf{G}(\mathbf{B})$ we can easily show that

$$\mathbf{G}(\mathbf{B}) = \mathbf{H}.$$

This completes the proof. □

Now we can prove the main result of this section.

Theorem 2.2.4. *Let* $\mathbf{A} \in \mathcal{UPA}$. *Then*

$$\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{S}_w(\mathbf{G}(\mathbf{A})), \qquad \mathbf{S}_r(\mathbf{A}) \simeq \mathbf{S}_r(\mathbf{G}(\mathbf{A})),$$
$$\mathbf{S}_s(\mathbf{A}) \simeq \mathbf{S}_s(\mathbf{G}(\mathbf{A})), \qquad \mathbf{S}_d(\mathbf{A}) \simeq \mathbf{S}_d(\mathbf{G}(\mathbf{A})).$$

P r o o f. (a): Let us take the function $\varphi\colon\ S_w(\mathbf{A}) \to S_w(\mathbf{G}(\mathbf{A}))$ such that

$$\varphi(\mathbf{B}) = \mathbf{G}(\mathbf{B}) \quad \text{for all} \quad \mathbf{B} \in \mathbf{S}_w(\mathbf{A}).$$

We want to prove that φ is the required isomorphism. P. 2.2.1 implies that φ is well defined, and from P. 2.2.2 and P. 2.2.3 we have that φ is a bijection. Since $\mathbf{S}_w(\mathbf{A})$ and $\mathbf{S}_w(\mathbf{G}(\mathbf{A}))$ are total algebras, we must only show that for all $\mathbf{B}_1, \mathbf{B}_2 \in S_w(\mathbf{A})$,

$$\mathbf{G}(\mathbf{B}_1 \wedge \mathbf{B}_2) = \mathbf{G}(\mathbf{B}_1) \wedge \mathbf{G}(\mathbf{B}_2) \;\; \text{and} \;\; \mathbf{G}(\mathbf{B}_1 \vee \mathbf{B}_2) = \mathbf{G}(\mathbf{B}_1) \vee \mathbf{G}(\mathbf{B}_2).$$

From D. 2.1.1 and the definition of the operation $\wedge$ in the lattices $\mathbf{S}_w(\mathbf{A})$ and $\mathbf{S}_w(\mathbf{G}(\mathbf{A}))$ we obtain

$$V^{\mathbf{G}(\mathbf{B}_1 \wedge \mathbf{B}_2)} = B_1 \cap B_2 = V^{\mathbf{G}(\mathbf{B}_1)} \cap V^{\mathbf{G}(\mathbf{B}_2)} = V^{\mathbf{G}(\mathbf{B}_1) \wedge \mathbf{G}(\mathbf{B}_2)}.$$

Secondly, for all $f \in K$ and $a, b \in A$ we have

$$\langle a, f, b\rangle \in E^{\mathbf{G}(\mathbf{B}_1 \wedge \mathbf{B}_2)} \Leftrightarrow \langle a, b\rangle \in f^{\mathbf{B}_1 \wedge \mathbf{B}_2} \Leftrightarrow \langle a, b\rangle \in f^{\mathbf{B}_1}$$
$$\wedge\ \langle a, b\rangle \in f^{\mathbf{B}_2} \Leftrightarrow \langle a, f, b\rangle \in E^{\mathbf{G}(\mathbf{B}_1)} \cap E^{\mathbf{G}(\mathbf{B}_2)} \Leftrightarrow \langle a, f, b\rangle \in E^{\mathbf{G}(\mathbf{B}_1) \wedge \mathbf{G}(\mathbf{B}_2)}.$$

So we have shown

$$V^{\mathbf{G}(\mathbf{B}_1 \wedge \mathbf{B}_2)} = V^{\mathbf{G}(\mathbf{B}_1) \wedge \mathbf{G}(\mathbf{B}_2)} \text{ and } E^{\mathbf{G}(\mathbf{B}_1 \wedge \mathbf{B}_2)} = E^{\mathbf{G}(\mathbf{B}_1) \wedge \mathbf{G}(\mathbf{B}_2)}.$$

From these facts and P.1.1.6 we obtain the first equality. The analogous proof of the second equality is left to the reader.

(b): We want to prove that $\varphi|_{S_r(A)}$ is the required isomorphism. Applying P. 2.2.1 and (a) we get that $\varphi|_{S_r(A)}$ is a bijection of $S_r(\mathbf{A})$ onto $S_r(\mathbf{G}(\mathbf{A}))$. Since $\mathbf{S}_r(\mathbf{A})$ and $\mathbf{S}_r(\mathbf{G}(\mathbf{A}))$ are total algebras, we must only show that for all $\mathbf{B}_1, \mathbf{B}_2 \in S_w(\mathbf{A})$,

$$\mathbf{G}(\mathbf{B}_1 \wedge \mathbf{B}_2) = \mathbf{G}(\mathbf{B}_1) \wedge \mathbf{G}(\mathbf{B}_2) \text{ and } \mathbf{G}(\mathbf{B}_1 \vee \mathbf{B}_2) = \mathbf{G}(\mathbf{B}_1) \vee \mathbf{G}(\mathbf{B}_2).$$

From D. 2.1.1 and the definition of the operation $\wedge$ in the lattices $\mathbf{S}_r(\mathbf{A})$ and $\mathbf{S}_r(\mathbf{G}(\mathbf{A}))$ we obtain

$$V^{\mathbf{G}(\mathbf{B}_1 \wedge \mathbf{B}_2)} = B_1 \cap B_2 = V^{\mathbf{G}(\mathbf{B}_1)} \cap V^{\mathbf{G}(\mathbf{B}_2)} = V^{\mathbf{G}(\mathbf{B}_1) \wedge \mathbf{G}(\mathbf{B}_2)}$$

and

$$V^{\mathbf{G}(\mathbf{B}_1 \vee \mathbf{B}_2)} = B_1 \cup B_2 = V^{\mathbf{G}(\mathbf{B}_1)} \cup V^{\mathbf{G}(\mathbf{B}_2)} = V^{\mathbf{G}(\mathbf{B}_1) \vee \mathbf{G}(\mathbf{B}_2)}.$$

From the above facts and P. 1.1.6 we obtain the desired equalities.

(c) and (d): P. 2.2.1 and (a) easily imply that $\varphi|_{S_s(A)}$ ($\varphi|_{S_d(A)}$) is a bijection of $S_s(\mathbf{A})$ ($S_d(\mathbf{A})$) onto $S_s(\mathbf{G}(\mathbf{A}))$ ($S_d(\mathbf{G}(\mathbf{A}))$). Since $\mathbf{A}$ is a unary partial algebra, $\mathbf{S}_s(\mathbf{A})$ ($\mathbf{S}_d(\mathbf{A})$) is a sublattice of the lattice $\mathbf{S}_w(\mathbf{A})$. Thus from P. 1.1.7 and (a) we obtain that $\varphi|_{S_s(A)}$ ($\varphi|_{S_d(A)}$) is an isomorphism of the lattices $\mathbf{S}_s(\mathbf{A})$ and $\mathbf{S}_s(\mathbf{G}(\mathbf{A}))$ ($\mathbf{S}_d(\mathbf{A})$ and $\mathbf{S}_d(\mathbf{G}(\mathbf{A}))$). This completes the proof of (c) and (d). □

Recall that for any $\mathbf{A} \in \mathcal{UPA}lg$ and $\emptyset \neq B \subseteq A$ we have the least strong subalgebra (initial segment) of $\mathbf{A}$ containing B. This strong subalgebra (initial segment) will be denoted by $\langle B\rangle^s_{\mathbf{A}}$ ($\langle B\rangle^d_{\mathbf{A}}$). More formally, we define

$$\langle B\rangle^s_{\mathbf{A}} := \bigwedge\{\mathbf{C} \leqslant_s \mathbf{A}\colon\ B \subseteq C\} \text{ and } \langle B\rangle^d_{\mathbf{A}} := \bigwedge\{\mathbf{C} \leqslant_d \mathbf{A}\colon\ B \subseteq C\}.$$

Now observe that the above theorem implies the following conclusion:

Proposition 2.2.5. *Let $\mathbf{A} \in \mathcal{UPA}lg$ and $\emptyset \neq B \subseteq A$. Then*

$$\mathbf{G}(\langle B\rangle^s_{\mathbf{A}}) = \langle B\rangle^s_{\mathbf{G}(\mathbf{A})} \quad \text{and} \quad \mathbf{G}(\langle B\rangle^d_{\mathbf{A}}) = \langle B\rangle^d_{\mathbf{G}(\mathbf{A})}.$$

P r o o f. Let φ be the lattice isomorphism from the proof of Th. 2.2.4(a). Then we easily obtain the following equalities:

$$\begin{aligned}\mathbf{G}(\langle B\rangle^s_{\mathbf{A}}) &= \varphi(\langle B\rangle^s_{\mathbf{A}}) = \varphi\Big(\bigwedge\{\mathbf{C}\leqslant_s \mathbf{A}\colon\ B\subseteq C\}\Big)\\ &= \bigwedge\{\varphi(\mathbf{C})\leqslant_s \varphi(\mathbf{A})\colon\ B\subseteq V^{\varphi(\mathbf{C})}\} = \bigwedge\{\mathbf{G}(\mathbf{C})\leqslant_s \mathbf{G}(\mathbf{A})\colon\ B\subseteq V^{\mathbf{G}(\mathbf{C})}\}\\ &= \bigwedge\{\mathbf{H}\leqslant_s \mathbf{G}(\mathbf{A})\colon\ B\subseteq V^{\mathbf{H}}\} =: \langle B\rangle^s_{\mathbf{G}(\mathbf{A})}\,.\end{aligned}$$

The analogous proof of the second part is omitted. □

Of course, isomorphic unary partial algebras have isomorphic lattices of subalgebras. Now Th. 2.2.5 implies the following stronger result:

Theorem 2.2.6. *Let $\mathbf{A},\mathbf{B}\in\mathcal{UPA}lg$ satisfy the condition*

$$\mathbf{G}(\mathbf{A})\simeq\mathbf{G}(\mathbf{B}).$$

Then

$$\mathbf{S}_w(\mathbf{A})\simeq\mathbf{S}_w(\mathbf{B}),\ \mathbf{S}_r(\mathbf{A})\simeq\mathbf{S}_r(\mathbf{B}),\ \mathbf{S}_s(\mathbf{A})\simeq\mathbf{S}_s(\mathbf{B}),\ \mathbf{S}_d(\mathbf{A})\simeq\mathbf{S}_d(\mathbf{B}).$$

Theorem 2.2.4 and the results from chapter one imply also that the lattices of weak and relative subalgebras of **A** are isomorphic to the lattices of weak and relative subgraphs of the graph $\mathbf{G}^*(\mathbf{A})$. More precisely, the following results are satisfied:

Theorem 2.2.7. *Let $\mathbf{A}\in\mathcal{UPA}lg$. Then*

$$\mathbf{S}_w(\mathbf{A})\simeq\mathbf{S}_w(\mathbf{G}^*(\mathbf{A}))\quad\textit{and}\quad\mathbf{S}_r(\mathbf{A})\simeq\mathbf{S}_r(\mathbf{G}^*(\mathbf{A})).$$

Proof is obtained from Th. 1.2.4 and Th. 2.2.4. Observe that these isomorphisms are provided by the function $\varphi\colon\ S_w(\mathbf{A})\longrightarrow S_w(\mathbf{G}^*(\mathbf{A}))$ such that

$$\varphi(\mathbf{B})=\mathbf{G}^*(\mathbf{B})\quad\text{for all}\quad\mathbf{B}\in\mathbf{S}_w(\mathbf{A})$$

and the function $\varphi|_{S_r(A)}$. This follows from the proofs of these two theorems.

The above theorem implies the following result, which is a stronger version of Th. 2.2.6 for the lattices of weak and relative subalgebras.

Theorem 2.2.8. *Let $\mathbf{A},\mathbf{B}\in\mathcal{UPA}lg$ satisfy the condition*

$$\mathbf{G}^*(\mathbf{A})\simeq\mathbf{G}^*(\mathbf{B}).$$

Then

$$\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{S}_w(\mathbf{B}) \quad and \quad \mathbf{S}_r(\mathbf{A}) \simeq \mathbf{S}_r(\mathbf{B}).$$

2.3.

In the first part of this section we give another proof of the following theorem (which has been proved first in [Bar1] by W. Bartol): for two unary partial algebras $\mathbf{A}$ and $\mathbf{B}$ their weak subalgebra lattices $\mathbf{S}_w(\mathbf{A})$ and $\mathbf{S}_w(\mathbf{B})$ are isomorphic iff their graphs $\mathbf{G}^*(\mathbf{A})$ and $\mathbf{G}^*(\mathbf{B})$ are isomorphic.

More precisely, applying Th. 2.2.7 we get that we must only prove the following result: for any two graphs $\mathbf{G}$ and $\mathbf{H}$ their weak subgraph lattices are isomorphic iff $\mathbf{G}$ and $\mathbf{H}$ are isomorphic. To this purpose we have to give a few facts about digraphs and graphs.

We recall yet a few concepts of the lattice theory (see e.g. [Grä2], [Jón]). We assume that the reader knows the definitions of complete, algebraic, distributive lattices, etc. For any lattice $\mathbf{L} = \langle L, \wedge, \vee \rangle$, $\leqslant_L$ is the lattice partial ordering of $\mathbf{L}$, i.e. $l \leqslant_L k \Leftrightarrow l = k \wedge l \Leftrightarrow k = l \vee k$.

Now let $\mathbf{L} = \langle L, \leqslant_L \rangle$ be a complete lattice and let 0 be the least element (i.e. $0 := \bigwedge L$). An element $a \in L$ is an atom iff for all $b \in L$ if $0 \leqslant_L b \leqslant_L a$, then $b = 0$ or $b = a$. An element $i \in L$ is join irreducible iff for all $l, k \in L$ if $i = l \vee k$, then $l = i$ or $k = i$. Let us introduce the following notation:

$\mathbf{L}^a$ is the set of all atoms of the lattice $\mathbf{L}$.

$\mathbf{L}^i$ is the set of all $i \in L$ such that $i \neq 0$, $i \notin \mathbf{L}^a$, i is join irreducible.

Now we describe the sets $\mathbf{S}_w(\mathbf{G})^a$ and $\mathbf{S}_w(\mathbf{G})^i$ for any graph $\mathbf{G}$. To this purpose we start with the following definition:

Definition 2.3.1. Let $\mathbf{G} \in \mathcal{AG}_n$, $v \in V^{\mathbf{G}}$ and $e \in E^{\mathbf{G}}$. Then:

(a) $\mathbf{G}(v)$ is the weak subgraph of $\mathbf{G}$ which has one vertex v only and no edges.

(b) $\mathbf{G}(e)$ is the weak subgraph of $\mathbf{G}$ which has one edge e only and its endpoints as the only vertices.

(c) $N_V^{\mathbf{G}} := \{\mathbf{G}(v) \in S_w(\mathbf{G})\colon\ v \in V^{\mathbf{G}}\}$, $N_E^{\mathbf{G}} := \{\mathbf{G}(e) \in S_w(\mathbf{G})\colon\ e \in E^{\mathbf{G}}\}$.

Lemma 2.3.2. *Let* $\mathbf{G} \in \mathcal{AG}_n$. *Then* $N_V^{\mathbf{G}} = \mathbf{S}_w(\mathbf{G})^a$ *and* $N_E^{\mathbf{G}} = \mathbf{S}_w(\mathbf{G})^i$.

The proof is obtained by a simple verification and is therefore omitted. Recall that the empty graph $\emptyset = \langle \emptyset, \emptyset, \emptyset \rangle$ belongs to the set $S_w(\mathbf{G})$ and it is the smallest element with respect to the relation $\leqslant_w$.

Now we can prove a result which describes the weak subgraph lattice of any graph.

Theorem 2.3.3. *Let* $\mathbf{G}, \mathbf{H} \in \mathcal{AG}_n$. *Then*

$$\mathbf{G} \simeq \mathbf{H} \quad \textit{iff} \quad \mathbf{S}_w(\mathbf{G}) \simeq \mathbf{S}_w(\mathbf{H}).$$

P r o o f. The implication "⇒" is obvious.

"⇐": Let $\Phi\colon \mathbf{S}_w(\mathbf{G}) \to \mathbf{S}_w(\mathbf{H})$ be an isomorphism of lattices and let

$$\Phi_V := \Phi|_{N_V^G}, \qquad \Phi_E := \Phi|_{N_E^G}.$$

The definition of lattice isomorphism and L. 2.3.2 imply that Φ_V (Φ_E) is a bijection from $N_V^{\mathbf{G}}$ $(N_E^{\mathbf{G}})$ onto $N_V^{\mathbf{H}}$ $(N_E^{\mathbf{H}})$.

Let $\psi_{V,G}\colon V^{\mathbf{G}} \to N_V^{\mathbf{G}}$ and $\psi_{V,H}\colon V^{\mathbf{H}} \to N_V^{\mathbf{H}}$ be functions such that

$$\psi_{V,G}(w) := \mathbf{G}(w) \ \text{ for all } \ w \in V^{\mathbf{G}} \quad \text{and} \quad \psi_{V,H}(u) := \mathbf{H}(u) \ \text{ for all } \ u \in V^{\mathbf{H}}.$$

Let $\psi_{E,G}\colon E^{\mathbf{G}} \to N_E^{\mathbf{G}}$ and $\psi_{E,H}\colon E^{\mathbf{H}} \to N_E^{\mathbf{H}}$ be functions such that

$$\psi_{E,G}(e) := \mathbf{G}(e) \ \text{ for all } \ e \in E^{\mathbf{G}} \quad \text{and} \quad \psi_{E,H}(h) := \mathbf{H}(h) \ \text{ for all } \ h \in E^{\mathbf{H}}.$$

By D. 2.3.1 we easily obtain that $\psi_{V,G}$, $\psi_{V,H}$, $\psi_{E,G}$, $\psi_{E,H}$ are bijections. Now let us take

$$\varphi_V := \psi_{V,H}^{-1} \circ \Phi_V \circ \psi_{V,G} \quad \text{and} \quad \varphi_E := \psi_{E,H}^{-1} \circ \Phi_E \circ \psi_{E,G}.$$

We want to prove that the pair $\varphi := \langle \varphi_V, \varphi_E \rangle$ is an isomorphism of $\mathbf{G}$ onto $\mathbf{H}$. Obviously we have that φ_V (φ_E) is a bijection from $V^{\mathbf{G}}$ $(E^{\mathbf{G}})$ onto $V^{\mathbf{H}}$ $(E^{\mathbf{H}})$. Thus we must only show

$$I^{\mathbf{H}}(\varphi_E(e)) = \varphi_V(I^{\mathbf{G}}(e)) \qquad \text{for all} \quad e \in E^{\mathbf{G}}.$$

Let us take $e \in E^{\mathbf{G}}$ and let $w_1, w_2 \in V^{\mathbf{G}}$, $u_1, u_2 \in V^{\mathbf{H}}$ be vertices such that $\{w_1, w_2\} = I^{\mathbf{G}}(e)$ and $\{u_1, u_2\} = I^{\mathbf{H}}(\varphi_E(e))$. Then (see D. 2.3.1)

$$\mathbf{G}(w_i) \leqslant_w \mathbf{G}(e) \ \text{ and } \ \mathbf{H}(u_i) \leqslant_w \mathbf{H}(\varphi_E(e)) \ \text{ for } i = 1, 2.$$

Since Φ is a lattice isomorphism, L. 2.3.2 and the above facts imply

$$\Phi_V(\mathbf{G}(w_i)) \leqslant_w \Phi_E(\mathbf{G}(e)) \ \text{ and } \ \Phi_V^{-1}(\mathbf{H}(u_i)) \leqslant_w \Phi_E^{-1}(\mathbf{H}(\varphi_E(e))) \ \text{ for } i = 1, 2,$$

so by the definitions of φ_V, φ_E we have

$$\mathbf{H}(\varphi_V(w_i)) \leqslant_w \mathbf{H}(\varphi_E(e)) \ \text{ and } \ \mathbf{G}(\varphi_V^{-1}(u_i)) \leqslant_w \mathbf{G}(e) \ \text{ for } i = 1, 2.$$

Thus we obtain (see D. 2.3.1)

$$\varphi_V(w_i) \in I^{\mathbf{H}}(\varphi_E(e)) \ \text{ and } \ \varphi_V^{-1}(u_i) \in I^{\mathbf{G}}(e), \ \text{ for } i = 1, 2.$$

Hence we obviously get

$$I^{\mathbf{H}}(\varphi_E(e)) = \varphi_V(I^{\mathbf{G}}(e)).$$

This completes the proof of our theorem. □

Theorem 2.3.4. *Let* $\mathbf{G}, \mathbf{H} \in \mathcal{AG}_d$. *Then*

$$\mathbf{S}_w(\mathbf{G}) \simeq \mathbf{S}_w(\mathbf{H}) \quad \textit{iff} \quad \mathbf{G}^* \simeq \mathbf{H}^*.$$

The proof is obtained from Th. 1.2.4 and Th. 2.3.3.

Theorem 2.3.5. (W. Bartol, 1989) *Let* $\mathbf{A}, \mathbf{B} \in \mathcal{UPA}lg$. *Then*

$$\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{S}_w(\mathbf{B}) \quad \textit{iff} \quad \mathbf{G}^*(\mathbf{A}) \simeq \mathbf{G}^*(\mathbf{B}).$$

The proof follows easily from Th. 2.2.7 and Th. 2.3.3 (the above theorem has been first formulated and proved by W. Bartol in [Bar1]. His proof does not use graph theory, but is based on the algebraic description of the weak subalgebra lattice which has been given in his paper). Observe also that unary partial algebras in this theorem can even be of different unary types.

In the second part of this section we will show first that the full algebraic characterization theorem of the weak subalgebra lattice of a unary partial algebra from [Bar1] may be generalized to the case of digraphs and graphs.

Next, we will show that for any lattice $\mathbf{L}$ which satisfies the conditions (d.1)–(d.4) from Th. 2.3.6 the graph from (c) may be constructed straightforward from $\mathbf{L}$ (D. 2.3.7 and Th. 2.3.8).

At the end of this chapter, we apply this result obtaining a characterization (in graph language) of the pairs $\langle \mathbf{L}, \mathbf{A} \rangle$, where $\mathbf{A}$ is a unary partial algebra and $\mathbf{L}$ is a lattice such that $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{L}$. We will also give analogous results for digraphs and graphs.

Theorem 2.3.6. *Let* $\mathbf{L} = \langle L, \leqslant_L \rangle$ *be an arbitrary lattice. Then the following conditions are equivalent:*

(a) *There exists* $\mathbf{A} \in \mathcal{UPA}$ *such that* $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{L}$.

(b) *There exists* $\mathbf{G} \in \mathcal{AG}_d$ *such that* $\mathbf{S}_w(\mathbf{G}) \simeq \mathbf{L}$.

(c) *There exists exactly one (up to isomorphism)* $\mathbf{G} \in \mathcal{AG}_n$ *such that* $\mathbf{S}_w(\mathbf{G}) \simeq \mathbf{L}$.

(d) *The lattice* $\mathbf{L} = \langle L, \leqslant_{\mathbf{L}} \rangle$ *satisfies the following conditions:*

- (d.1) $\mathbf{L}$ *is algebraic and distributive,*
- (d.2) *for all* $l \in L$, $l = \bigvee\{k \in \mathbf{L}^a \cup \mathbf{L}^i \cup \{0\} : k \leqslant_L l\}$,
- (d.3) *for each* $l \in \mathbf{L}^i$, $1 \leqslant |\{k \in \mathbf{L}^a : k \leqslant_L l\}| \leqslant 2$,
- (d.4) $\mathbf{L}^i$ *is an antichain with respect to the lattice ordering* $\leqslant_L$.

The proof of the equivalence $(a) \Leftrightarrow (d)$ is given in the paper [Bar1]. The equivalence $(a) \Leftrightarrow (b)$ is obtained from Cor. 2.1.4 and Th. 2.2.5. The equivalence $(b) \Leftrightarrow (c)$ follows from Th. 1.2.4, P. 2.1.5 and Th. 2.3.3.

Now we show that the graph from (c) can be constructed straightforward from $\mathbf{L}$. More precisely,

Definition 2.3.7. Let a lattice $\mathbf{L} = \langle L, \leqslant_L \rangle$ satisfy the conditions (d.1)–(d.4) from Th. 2.3.6. Then $\mathbf{G}(\mathbf{L})$ is the graph such that

$$V^{\mathbf{G}(\mathbf{L})} := \mathbf{L}^a, \qquad E^{\mathbf{G}(\mathbf{L})} := \mathbf{L}^i$$

and for all $e \in E^{\mathbf{G}(\mathbf{L})}$,

$$I^{\mathbf{G}(\mathbf{L})}(e) := \{v \in V^{\mathbf{G}(\mathbf{L})} \colon \ v \leqslant_L e\}.$$

A simple verification of the above definition (that $\mathbf{G}(\mathbf{L})$ is indeed a graph) is left to the reader. Secondly, it is easily shown that for any two lattices $\mathbf{L}$ and $\mathbf{K}$ if $\mathbf{L} \simeq \mathbf{K}$ then $\mathbf{G}(\mathbf{L}) \simeq \mathbf{G}(\mathbf{K})$. The following result is also true:

Theorem 2.3.8. *Let a lattice* $\mathbf{L}$ *satisfy* (d.1)–(d.4) *from Th.* 2.3.6. *Then*

$$\mathbf{S}_w(\mathbf{G}(\mathbf{L})) \simeq \mathbf{L}.$$

P r o o f. Th. 2.3.6 implies that there exists $\mathbf{G} \in \mathcal{AG}_n$ such that $\mathbf{S}_w(\mathbf{G}) \simeq \mathbf{L}$. So we must only show that

$$\mathbf{G}(\mathbf{L}) \simeq \mathbf{G}.$$

Let $\Phi \colon \ \mathbf{S}_w(\mathbf{G}) \to \mathbf{L}$ be a lattice isomorphism and let (see D. 2.3.1)

$$\Phi_V := \Phi|_{N_V^G}, \qquad \Phi_E := \Phi|_{N_E^G}.$$

The definition of a lattice isomorphism and L. 2.3.2 imply that Φ_V (Φ_E) is a bijection from $N_V^{\mathbf{G}}$ ($N_E^{\mathbf{G}}$) onto $\mathbf{L}^a$ ($\mathbf{L}^i$).

Let $\psi_V \colon \ V^{\mathbf{G}} \to N_V^G$ and $\psi_E \colon \ E^{\mathbf{G}} \to N_E^G$ be functions such that

$$\psi_V(v) := \mathbf{G}(v) \quad \text{and} \quad \psi_E(e) := \mathbf{G}(e) \quad \text{for all } v \in V^{\mathbf{G}},\ e \in E^{\mathbf{G}}.$$

Then ψ_V (ψ_E) is a bijection from $V^{\mathbf{G}}$ ($E^{\mathbf{G}}$) onto $N_V^{\mathbf{G}}$ ($N_E^{\mathbf{G}}$) (see D. 2.3.1). Now let us take

$$\varphi_V := \Phi_V \circ \psi_V \quad \text{and} \quad \varphi_E := \Phi_E \circ \psi_E.$$

We want to show that $\varphi = \langle \varphi_V, \varphi_E \rangle$ is an isomorphism of $\mathbf{G}$ and $\mathbf{G}(\mathbf{L})$. First observe that from the above facts and the definition of $\mathbf{G}(\mathbf{L})$ we obtain that φ_V (φ_E) is a bijection from $V^{\mathbf{G}}$ ($E^{\mathbf{G}}$) onto $V^{\mathbf{G}(\mathbf{L})}$ ($E^{\mathbf{G}(\mathbf{L})}$). So we must only prove

$$\varphi_V(I^{\mathbf{G}}(e)) = I^{\mathbf{G}(\mathbf{L})}(\varphi_E(e)) \quad \text{for all } \ e \in E^{\mathbf{G}}.$$

Let us take $e \in E^{\mathbf{G}}$. D. 2.3.1 and D. 2.3.7 imply

$$v \in I^{\mathbf{G}}(e) \Rightarrow \psi_V(v) \leqslant_w \psi_E(e) \Rightarrow \Phi_V(\psi_V(e)) \leqslant_L \Phi_E(\psi_E(e))$$
$$\Rightarrow \varphi_V(v) \leqslant_L \varphi_E(e) \Rightarrow \varphi_V(v) \in I^{\mathbf{G}(\mathbf{L})}(\varphi_E(e)).$$

Thus we have shown

$$\varphi_V(I^{\mathbf{G}}(e)) \subseteq I^{\mathbf{G}(\mathbf{L})}(\varphi_E(e)).$$

Now let $I^{\mathbf{G}(\mathbf{L})}(\varphi_E(e)) = \{u_1, u_2\}$, i.e. $u_1, u_2 \leqslant_L \varphi_E(e)$ (see D. 2.3.7). Since Φ is a lattice isomorphism, we obtain

$$\Phi_V^{-1}(u_i) \leqslant_w \Phi_E^{-1}(\varphi_E(e)) = \psi_E(e) \quad \text{for} \;\; i = 1, 2.$$

Hence and by D. 2.3.1 and L. 2.3.2 we get

$$\psi_V^{-1} \circ \Phi_V^{-1}(u_i) \in I^{\mathbf{G}}(e) \text{ for } i = 1, 2.$$

Since $\varphi_V^{-1} = \psi_V^{-1} \circ \Phi_V^{-1}$ and $\varphi_E^{-1} = \psi_E^{-1} \circ \Phi_E^{-1}$, we obtain from the above fact that

$$\varphi_V^{-1}(u_i) \in I^{\mathbf{G}}(e), \quad \text{for} \;\; i = 1, 2.$$

Thus we have shown that

$$I^{\mathbf{G}(\mathbf{L})}(\varphi_E(e)) \subseteq \varphi_V(I^{\mathbf{G}}(e)).$$

These two inclusions imply the desired equality. This completes our proof. □

Theorem 2.3.9. *Let* $\mathbf{G} \in \mathcal{AG}_d$ ($\mathbf{G} \in \mathcal{AG}_n$). *Then*

$$\mathbf{G}(\mathbf{S}_w(\mathbf{G})) \simeq \mathbf{G}^* \quad (\mathbf{G}(\mathbf{S}_w(\mathbf{G})) \simeq \mathbf{G}).$$

The proof follows from Th. 1.2.4, Th. 2.3.3 and Th. 2.3.8.

Theorem 2.3.10. *Let* $\mathbf{A} \in \mathcal{UPA}lg$. *Then*

$$\mathbf{G}(\mathbf{S}_w(\mathbf{A})) \simeq \mathbf{G}^*(\mathbf{A}).$$

The proof is obtained from Th. 2.2.7 and Th. 2.3.9.

Now we can give a characterization (in the graph language) of the pairs $\langle \mathbf{L}, \mathbf{G} \rangle$, where $\mathbf{L}$ is a lattice and $\mathbf{G}$ is a digraph (graph) such that $\mathbf{S}_w(\mathbf{G}) \simeq \mathbf{L}$. Moreover, we also prove the analogous result for the pairs $\langle \mathbf{L}, \mathbf{A} \rangle$, where $\mathbf{L}$ is a lattice and $\mathbf{A}$ is a unary partial algebra such that $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{L}$.

Theorem 2.3.11. *Let a lattice* $\mathbf{L}$ *satisfy* (d.1)–(d.4) *from Th.* 2.3.6 *and let* $\mathbf{G} \in \mathcal{AG}_d$ ($\mathbf{G} \in \mathcal{AG}_n$). *Then the following conditions are equivalent:*

(a) $\mathbf{S}_w(\mathbf{G}) \simeq \mathbf{L}$.
(b) $\mathbf{G}^* \simeq \mathbf{G}(\mathbf{L})$ ($\mathbf{G} \simeq \mathbf{G}(\mathbf{L})$).

The proof is obtained from Th. 1.2.4, Th. 2.3.3 and Th. 2.3.8.

Theorem 2.3.12. *Let a lattice* $\mathbf{L}$ *satisfy* (d.1)–(d.4) *from Th.* 2.3.6 *and let* $\mathbf{A} \in \mathcal{UPA}$*. Then the following conditions are equivalent:*

(a) $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{L}$.

(b) $\mathbf{S}_w(\mathbf{G}^*(\mathbf{A})) \simeq \mathbf{L}$.

(c) $\mathbf{G}^*(\mathbf{A}) \simeq \mathbf{G}(\mathbf{L})$.

The proof follows straightforward from Th. 2.2.7 and Th. 2.3.11.

Theorem 2.3.13. *Let a lattice* $\mathbf{L}$ *satisfy* (d.1)–(d.4) *from Th.* 2.3.6 *and let* K *be a unary algebraic type (i.e.* K *is a set of unary operation symbols). Then the following conditions are equivalent:*

(a) *There exists* $\mathbf{A} \in \mathcal{UPA}lg(K)$ *such that* $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{L}$.

(b) *There exists* $\mathbf{A} \in \mathcal{UPA}lg(K)$ *such that* $\mathbf{G}^*(\mathbf{A}) \simeq \mathbf{G}(\mathbf{L})$.

(c) *There exists* $\mathbf{G} \in \mathcal{AG}_d(|K|)$ *such that* $\mathbf{S}_w(\mathbf{G}) \simeq \mathbf{L}$.

(d) *There exists* $\mathbf{G} \in \mathcal{AG}_d(|K|)$ *such that* $\mathbf{G}^* \simeq \mathbf{G}(\mathbf{L})$.

The equivalence $(a) \Leftrightarrow (b)$ $((c) \Leftrightarrow (d))$ follows from Th. 2.3.12 (Th. 2.3.11), and the equivalence $(a) \Leftrightarrow (c)$ is obtained from P. 2.1.2, P. 2.1.3 and Th. 2.2.4(a).

Applying this result (and also other results from this paper) we will characterize in a subsequent paper the pairs $\langle \mathbf{L}, K \rangle$, where $\mathbf{L}$ is a lattice and K is a unary algebraic type, such that there exists a unary partial algebra $\mathbf{A}$ of the unary type K with the weak subalgebra lattice $\mathbf{S}_w(\mathbf{A})$ isomorphic to $\mathbf{L}$.

Recall that such a characterization for arbitrary algebraic lattices and arbitrary types in the case of total algebras is an important problem of Universal Algebra (see e.g. [Jón]) which is not completely solved yet. But for weak subalgebra lattices of unary partial algebras we can give a complete solution.

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