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INVARIANT SUBSPACES IN HIGHER ORDER JET PROLONGATIONS OF A FIBRED MANIFOLD

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Abstract. We present a generalization of the concept of semiholonomic jets within the framework of higher order prolongations of a fibred manifold. In this respect, a compilation of our 2-fibred manifold approach with the methods of natural operators theory is used.

Keywords: 2-fibred manifold, jet prolongation, semiholonomic jets, natural transformation, connection

MSC 2000: 58A20, 53A55, 53C05

1. Introduction

Let $\pi\colon Y \to X$ be a fibred manifold and $\pi_1\colon J^1\pi \to X$ its first prolongation. The concept of *semiholonomic jets* creating an invariant subspace (in fact, an affine subbundle) $\widehat{J}^2\pi$ in the space $J^1\pi_1$ of *repeated jets* is well-known and widely used (e.g. [5], [8] and [1], [2]). The higher-order generalization of this concept was studied e.g. in [7] and recently also by the second author in [10]. It appears that it represents an important background for understanding both the internal structure of jet prolongations and the higher-order connections as differential equations.

It was the research on relations between various types of connections which motivated the developement of a new approach using the framework of 2-*fibred manifolds* in [3]. This method was essentialy applied also in [10], resulting among other in a definition of $\pi_{k+r,k}$-*semiholonomic jets* useful in the theory of prolongations of higher-order equations represented by connections.

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In this paper, we prove that our approach can stand for a powerful tool in the study of invariant subspaces in the most general higher-order situation and that it is *natural* in the sense of [5] and [6]. In Section 2 we recall the crucial tool we work with—a 2-fibred manifold—and the role of a specific morphism Φ within the first prolongations; for more details we refer to [3]. Section 3 describes the mechanism of our approach just for the most known situation of semiholonomic jets. Moreover, the direction for futher generalization is indicated. Section 4 deals with the situation already described in [10]; in adition, the naturality of the results is discussed. The top of our story is presented in Section 5, where we study invariant subspaces in $J^s\pi_{k+r}$. For this purpose, we generalize our approach by prolonging the underlying 2-fibred manifold. As a result, we obtain a family of invariant subspaces generalizing the spaces of $\pi_{k+r,k}$-semiholonomic jets from the previous discussion. Again, their naturality is mentioned.

2. 2-fibred manifold

A 2-fibred manifold is by [4] a quintuple $Z \xrightarrow{\varrho} Y \xrightarrow{\pi} X$, where $\pi\colon Y \to X$ and $\varrho\colon Z \to Y$ (and thus also $\pi\circ\varrho\colon Z \to X$) are fibred manifolds. Following the standard notation of jet prolongations of fibred manifolds and fibred morphisms [9], the first prolongation together with the crucial underlying structures can be described by the following diagram:

$$
\begin{array}{ccccccc}
X & \xleftarrow{J^1(\pi,\mathrm{id}_X)\equiv\pi_1} & J^1\pi & \xleftarrow{J^1(\varrho,\mathrm{id}_X)} & J^1(\pi\circ\varrho) & & \\
\downarrow{\scriptstyle \mathrm{id}_X} & & {\scriptstyle \pi_{1,0}}\downarrow & & {\scriptstyle (\pi\circ\varrho)_{1,0}}\downarrow & & \\
X & \xleftarrow{\pi} & Y & \xleftarrow{\varrho} & Z & \xleftarrow{\varrho_{1,0}} & J^1\varrho \\
\downarrow{\scriptstyle \mathrm{id}_X} & & {\scriptstyle \pi}\downarrow & & {\scriptstyle \pi\circ\varrho}\downarrow & & \\
X & \xleftarrow{\mathrm{id}_X} & X & \xleftarrow{\mathrm{id}_X} & X. & &
\end{array} \tag{1}
$$

In [3], we introduced the idea of a fibred morphism

$$\Phi\colon Z \to J^1\pi \tag{2}$$

between ϱ and $\pi_{1,0}$ over Y and we studied its role in geometrical relations between connections. Namely, there is a canonical fibred morphism

$$k\colon J^1\pi \times_Y J^1\varrho \to J^1(\pi\circ\varrho)$$

between $\pi_{1,0} \times_Y \varrho_1$ ($\varrho_1 \colon J^1\varrho \to Y$) and $\varrho \circ (\pi \circ \varrho)_{1,0}$ over Y, defined in terms of the corresponding sections by

$$k(j^1_x\gamma, j^1_{\gamma(x)}\psi) = j^1_x(\psi \circ \gamma).$$

Then an arbitrary fibred morphism Φ (2) induces the affine bundle morphism

$$k_\Phi \colon J^1\varrho \to J^1(\pi \circ \varrho)$$

between $\varrho_{1,0}$ and $(\pi \circ \varrho)_{1,0}$ over Z by the composition

$$J^1\varrho \xrightarrow{\varrho_{1,0}\times \mathrm{id}} Z \times_Y J^1\varrho \xrightarrow{\Phi\times \mathrm{id}} J^1\pi \times_Y J^1\varrho \xrightarrow{k} J^1(\pi \circ \varrho).$$

This k_Φ can be then composed with a connection on ϱ (section of $\varrho_{1,0}$) to get a connection on $\pi \circ \varrho$ (section of $(\pi \circ \varrho)_{1,0}$). For more details and various examples we refer to [3].

Here, we will be interested in another object related to a morphism Φ. Put

$$A_\Phi = \{j^1_x\xi \in J^1(\pi \circ \varrho);\ \Phi \circ (\pi \circ \varrho)_{1,0}(j^1_x\xi) = J^1(\varrho, \mathrm{id}_X)(j^1_x\xi)\}.$$

It is easy to see that A_Φ is an affine subbundle in $J^1(\pi \circ \varrho)$ such that $\mathrm{Im}\, k_\Phi \subset A_\Phi \subset J^1(\pi \circ \varrho)$. In fact, $A_\Phi := \ker \mathrm{Sp}_\Phi$, where

$$\mathrm{Sp}_\Phi \colon J^1(\pi \circ \varrho) \to V_\pi Y \otimes \pi^*(T^*X)$$

can be on the lines of the Spencer operator (see e.g. [9]) defined in such a way that $\mathrm{Sp}_\Phi(j^1_x\xi)$ is a vector such that

$$J^1(\varrho, \mathrm{id}_X)(j^1_x\xi) + \mathrm{Sp}_\Phi(j^1_x\xi) = \Phi \circ (\pi \circ \varrho)_{1,0}(j^1_x\xi).$$

The vector bundle $\overline{A}_\Phi$ associated to A_Φ is (for each Φ)

$$\overline{A}_\Phi = V_\varrho Z \otimes (\pi \circ \varrho)^*(T^*X) \subset V_{(\pi\circ\varrho)} Z \otimes (\pi \circ \varrho)^*(T^*X),$$

which in general does not split except for ϱ being an affine or vector bundle.

Consider first a 2-fibred manifold $J^1\pi \xrightarrow{\pi_{1,0}} Y \xrightarrow{\pi} X$ with the corresponding diagram

$$
\begin{array}{ccccccc}
X & \xleftarrow{\pi_1} & J^1\pi & \xleftarrow{J^1(\pi_{1,0},\mathrm{id}_X)} & J^1\pi_1 & & \\
\big\downarrow{\scriptstyle \mathrm{id}_X} & & {\scriptstyle \pi_{1,0}}\big\downarrow & & {\scriptstyle (\pi_1)_{1,0}}\big\downarrow & & \\
X & \xleftarrow{\pi} & Y & \xleftarrow{\pi_{1,0}} & J^1\pi & \xleftarrow{(\pi_{1,0})_{1,0}} & J^1\pi_{1,0} \\
\big\downarrow{\scriptstyle \mathrm{id}_X} & & {\scriptstyle \pi}\big\downarrow & & {\scriptstyle \pi_1}\big\downarrow & & \\
X & \xleftarrow{\mathrm{id}_X} & X & \xleftarrow{\mathrm{id}_X} & X & &
\end{array}
\tag{3}
$$

and let $\Phi\colon\ J^1\pi \to J^1\pi$ be a fibred morphism over Y. Denoting by (x^i, y^σ) the canonical coordinates on Y, the induced coordinates on $J^1\pi$ or on $J^1\pi_1$ are $(x^i, y^\sigma, y^\sigma_i)$ or $(x^i, y^\sigma, y^\sigma_i, y^\sigma_{;i}, y^\sigma_{i;j})$, respectively. The morphism Φ is then locally expressed by

$$(x^i, y^\sigma, y^\sigma_i) \overset{\Phi}{\mapsto} (x^i, y^\sigma, \Phi^\sigma_i(x^j, y^\lambda, y^\lambda_k))$$

and the corresponding invariant subspace A_Φ can be locally characterized by

$$y^\sigma_{;i} = \Phi^\sigma_i(x^j, y^\lambda, y^\lambda_k). \tag{4}$$

The associated vector subbundle is in this case

$$\overline{A}_\Phi = V_{\pi_{1,0}} J^1\pi \otimes \pi_1^*(T^*X) \subset V_{\pi_1} J^1\pi \otimes \pi_1^*(T^*X).$$

In particular, if $\Phi = \mathrm{id}_{J^1\pi}$, then A_Φ coincides with the subbundle $\widehat{J}^2\pi$ of *semiholonomic jets*, the equations of which locally read

$$y^\sigma_{;i} = y^\sigma_i.$$

Recall here that there is a splitting

$$\widehat{J}^2\pi \cong J^2\pi \times_{J^1\pi} \pi_{1,0}^*(V_\pi Y \otimes \pi^*(\Lambda^2 T^*X))$$

—we refer to [3] for more details.

This construction of semiholonomic jets leads to the first task: to determine all canonical morphisms Φ and consequently to classify all the corresponding invariant subspaces of $J^1\pi_1$. The result is as follows.

Proposition 1. *The morphism* $\Phi = \mathrm{id}_{J^1\pi}$ *is the only natural transformation* $J^1\pi \to J^1\pi$ *over the identity of* Y.

P r o o f. Denote by $G^r_{n,m}$ the group of all r-jets at the origin of the diffeomorphisms $\overline{x}^i = \overline{x}^i(x)$, $\overline{y}^\sigma = \overline{y}^\sigma(x,y)$ of $\mathbb{R}^{n+m}$ preserving the origin and the canonical fibration $\mathbb{R}^{n+m} \to \mathbb{R}^n$. The canonical coordinates in $G^1_{n,m}$ will be denoted by $(a^i_j, a^\sigma_\lambda, a^\sigma_i)$, while the coordinates of the inverse element will be denoted by a tilde. By the general theory [5], natural transformations $\Phi\colon J^1\pi \to J^1\pi$ over id_π correspond to the $G^1_{n,m}$-equivariant maps

$$r^\sigma_i = r^\sigma_i(x^i, y^\sigma, y^\sigma_i)$$

of standard fibres, which express the coordinate form of Φ. The following transformation laws, which describe the action of $G^1_{n,m}$ on standard fibres, can be easily computed by direct calculations:

$$\begin{aligned} \overline{r}^\sigma_i &= a^\sigma_\lambda r^\lambda_j \tilde{a}^j_i + a^\sigma_j \tilde{a}^j_i, \\ \overline{y}^\sigma_i &= a^\sigma_\lambda y^\lambda_j \tilde{a}^j_i + a^\sigma_j \tilde{a}^j_i. \end{aligned}$$

Using homotheties we have $r^\sigma_i = k y^\sigma_i$, $k \in \mathbb{R}$. Then the equivariance yields

$$k y^\sigma_i + a^\sigma_i = k(y^\sigma_i + a^\sigma_i),$$

which implies $k = 1$. Hence $r^\sigma_i = y^\sigma_i$, so that the only natural transformation in question is the identity of $J^1\pi$. □

By Proposition 1, if we identify the invariant subspaces A_Φ with canonical morphisms Φ, then the semiholonomic jets $\widehat{J}^2\pi$ form the only canonical subspace of $J^1\pi_1$.

The goals for futher investigation are straightforward:

(1) To define certain analogues of semiholonomic jets in the case of higher order prolongations of a fibred manifold $\pi\colon Y \to X$ by means of an appropriate morphism Φ.
(2) To classify all invariant subspaces from item (1).

We remark that the concept of a geometrical (or a canonical) construction has been reflected as a natural differential operator or a natural transformation, see [5].

4. $\pi_{k+r,k}$-semiholonomic jets

In this section we show that there is an analogue of Proposition 1 for 2-fibred manifolds $J^{k+r}\pi \xrightarrow{\pi_{k+r,k}} J^k\pi \xrightarrow{\pi_k} X$, $r \geqslant 1$. We separate the cases of $r = 1$ and $r \geqslant 2$. The reason is that $\pi_{k+1,k}\colon\ J^{k+1}\pi \to J^k\pi$ is an affine bundle, which is not the case of a general $\pi_{k+r,k}\colon\ J^{k+r}\pi \to J^k\pi$ with $r \geqslant 2$.

The first situation has the corresponding diagram

(5)
$$\begin{array}{ccccccc}
X & \xleftarrow{(\pi_k)_1} & J^1\pi_k & \xleftarrow{J^1(\pi_{k+1,k},\mathrm{id}_X)} & J^1\pi_{k+1} & & \\
\Big\downarrow \mathrm{id}_X & & (\pi_k)_{1,0}\Big\downarrow & & (\pi_{k+1})_{1,0}\Big\downarrow & & \\
X & \xleftarrow{\pi_k} & J^k\pi & \xleftarrow{\pi_{k+1,k}} & J^{k+1}\pi & \xleftarrow{(\pi_{k+1,k})_{1,0}} & J^1\pi_{k+1,k} \\
\Big\downarrow \mathrm{id}_X & & \pi_k\Big\downarrow & & \pi_{k+1}\Big\downarrow & & \\
X & \xleftarrow{\mathrm{id}_X} & X & \xleftarrow{\mathrm{id}_X} & X. & &
\end{array}$$

For an arbitrary fibred morphism $\Phi\colon\ J^{k+1}\pi \to J^1\pi_k$ define

$$A_\Phi = \{z \in J^1\pi_{k+1}\,;\ J^1(\pi_{k+1,k}, \mathrm{id}_X)(z) = \Phi \circ (\pi_{k+1})_{1,0}(z)\}.$$

By the general theory, A_Φ is an affine subbundle of $J^1\pi_{k+1}$ (with respect to the fibration $(\pi_{k+1})_{1,0}$).

Here, there is a canonical embedding

$$\iota_{1,k}\colon\ J^{k+1}\pi \hookrightarrow J^1\pi_k$$

defined by

$$\iota_{1,k}(j^{k+1}_x\gamma) = j^1_x(j^k\gamma).$$

The coordinate expression of this canonical morphism is

(6)
$$y^\sigma_{;i} = y^\sigma_i, \dots, y^\sigma_{j_1\dots j_k;i} = y^\sigma_{j_1\dots j_k i}.$$

This canonical embedding $\iota_{1,k}$ induces an invariant subspace $A_{\iota_{1,k}}$. It is easy to see that

$$A_{\iota_{1,k}} \equiv \widehat{J}^{k+2}\pi \subset J^1\pi_{k+1},$$

where the elements of $\widehat{J}^{k+2}\pi$ are called $(k+2)$-*semiholonomic jets*. The local equations for them are just (6), expressing the fact that while for $(k+2)$-*holonomic jets* from $J^{k+2}\pi$ all derivative coordinates are totally symmetric, those on $\widehat{J}^{k+2}\pi$ are totally symmetric except for the highest-order ones. Obviously,

$$\iota_{1,k+1}(J^{k+2}\pi) \subset \widehat{J}^{k+2}\pi \subset J^1\pi_{k+1}.$$

By the following assertion, this subspace is the only canonical one, if we again identify invariant subspaces $A_\Phi \subset J^1\pi_{k+1}$ with natural transformations

$$\Phi\colon\ J^{k+1}\pi \to J^1\pi_k.$$

Proposition 2. *The morphism $\iota_{1,k}$ is the only natural transformation $J^{k+1}\pi \to J^1\pi_k$ over the identity of $J^k\pi$.*

Proof. The proof is quite similar to that of Proposition 1, so that we sketch the basic steps only. In general, the whole proof reduces to determining all $G^{k+1}_{n,m}$-equivariant maps of the form

$$\begin{aligned} r^\sigma_{;i} &= r^\sigma_{;i}(y^\sigma_j, \dots, y^\sigma_{j_1\dots j_k}, y^\sigma_{j_1\dots j_k i}),\\ &\dots\\ r^\sigma_{j_1\dots j_k;i} &= r^\sigma_{j_1\dots j_k;i}(y^\sigma_j, \dots, y^\sigma_{j_1\dots j_k}, y^\sigma_{j_1\dots j_k i}).\end{aligned}$$

Using homotheties we find $r^\sigma_{;i} = a_0 y^\sigma_i$, $r^\sigma_{j;i} = a_1 y^\sigma_{ji}$, $\dots$,$r^\sigma_{j_1\dots j_k;i} = a_k y^\sigma_{j_1\dots j_k i}$ with arbitrary $a_0, \dots, a_k \in \mathbb{R}$. Using equivariances we directly prove that $a_0 = a_1 = \dots = 1$, which is the coordinate form of $\iota_{1,k}$. □

In accordance with the affine structure of $\pi_{k+1,k}$, there is a possibility of deeper analysis of higher-order semiholonomic jets, reflecting the classical situation of $\widehat{J}^2\pi$. It can be shown that $\widehat{J}^{k+1}\pi$ is a submanifold of $J^1\pi_k$ which can be defined as the kernel of the *k-jet Spencer operator*

$$\mathrm{Sp}_k\colon\ J^1\pi_k \to V_{\pi_{k-1}} J^{k-1}\pi \otimes \pi^*_{k-1}(T^*X).$$

This is defined by the requirement on $\mathrm{Sp}_k(j^1_x\psi)$ to be just the element (of the total space of the vector bundle associated to $(\pi_{k-1})_{1,0}$) such that

$$J^1(\pi_{k,k-1}, \mathrm{id}_X)(j^1_x\psi) + \mathrm{Sp}_k(j^1_x\psi) = \iota_{1,k-1} \circ (\pi_k)_{1,0}(j^1_x\psi)$$

with respect to the affine structure. In addition,

$$\widehat{\pi}_{k+1,k} := (\pi_k)_{1,0}\colon\ J^1\pi_k \supset \widehat{J}^{k+1}\pi \to J^k\pi$$

is an affine subbundle of $(\pi_k)_{1,0}$ with the associated vector bundle (over $J^k\pi$) whose total space is

$$\pi^*_{k,0}(V_\pi Y) \otimes \pi^*_k\left(S^kT^*X \otimes T^*X\right) \cong V_{\pi_{k,k-1}} J^k\pi \otimes \pi^*_k(T^*X) \subset V_{\pi_k} J^k\pi \otimes \pi^*_k(T^*X).$$

Moreover, one gets a canonical splitting of the affine bundle $\widehat{\pi}_{k+1,k}$, expressed in terms of the total spaces by

$$\widehat{J}^{k+1}\pi \cong J^{k+1}\pi \times_{J^k\pi} \pi^*_{k,0}(V_\pi Y \otimes \pi^*(\Diamond^2_{k-1}T^*X)),$$

which gives rise to natural projections

$$\begin{aligned} s_k \colon\ & \widehat{J}^{k+1}\pi \to J^{k+1}\pi, \\ r_k \colon\ & \widehat{J}^{k+1} \to \pi^*_{k,0}(V_\pi Y \otimes \pi^*(\Diamond^2_{k-1}T^*X)), \end{aligned}$$

expressing the totally symmetric or asymmetric part of every highest-order derivative coordinate $y^\sigma_{j_1\dots j_k;i}$, respectively. Namely, $\Diamond^2_{k-1}T^*X$ is in accordance with [7] defined by

$$\Diamond^2_{k-1}T^*X = A(T^*X \otimes S^kT^*X),$$

where $A := \mathrm{id} - s$ with $s\colon \otimes^k T^*X \to S^kT^*X$ is the symmetrization linear projector.

Remark 1. This decomposition can be used for a construction generalizing the idea of the formal curvature map R, introduced in [1]. Here,

$$R\colon J^1\pi_{k+1,k} \to \pi^*_{k+1,k}\left(V_{\pi_k}J^k\pi \otimes \pi^*_k\left(\Lambda^2T^*X\right)\right)$$

is defined for each $j^1_{j^k_x\gamma}\chi \in J^1\pi_{k+1,k}$ by

$$R(j^1_{j^k_x\gamma}\chi) = r_{k+1} \circ J^1(\chi, \mathrm{id}_X) \circ \iota_{1,k} \circ \chi(j^k_x\gamma).$$

This concept naturally leads to a transparent description of the curvature of a higher order connection on π. Namely, for any $\Gamma^{(k+1)}\colon J^k\pi \to J^{k+1}\pi$, one can easily see that

$$\begin{aligned} &R_{\Gamma^{(k+1)}} = -\mathrm{pr}_2 \circ R \circ j^1\Gamma^{(k+1)} \\ &= -\mathrm{pr}_2 \circ r_{k+1} \circ J^1(\Gamma^{(k+1)}, \mathrm{id}_X) \circ \iota_{1,k} \circ \Gamma^{(k+1)}\colon J^k\pi \to V_{\pi_{k,k-1}}J^k\pi \otimes \pi^*_k(\Lambda^2T^*X). \end{aligned}$$

We refer to [10] for a discussion on $\Diamond^2_{k-1}T^*X$ and other details.

Consider finally the 2-fibred manifold $J^{k+r}\pi \xrightarrow{\pi_{k+r,k}} J^k\pi \xrightarrow{\pi_k} X$, $r \geqslant 2$. The corresponding diagram now is

$$\begin{array}{ccccccc}
X & \xleftarrow{(\pi_k)_1} & J^1\pi_k & \xleftarrow{J^1(\pi_{k+r,k},\mathrm{id}_X)} & J^1\pi_{k+r} & & \\
\Big\downarrow{\scriptstyle \mathrm{id}_X} & & \Big\downarrow{\scriptstyle (\pi_k)_{1,0}} & & \Big\downarrow{\scriptstyle (\pi_{k+r})_{1,0}} & & \\
X & \xleftarrow{\pi_k} & J^k\pi & \xleftarrow{\pi_{k+r,k}} & J^{k+r}\pi & \xleftarrow{(\pi_{k+r,k})_{1,0}} & J^1\pi_{k+r,k} \\
\Big\downarrow{\scriptstyle \mathrm{id}_X} & & \Big\downarrow{\scriptstyle \pi_k} & & \Big\downarrow{\scriptstyle \pi_{k+r}} & & \\
X & \xleftarrow{\mathrm{id}_X} & X & \xleftarrow{\mathrm{id}_X} & X. & &
\end{array} \tag{7}$$

As before, for an arbitrary fibred morphism $\Phi\colon J^{k+r}\pi \to J^1\pi_k$ over the identity of $J^k\pi$,

$$A_\Phi = \{z \in J^1\pi_{k+r}\,;\; J^1(\pi_{k+r,k}, \mathrm{id}_X)(z) = \Phi \circ (\pi_{k+r})_{1,0}(z)\}$$

is an affine subbundle with respect to $(\pi_{k+r})_{1,0}$. Denote by

(8) $$\Phi_0 = \iota_{1,k} \circ \pi_{k+r,k+1}\colon\; J^{k+r}\pi \to J^1\pi_k$$

the composition whose coordinate expression coincides with (6). Quite analogously to Proposition 2 we can prove the following assertion.

Proposition 3. *The morphism Φ_0 defined by* (8) *is the only natural transformation $J^{k+r}\pi \to J^1\pi_k$ over the identity of $J^k\pi$.*

Denote $A_{\pi_{k+r,k}} = A_{\Phi_0}$. This space consists of the points $z \in J^1\pi_{k+r}$ satisfying

(9) $$\iota_{1,k} \circ \pi_{k+r,k+1} \circ (\pi_{k+r})_{1,0}(z) = J^1(\pi_{k+r,k}, \mathrm{id}_X)(z).$$

Following the above terminology, such elements can be called $\pi_{k+r,k}$-*semiholonomic jets*; the local expression of (8) is again just (6). Consequently, there is a canonical inclusion

$$J^{k+r+1}\pi \subset \widehat{J}^{k+r+1}\pi \subset A_{\pi_{k+r,k}},$$

which corresponds to the associated vector bundle

$$\overline{A}_{\pi_{k+r,k}} = V_{\pi_{k+r,k}}J^{k+r}\pi \otimes \pi^*_{k+r}(T^*X) \subset V_{\pi_{k+r}}J^{k+r}\pi \otimes \pi^*_{k+r}(T^*X).$$

Remark 2. Here there is no an equivalent of the constructions mentioned in Remark 1. Nevertheless, certain ideas related to general jet fields can be studied, as shown in [10].

5. Invariant subspaces in higher order jet prolongations of a fibred manifold

Let $s \geqslant 1$ be fixed. This section is devoted to the study of invariant subspaces in $J^s\pi_{k+r}$ with $1 \leqslant s \leqslant r$ and $k + r = \text{const}$. Roughly speaking, we will define invariant subspaces of $J^s\pi_{k+r}$ which can be considered generalizations of $\pi_{k+r,k}$-semiholonomic jets in the case $s = 1$. Here, we show that there is a family of such spaces according to the "degree of freedom" available in the "parameters" k and r.

A general framework for this situation is the 2-fibred manifold

$$J^{k+r}\pi \xrightarrow{\pi_{k+r,k}} J^k\pi \xrightarrow{\pi_k} X,$$

which will be now prolonged to the s-th order, as described in the following diagram:

$$\begin{array}{ccccccc}
X & \xleftarrow{(\pi_k)_s} & J^s\pi_k & \xleftarrow{J^s(\pi_{k+r,k},\mathrm{id}_X)} & J^s\pi_{k+r} & & \\
\Big\downarrow \mathrm{id}_X & & (\pi_k)_{s,0}\Big\downarrow & & (\pi_{k+r})_{s,0}\Big\downarrow & & \\
X & \xleftarrow{\pi_k} & J^k\pi & \xleftarrow{\pi_{k+r,k}} & J^{k+r}\pi & \xleftarrow{(\pi_{k+r,k})_{s,0}} & J^s\pi_{k+r,k} \\
\Big\downarrow \mathrm{id}_X & & \pi_k\Big\downarrow & & \pi_{k+r}\Big\downarrow & & \\
X & \xleftarrow{\mathrm{id}_X} & X & \xleftarrow{\mathrm{id}_X} & X. & &
\end{array}$$

As usual, we start with a general fibred morphism

$$\Phi\colon\ J^{k+r}\pi \to J^s\pi_k$$

between $\pi_{k+r,k}$ and $(\pi_k)_{s,0}$ over the identity of $J^k\pi$. Define

$$A_\Phi = \{z \in J^s\pi_{k+r}\,;\ J^s(\pi_{k+r,k},\mathrm{id}_X)(z) = \Phi \circ (\pi_{k+r})_{s,0}(z)\}.$$

According to the geometric nature of the definition, A_Φ is an invariant subspace of $J^s\pi_{k+r}$. Since $(\pi_{k+r})_{s,0}$ is not an affine bundle for $s > 1$, the set A_Φ cannot be defined as the kernel of any affine bundle morphism. Nevertheless, analogously to the canonical affine morphism Φ_0 (8), the composition of $\iota_{s,k}\colon\ J^{k+1}\pi \to J^s\pi_k$ defined by

$$\iota_{s,k}(j^{k+1}_x\gamma) = j^s_x(j^k\gamma)$$

with the jet projection $\pi_{k+r,k+s}\colon\ J^{k+r}\pi \to J^{k+s}\pi$ defines a canonical map

$$\Phi^s_{k,r} = \iota_{s,k} \circ \pi_{k+r,k+1}\colon\ J^{k+r}\pi \to J^s\pi_k. \tag{10}$$

Consequently, $\Phi^s_{k,r}(j^{k+r}_x\gamma) = j^s_x(j^k\gamma)$. Then it is easy to see that

$$A_{\Phi^s_{m,n}} \subset A_{\Phi^s_{m-1,n+1}}.$$

In fact, for any $z \in A_{\Phi^s_{m,n}}$ we have $J^s(\pi_{m+n,m-1},\mathrm{id}_X)(z) = J^s(\pi_{m,m-1},\mathrm{id}_X) \circ J^s(\pi_{m+n,m},\mathrm{id}_X)(z) = J^s(\pi_{m,m-1},\mathrm{id}_X) \circ \iota_{s,m} \circ \pi_{m+n,m+1} \circ (\pi_{m+n})_{s,0}(z) = \iota_{s,m-1} \circ \pi_{m+1,m} \circ \pi_{m+n,m+1} \circ (\pi_{m+n})_{s,0}(z) = \iota_{s,m-1} \circ \pi_{m+n,m} \circ (\pi_{m+n})_{s,0}(z)$. Consequently,

$$J^{k+r+s}\pi \subset A_{\Phi^s_{m,n}} \subset A_{\Phi^s_{m-1,n+1}} \subset A_{\Phi^s_{m-2,n+2}} \subset \ldots \subset A_{\Phi^s_{0,k+r}} \subset J^s\pi_{k+r}.$$

Hence there is a family of invariant subspaces in $J^s\pi_{k+r}$ given by all combinations of k and r such that $k + r = \mathrm{const}$, $1 \leqslant s \leqslant r$.

Example 1. The space $J^1\pi_3$ has the invariant subspaces

$$J^4\pi \subset A_{\Phi^1_{2,1}} \subset A_{\Phi^1_{1,2}} \subset A_{\Phi^1_{0,3}}.$$

The coordinate description of these invariant subspaces is given by the following table:

$$\begin{aligned} A_{\Phi^1_{0,3}}: &\quad y^\sigma_{;i} = y^\sigma_i, \\ A_{\Phi^1_{1,2}}: &\quad y^\sigma_{;i} = y^\sigma_i, \quad y^\sigma_{j;i} = y^\sigma_{ji}, \\ \widehat{J}^4\pi \equiv A_{\Phi^1_{2,1}}: &\quad y^\sigma_{;i} = y^\sigma_i, \quad y^\sigma_{j;i} = y^\sigma_{ji}, \quad y^\sigma_{j_1j_2;i} = y^\sigma_{j_1j_2i}, \\ J^4\pi: &\quad y^\sigma_{;i} = y^\sigma_i, \quad y^\sigma_{j;i} = y^\sigma_{ji}, \quad y^\sigma_{j_1j_2;i} = y^\sigma_{j_1j_2i}, \quad y^\sigma_{j_1j_2j_3;i} = y^\sigma_{j_1j_2j_3i}. \end{aligned}$$

Example 2. There are three invariant subspaces in the space $J^2\pi_3$:

$$J^5\pi \subset A_{\Phi^2_{1,2}} \subset A_{\Phi^2_{0,3}}.$$

In coordinates,

$$\begin{aligned} A_{\Phi^2_{0,3}}: &\quad y^\sigma_{;i} = y^\sigma_i, \quad y^\sigma_{;i_1i_2} = y^\sigma_{i_1i_2}, \\ A_{\Phi^2_{1,2}}: &\quad y^\sigma_{;i} = y^\sigma_i, \quad y^\sigma_{;i_1i_2} = y^\sigma_{i_1i_2}, \quad y^\sigma_{j;i} = y^\sigma_{ji}, \quad y^\sigma_{j;i_1i_2} = y^\sigma_{ji_1i_2}. \end{aligned}$$

Example 3. The last example is $J^3\pi_3$ with two invariant subspaces

$$J^6\pi \subset A_{\Phi^3_{0,3}},$$

where $A_{\Phi^3_{0,3}}$ is locally given by

$$y^\sigma_{;i} = y^\sigma_i, \quad y^\sigma_{;i_1i_2} = y^\sigma_{i_1i_2}, \quad y^\sigma_{;i_1i_2i_3} = y^\sigma_{i_1i_2i_3}.$$

There is a natural question of the full classification of all invariant subspaces in $J^s\pi_{k+r}$. Taking into account the identification of A_Φ with $\Phi\colon J^{k+r}\pi \to J^s\pi_k$, we can reduce this question to determining all canonical morphisms Φ. We have

Proposition 4. *The canonical morphism $\Phi^s_{k,r}$ defined by* (10) *is the only natural transformation $J^{k+r}\pi \to J^s\pi_k$ over the identity of $J^k\pi$.*

Proof. Denote by $(x^i, y^\sigma, y^\sigma_{j_1}, \dots, y^\sigma_{j_1\dots j_k}, y^\sigma_{j_1\dots j_k\ell_1}, \dots, y^\sigma_{j_1\dots j_k\ell_1\dots\ell_r})$ the local coordinates on $J^{k+r}\pi$ and by $(x^i, y^\sigma, y^\sigma_{j_1}, \dots, y^\sigma_{j_1\dots j_k}, y^\sigma_{;i_1}, \dots, y^\sigma_{;i_1\dots i_s}, y^\sigma_{j_1;i_1}, \dots, y^\sigma_{j_1;i_1\dots i_s}, \dots, y^\sigma_{j_1\dots j_k;i_1}, \dots, y^\sigma_{j_1\dots j_k;i_1\dots i_s})$ the local coordinates on $J^s\pi_k$. Analogously to the proof of Propositions 1 and 2, we have to determine certain $G^{k+s}_{n,m}$-equivariant maps which express the coordinate form of natural transformations in question. Using homotheties and equivariances we prove that $y^\sigma_{;i_1} = y^\sigma_{i_1}$, $y^\sigma_{j_1;i_1} = y^\sigma_{j_1i_1}$, …, $y^\sigma_{j_1\dots j_k;i_1} = y^\sigma_{j_1\dots j_ki_1}$ and $y^\sigma_{;i_1i_2} = y^\sigma_{i_1i_2}$,…, $y^\sigma_{;i_1\dots i_s} = y^\sigma_{i_1\dots i_s}$, …, $y^\sigma_{j_1\dots j_k;i_1\dots i_s} = y^\sigma_{j_1\dots j_ki_1\dots i_s}$. □

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