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A BARRIER METHOD FOR QUASILINEAR ORDINARY DIFFERENTIAL EQUATIONS OF THE CURVATURE TYPE

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0. Introduction

This paper is concerned with quasilinear equations of the form

$$\left(\frac{y'}{\sqrt{1+(y')^2}}\right)' = f(t,y), \tag{1}$$

where f is of class $C([t_0,\infty)\times\mathbb{R})$. For simplicity, we often express (1) as

$$(\psi(y'))' = f(t,y),$$

where

$$\psi(s) = \frac{s}{\sqrt{1+s^2}}, \quad s\in\mathbb{R}.$$

The leading term of (1) denotes the curvature of the solution curve $y = y(t)$. We note that (1) can be rewritten as

$$y'' = (1+(y')^2)^{3/2} f(t,y). \tag{2}$$

Let us consider general quasilinear ordinary differential equations of the type

$$y'' = g(t,y,y'), \quad t\in I,$$

where $g\in C(I\times\mathbb{R}\times\mathbb{R})$ and I is an interval in $\mathbb{R}$. Our equation (1) belongs to this type as is seen from its equivalent form (2). This equation (or the nonlinear term g) is defined to satisfy Nagumo's condition if for some $G\in C(0,\infty)$ we have $G(v)>0$, $v\geqslant 0$,

$$|g(t,y,z)|\leqslant G(|z|) \quad \text{on} \quad I\times\mathbb{R}\times\mathbb{R} \quad \text{and} \quad \int^{\infty}\frac{s\,\mathrm{d}s}{G(s)} = \infty.$$

It is wellknown that, for equations satisfying Nagumo's condition, boundary value problems and initial value problems are solvable provided there are suitable supersolutions and subsolutions. Such theory is often called the barrier method briefly. However, noting the expression (2), we find that Nagumo's condition is violated for our equation (1). Accordingly, we cannot obtain information about the existence of solutions of equation (1) directly from standard barrier method. A more precise formulation and refinements for Nagumo's condition are found in [1,3].

Motivated by this fact, in the present paper we try to deduce existence theorems for (1) from the existence of appropriate supersolutions and subsolutions. This is the main purpose of the paper. As seen from the explicit formula $\psi^{-1}(s) = \frac{s}{\sqrt{1-s^2}}$, $s \in (-1, 1)$, $\psi^{-1}(s)$ is not defined for $|s| \geqslant 1$. Our main difficulty comes about from this fact. But a careful inspection of known methods enable us to find an existence theorem for initial value problems on infinite intervals. Related results are found in [2,4].

The plan of the paper is as follows. In §1 we give preparatory results for boundary value problems on finite intervals. The main result (Theorem 4) is stated and proved in §2. Some illustrative examples are given in §3.

1. Preliminaries

As a first step, we consider the simple two-point boundary value problem

$$\begin{cases} \Big(\frac{y'}{\sqrt{1+(y')^2}}\Big)' = h(t), \qquad a \leqslant t \leqslant b, \\ y(a) = A, \quad y(b) = B, \end{cases} \tag{3}$$

where $a > 0, b, A$ and B are given constants, and $h \in C[a, b]$.

Lemma 1. *Suppose that there is a* $\delta > 0$ *satisfying*

$$1 - \frac{2M}{\delta a^\delta} > \pm\psi\Big(\frac{B-A}{b-a}\Big), \tag{4}$$

where

$$M \equiv \max_{a \leqslant t \leqslant b} t^{1+\delta} |h(t)|.$$

Then, problem (3) *has a unique solution.*

P r o o f. (i) (Uniqueness) Let y_1 and y_2 be two distinct solutions of problem (3). We have

$$\begin{cases} (\psi(y_1'))' = (\psi(y_2'))', \qquad a \leqslant t \leqslant b, \\ y_1(a) = y_2(a), \quad y_1(b) = y_2(b). \end{cases}$$

The first identity of the above shows that $\psi(y_1') \equiv \psi(y_2') + c_1$ in $[a,b]$ for some constant c_1. Since for some $t_0 \in (a,b)$ we have $y_1'(t_0) = y_2'(t_0)$, we know $c_1 = 0$. (Consider the points at which $y_1 - y_2$ takes extrema.) Accordingly, $y_1' \equiv y_2'$ on $[a,b]$. Since $y_1(a) = y_2(a)$, we find that $y_1 \equiv y_2$. This contradiction proves the uniqueness.

(ii) (Existence) It is evident that if we can find a constant $\widehat{c}$ such that

$$\left|\widehat{c} + \int_a^t h(s)\,\mathrm{d}s\right| < 1, \quad a \leqslant t \leqslant b \tag{5}$$

and

$$\int_a^b \psi^{-1}\Big(\widehat{c} + \int_a^s h(r)\,\mathrm{d}r\Big)\,\mathrm{d}s = B - A, \tag{6}$$

then the function

$$y(t) = A + \int_a^t \psi^{-1}\Big(\widehat{c} + \int_a^s h(r)\,\mathrm{d}r\Big)\,\mathrm{d}s, \quad a \leqslant t \leqslant b,$$

solves problem (3). Consider the function H of λ defined by

$$H(\lambda) = \int_a^b \psi^{-1}\Big(\lambda + \int_a^s h(r)\,\mathrm{d}r\Big)\,\mathrm{d}s$$

on the interval $I = (-1 + \frac{M}{\delta a^\delta}, 1 - \frac{M}{\delta a^\delta})$. Since for $\lambda \in I$ we have

$$\begin{aligned}\left|\lambda + \int_a^s h(s)\,\mathrm{d}s\right| &\leqslant |\lambda| + \int_a^b s^{1+\delta}|h(s)|s^{-1-\delta}\,\mathrm{d}s \\ &< 1 - \frac{a^{-\delta}}{\delta}M + M\int_a^b s^{-1-\delta}\,\mathrm{d}s = 1 - \frac{Mb^{-\delta}}{\delta} < 1, \quad s \in [a,b],\end{aligned}$$

H is well-defined on I, and clearly, it is continuous and strictly increasing there. Noting that condition (4) is equivalent to $(b-a)\psi^{-1}(1 - \frac{2M}{\delta a^\delta}) > \pm(B-A)$, we can find a sufficiently small $\varepsilon \in (0,1)$ satisfying

$$(b-a)\psi^{-1}\Big(1 - \frac{2M}{\delta a^\delta} - \varepsilon\Big) > \pm(B-A).$$

Put $\lambda(\varepsilon) = 1 - \frac{M}{\delta a^\delta} - \varepsilon$. Then we have

$$\begin{aligned}H(\lambda(\varepsilon)) &= \int_a^b \psi^{-1}\Big(1 - \frac{M}{\delta a^\delta} - \varepsilon + \int_a^s r^{1+\delta}f(r)r^{-1-\delta}\,\mathrm{d}r\Big)\,\mathrm{d}s \\ &\geqslant \int_a^b \psi^{-1}\Big(1 - \frac{M}{\delta a^\delta} - \varepsilon - M\int_a^s r^{-1-\delta}\,\mathrm{d}r\Big)\,\mathrm{d}s \\ &> \int_a^b \psi^{-1}\Big(1 - \frac{2M}{\delta a^\delta} - \varepsilon\Big)\,\mathrm{d}s \\ &= (b-a)\psi^{-1}\Big(1 - \frac{2M}{\delta a^\delta} - \varepsilon\Big) > B - A,\end{aligned}$$

and similarly, $H(-\lambda(\varepsilon)) < B - A$. Hence, there is a unique $\widehat{c}$ in the interval $[-\lambda(\varepsilon), \lambda(\varepsilon)]$ satisfying (5) and (6). The proof is complete. $\square$

Lemma 2. *Let $f \in C([a,b] \times \mathbb{R})$, $a > 0$. Suppose that there is a $\delta > 0$ satisfying*

$$1 - \frac{2M}{\delta a^\delta} > \pm\psi\Big(\frac{B-A}{b-a}\Big),$$

where

$$M \equiv \sup_{a \leqslant t \leqslant b,\, y \in \mathbb{R}} t^{1+\delta}|f(t,y)| < \infty.$$

Then the boundary value problem

$$\text{(7)} \qquad \begin{cases} \Big(\frac{y'}{\sqrt{1+(y')^2}}\Big)' = f(t,y), & a \leqslant t \leqslant b \\ y(a) = A, \quad y(b) = B, \end{cases}$$

has a solution.

P r o o f. It follows from our assumption that for sufficiently small $\varepsilon > 0$ we have

$$(b-a)\psi^{-1}\Big(1 - \frac{2M}{\delta a^\delta} - \varepsilon\Big) > \pm(B-A).$$

Let

$$L \equiv \max_{|u| \leqslant 1-\varepsilon} |\psi^{-1}(u)|$$

and consider the non-empty closed convex subset Y of the Banach space $C[a,b]$ equipped with the usual maximum norm given by

$$Y = \Big\{y \in C[a,b]\colon\ |y(t)| \leqslant |A| + L(b-a) \quad \text{on} \quad [a,b]\Big\}.$$

Then, as in the proof of Lemma 1, with each $y \in Y$ we can associate a *unique* number $c(y)$ satisfying

$$\text{(8)} \qquad -1 + \frac{M}{\delta a^\delta} + \varepsilon \leqslant c(y) \leqslant 1 - \frac{M}{\delta a^\delta} - \varepsilon$$

and

$$\text{(9)} \qquad \int_a^b \psi^{-1}\Big(c(y) + \int_a^s f(r, y(r))\,\mathrm{d}r\Big)\,\mathrm{d}s = B - A.$$

It is easy to see that problem (7) is equivalent to the integral equation

$$\text{(10)} \qquad y(t) = A + \int_a^t \psi^{-1}\Big(c(y) + \int_a^s f(r, y(r))\,\mathrm{d}r\Big)\,\mathrm{d}s, \quad a \leqslant t \leqslant b.$$

For $y \in Y$ we define $\mathscr{F}y$ by the right hand side of (10). We will prove the existence of a fixed element of the operator $\mathscr{F}\colon Y \to Y$ via the Schauder fixed point theorem.

(i) *$\mathscr{F}$ maps $\mathscr{F}$ itself.* Let $y \in Y$. Since

$$
\begin{aligned}
\left| c(y) + \int_a^t f(s, y(s))\,\mathrm{d}s \right| &\leqslant |c(y)| + \int_a^t s^{-1-\delta} s^{1+\delta} |f(s, y(s))|\,\mathrm{d}s \\
&\leqslant 1 - \frac{M}{\delta a^{\delta}} - \varepsilon + M \int_a^t s^{-1-\delta}\,\mathrm{d}s \\
&= 1 - \frac{M t^{-\delta}}{\delta} - \varepsilon < 1 - \varepsilon, \qquad a \leqslant t \leqslant b,
\end{aligned} \tag{11}
$$

it follows from the definition of L that

$$
\begin{aligned}
|\mathscr{F}y(t)| &\leqslant |A| + \int_a^t \left| \psi^{-1}\left(c(y) + \int_a^s f(r, y(r))\,\mathrm{d}r \right) \right| \mathrm{d}s \\
&\leqslant |A| + L \int_a^t \mathrm{d}s \leqslant |A| + L(b-a), \quad a \leqslant t \leqslant b,
\end{aligned}
$$

implying that $\mathscr{F}y \in Y$.

(ii) *$\mathscr{F}$ is continuous.* Let $\{y_n\} \subset Y$ be a sequence satisfying $\lim\limits_{n\to\infty} y_n(t) = y(t)$ for some $y \in Y$ uniformly on $[a, b]$. We must show that $\lim\limits_{n\to\infty} \mathscr{F}y_n(t) = \mathscr{F}y(t)$ uniformly on $[a, b]$.

As a first step, we show that $\lim\limits_{n\to\infty} c(y_n) = c(y)$. To this end, suppose the contrary that $\{c(y_n)\}$ does not converge to $c(y)$. Since $\{c(y_n)\}$ is bounded by (8), we find that $\lim\limits_{n_i\to\infty} c(y_{n_i}) = \xi \neq c(y)$ for a subsequence $\{c(y_{n_i})\}$. Noting that

$$
|f(t, y_n(t))| = t^{-1-\delta} t^{1+\delta} |f(t, y_n(t))| \leqslant M t^{-1-\delta}, \ t \in [a, b], \ n \in \mathbb{N},
$$

and that (9) (with y replaced by y_n) holds, we know via the Lebesgue dominated convergence theorem that

$$
\begin{aligned}
B - A &\equiv \lim_{n_i\to\infty} \int_a^b \psi^{-1}\left(c(y_{n_i}) + \int_a^s f(r, y_{n_i}(r))\,\mathrm{d}r \right) \mathrm{d}s \\
&= \int_a^b \psi^{-1}\left(\xi + \int_a^s f(r, y(r))\,\mathrm{d}r \right) \mathrm{d}s.
\end{aligned}
$$

This contradicts the uniqueness of the number $c(y)$ satisfying (9) (and (8)). Therefore, $\lim\limits_{n\to\infty} c(y_n) = c(y)$. It follows from this fact and the dominated convergence theorem, again, that $\lim\limits_{n\to\infty} \mathscr{F}y_n(t) = \mathscr{F}y(t)$ uniformly on $[a, b]$.

(iii) $\overline{\mathscr{F}Y}$ *is compact.* Since $\mathscr{F}Y \subset Y$, $\mathscr{F}Y$ is uniformly bounded on $[a,b]$. Let $y \in Y$. Then by (11) we obtain

$$|(\mathscr{F}y)'(t)| \leqslant \left|\psi^{-1}\Big(c(y) + \int_a^t f(s, y(s))\,\mathrm{d}s\Big)\right| \leqslant L, \quad a \leqslant t \leqslant b.$$

This implies that $\mathscr{F}Y$ is equicontinuous. Consequently, $\overline{\mathscr{F}Y}$ is compact.

From the above observation we know that $\mathscr{F}$ has a fixed element in Y which gives rise to a desired solution of BVP (7). The proof is complete. □

Now, for completeness, we give the definition of supersolutions and subsolutions:

Definition. Let I be an interval in $\mathbb{R}$ (possibly unbounded), and let f be of class $C(I \times \mathbb{R})$. A function $\bar{\omega} \in C^2(I)$ is called a supersolution of the equation

$$\Big(\frac{y'}{\sqrt{1+(y')^2}}\Big)' = f(t, y) \tag{1}$$

on I if the inequality

$$\Big(\frac{\bar{\omega}'}{\sqrt{1+(\bar{\omega}')^2}}\Big)' \leqslant f(t, \bar{\omega}), \qquad t \in I,$$

holds. Conversely, if the inequality

$$\Big(\frac{\underline{\omega}'}{\sqrt{1+(\underline{\omega}')^2}}\Big)' \geqslant f(t, \underline{\omega}), \qquad t \in I,$$

holds, $\underline{\omega} \in C^2(I)$ is called a subsolution of (1) on I.

Lemma 3. *Let $f \in C([a,b] \times \mathbb{R})$, $a > 0$. Suppose that there are a supersolution $\bar{\omega} \in C^2[a,b]$ and a subsolution $\underline{\omega} \in C^2[a,b]$ of (0.1) on $[a,b]$ satisfying*

$$\underline{\omega}(t) \leqslant \bar{\omega}(t), \qquad a \leqslant t \leqslant b,$$

and

$$\underline{\omega}(a) \leqslant A \leqslant \bar{\omega}(a), \quad \underline{\omega}(b) \leqslant B \leqslant \bar{\omega}(b).$$

Suppose moreover that for some $\delta > 0$

$$1 - \frac{2M}{\delta a^{\delta}} > \pm\psi\Big(\frac{B-A}{b-a}\Big), \tag{12}$$

where

$$M \equiv \sup_{a \leqslant t \leqslant b, \underline{\omega}(t) \leqslant y \leqslant \bar{\omega}(t)} t^{1+\delta} |f(t,y)|.$$

Then, BVP (7) *has a solution* $y \in C^2[a,b]$ *satisfying*

$$\underline{\omega}(t) \leqslant y(t) \leqslant \bar{\omega}(t), \quad a \leqslant t \leqslant b.$$

P r o o f. We adapt the method in [1,§1]. Let $K > 0$ be a constant satisfying $|\bar{\omega}(t)|$, $|\underline{\omega}(t)| \leqslant K$ on $[a,b]$, and let $\varepsilon > 0$ be a sufficiently small constant satisfying

$$\frac{2M}{\delta a^\delta} + \varepsilon b^{1+\delta}\Big(K + \frac{1}{2}\Big) \cdot \frac{2}{\delta a^\delta} < 1 \pm \psi\Big(\frac{B-A}{b-a}\Big), \tag{13}$$

which is possible by assumption (12). Define a modified function $\widetilde{f}$ of f by

$$\widetilde{f}(t,y) \equiv \begin{cases} f(t,\bar{\omega}(t)) + \varepsilon \cdot \frac{y-\bar{\omega}(t)}{1+y^2} & \text{on} \quad [a,b] \times [\bar{\omega}(t), \infty), \\ f(t,y) & \text{on} \quad [a,b] \times [\underline{\omega}(t), \bar{\omega}(t)], \\ f(t,\underline{\omega}(t)) + \varepsilon \cdot \frac{y-\underline{\omega}(t)}{1+y^2} & \text{on} \quad [a,b] \times (-\infty, \underline{\omega}(t)]. \end{cases}$$

Then $\widetilde{f} \in C([a,b] \times \mathbb{R})$. Put

$$\sup_{a \leqslant t \leqslant b, y \in \mathbb{R}} t^{1+\delta} |\widetilde{f}(t,y)| = \widetilde{M}.$$

By the definition of $\widetilde{f}$ we find that

$$\widetilde{M} \leqslant M + \varepsilon b^{1+\delta}\Big(K + \frac{1}{2}\Big).$$

Hence, taking account of (13), we have

$$1 - \frac{2\widetilde{M}}{\delta a^\delta} > \pm\psi\Big(\frac{B-A}{b-a}\Big),$$

which together with Lemma 2, implies that the (modified) boundary value problem

$$\begin{cases} \Big(\frac{y'}{\sqrt{1+(y')^2}}\Big)' = \widetilde{f}(t,y), \quad a \leqslant t \leqslant b \\ y(a) = A, \quad y(b) = B, \end{cases}$$

has a solution $y(t)$. It suffices for our purpose to show that

$$\underline{\omega}(t) \leqslant y(t) \leqslant \bar{\omega}(t) \quad \text{on} \quad [a,b]. \tag{14}$$

To establish the first inequality of (14), we prove that $z(t) \equiv y(t) - \underline{\omega}(t) \geqslant 0$ on $[a, b]$. If this is not the case, there is a $t_0 \in (a, b)$ satisfying $z(t_0) = \min\limits_{a \leqslant t \leqslant b} z(t) < 0$. Obviously,

$$y(t_0) < \underline{\omega}(t_0),\ y'(t_0) = \underline{\omega}'(t_0) \quad \text{and} \quad y''(t_0) \geqslant \underline{\omega}''(t_0).$$

Now, let us consider the linear ordinary differential operator

$$\mathscr{L} = \frac{1}{(1 + (y'(t_0))^2)^{3/2}} \cdot \frac{d^2}{\mathrm{d}t^2}.$$

We know that $\mathscr{L} z(t_0) \geqslant 0$. However, another computation shows that

$$\begin{aligned}
\mathscr{L} z(t_0) &= (\psi(y'))'(t_0) - (\psi(\underline{\omega}'))'(t_0) \\
&= \widetilde{f}(t_0, y(t_0)) - f(t_0, \underline{\omega}(t_0)) \\
&= f(t_0, \underline{\omega}(t_0)) + \varepsilon \cdot \frac{y(t_0) - \underline{\omega}(t_0)}{1 + [y(t_0)]^2} - f(t_0, \underline{\omega}(t_0)) < 0.
\end{aligned}$$

This contradiction proves that $z(t) \geqslant 0$ on $[a, b]$, and hence the first inequality of (14) holds. The second inequality can be proved in the same fashion. The proof is complete. $\square$

2. Main result

We are now in a position to state and prove the main result.

Theorem 4. *Let $f \in C([a, \infty) \times \mathbb{R})$, $a > 0$, and let $\bar{\omega}$, $\underline{\omega} \in C^2[a, \infty)$ be a supersolution and a subsolution, respectively, of equation* (1) *satisfying*

$$\begin{aligned}
&\underline{\omega}(t) \leqslant \bar{\omega}(t), \qquad t \geqslant a; \\
&\underline{\omega}(a) \leqslant A \leqslant \bar{\omega}(a); \\
&\underline{\omega}(t) = o(t) \quad \text{or} \quad \bar{\omega}(t) = o(t) \quad \text{as} \quad t \to \infty.
\end{aligned} \tag{15}$$

Suppose moreover that for some $\delta > 0$

$$2\Big(\sup_{t \geqslant a, \underline{\omega}(t) \leqslant y \leqslant \bar{\omega}(t)} t^{1+\delta} |f(t, y)|\Big) < \delta a^\delta. \tag{16}$$

Then the initial value problem

$$\begin{cases} \left(\frac{y'}{\sqrt{1 + (y')^2}}\right)' = f(t, y), \quad t \geqslant a, \\ y(a) = A, \end{cases} \tag{17}$$

has a solution $y \in C^2[a,\infty)$ satisfying

$$\underline{\omega}(t) \leqslant y(t) \leqslant \bar{\omega}(t), \quad t \geqslant a. \tag{18}$$

Remark 5. A close look at the forthcoming proof shows that condition (15) can be weakened to the condition that there is a sequence $\{b_n\}$ satisfying $\lim\limits_{n\to\infty} b_n = +\infty$ and either

$$\lim_{n\to\infty} \frac{\underline{\omega}(b_n)}{b_n} = 0 \quad \text{or} \quad \lim_{n\to\infty} \frac{\bar{\omega}(b_n)}{b_n} = 0.$$

P r o o f of Theorem 4. We may assume that $\bar{\omega}(t) = o(t)$ as $t \to \infty$.
Let

$$\begin{aligned} B_n &= \bar{\omega}(a+n); \text{ and} \\ M_n &= \max_{a\leqslant t\leqslant a+n, \underline{\omega}(t)\leqslant y\leqslant \bar{\omega}(t)} t^{1+\delta}|f(t,y)|, \quad n = 1,2,\dots \end{aligned}$$

Then we know that

$$M_1 \leqslant M_2 \leqslant \ldots \leqslant M_n \leqslant \ldots \leqslant M \equiv \sup_{t\geqslant a, \underline{\omega}(t)\leqslant y\leqslant \bar{\omega}(t)} t^{1+\delta}|f(t,y)|,$$

and hence we can find a small $c > 0$ satisfying

$$1 - \frac{2M_n}{\delta a^\delta} \geqslant 1 - \frac{2M}{\delta a^\delta} > c > \pm\psi\Big(\frac{B_n - A}{n}\Big) \quad \text{for all large } n.$$

Consequently, for sufficiently small $\varepsilon > 0$ (not depending on n) and sufficiently large $n_0 \geqslant 1$ we have

$$\psi^{-1}\Big(1 - \frac{2M_n}{\delta a^\delta} - \varepsilon\Big) > \pm\frac{B_n - A}{n} \quad \text{for} \quad n \geqslant n_0. \tag{19}$$

Put

$$L = \max_{|u|\leqslant 1-\varepsilon} |\psi^{-1}(u)|.$$

Lemma 3 together with (19) implies that for each $n \geqslant n_0$ the BVP

$$\begin{cases} \Big(\frac{y'}{\sqrt{1+(y')^2}}\Big)' = f(t,y), \quad a \leqslant t \leqslant a+n, \\ y(a) = A, \ y(a+n) = B_n, \end{cases}$$

has a solution y_n satisfying

$$\underline{\omega}(t) \leqslant y_n(t) \leqslant \bar{\omega}(t), \quad a \leqslant t \leqslant a+n, \quad \text{for} \quad n \geqslant n_0.$$

We recall that y_n, $n \geqslant n_0$, satisfy

$$y_n(t) = A + \int_a^t \psi^{-1}\Big(c_n + \int_a^s f(r, y_n(r))\,\mathrm{d}r\Big)\,\mathrm{d}s, \quad a \leqslant t \leqslant a+n, \tag{20}$$

where c_n is a suitable number satisfying

$$-1 + \frac{M_n}{\delta a^\delta} + \varepsilon \leqslant c_n \leqslant 1 - \frac{M_n}{\delta a^\delta} - \varepsilon. \tag{21}$$

We will show that the sequence $\{y_n\}_{n\geqslant n_0}$ contains a subsequence which converges to the desired solution of IVP (17).

First, we find that if $m \geqslant n$ ($\geqslant n_0$), then

$$|y_m(t)| \leqslant \max\Big\{\max_{a\leqslant t\leqslant a+n} |\underline{\omega}(t)|, \max_{a\leqslant t\leqslant a+n} |\overline{\omega}(t)|\Big\}, \; a \leqslant t \leqslant a+n.$$

This means that $\{y_n\}_{n\geqslant n_0}$ is uniformly bounded on each compact subset of $[a, \infty)$. Since for $n \geqslant n_0$

$$\begin{aligned}\Big|c_n + \int_a^t f(s, y_n(s))\,\mathrm{d}s\Big| &\leqslant |c_n| + \int_a^t s^{-1-\delta} s^{1+\delta} |f(s, y_n(s))|\,\mathrm{d}s \\ &\leqslant 1 - \frac{M_n}{\delta a^\delta} - \varepsilon + M_n \int_a^t s^{-1-\delta}\,\mathrm{d}s \\ &\leqslant 1 - \varepsilon, \quad a \leqslant t \leqslant a+n,\end{aligned}$$

differentiating (20) we have

$$|y_n'(t)| \leqslant \Big|\psi^{-1}\Big(c_n + \int_a^t f(s, y_n(s))\,\mathrm{d}s\Big)\Big| \leqslant L, \quad a \leqslant t \leqslant a+n.$$

This means that $\{y_n'\}_{n\geqslant n_0}$ is uniformly bounded on each compact subset of $[a, \infty)$. Hence, there is a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ which converges uniformly to a function $y \in C[a, \infty)$ on each compact subset of $[a, \infty)$. Let $n_i \geqslant n_0$ be fixed arbitrarily. Then

$$y_{n_k}(t) = A + \int_a^t \psi^{-1}\Big(c_{n_k} + \int_a^s f(r, y_{n_k}(r))\,\mathrm{d}r\Big)\,\mathrm{d}s, \quad a \leqslant t \leqslant a+n_i, \tag{22}$$

if $n_k \geqslant n_i$. Here we may assume from (21) that the sequence $\{c_{n_i}\}$ converges to a constant $c \in \mathbb{R}$. Letting $n_k \to \infty$ in (22), we have via the Lebesgue dominated convergence theorem

$$y(t) = A + \int_a^t \psi^{-1}\Big(c + \int_a^s f(r, y(r))\,\mathrm{d}r\Big)\,\mathrm{d}s, \quad a \leqslant t \leqslant a+n_i.$$

Since n_i is arbitrary, differentiating the both sides we find that y is a solution of IVP (17) satisfying (18). The proof is complete. $\square$

Remark 6. (i) Roughly speaking, condition (16) requires that $f(t,y) = O(t^{-1-\delta})$, $\delta > 0$, as $t \to \infty$ uniformly in y. In general, such decay conditions seem to be needed in order to construct solutions of equation (1) on infinite intervals. Furthermore, generally this condition can not be weakened to $O(t^{-1})$. For example, the simple equation $(\frac{y'}{\sqrt{1+(y')^2}})' = \pm t^{-1}$, $t \geqslant 1$, clearly has no solutions on $[1,\infty)$.

(ii) However, there also exist some types of $f(t,y)$ enjoying the property that equation (1) may have solutions on infinite intervals without the condition $f(t,y) = O(t^{-1-\delta})$, $\delta > 0$, as $t \to \infty$. For example, a unique solution of the IVP

$$\begin{cases} \left(\frac{y'}{\sqrt{1+(y')^2}}\right)' + k^2 y = 0, \quad k > 0, \\ y(0) = 0,\ y'(0) = \beta \neq 0, \end{cases}$$

does exist on $\mathbb{R}$, and is periodic.

3. Examples

We give examples to which our barrier method is applicable.

3.1. Consider the singular boundary value problem

$$\begin{cases} \left(\frac{y'}{\sqrt{1+(y')^2}}\right)' = \frac{y}{3t^2}, \quad t \geqslant 1, \\ y(1) = \alpha, \quad \lim\limits_{t\to\infty} y(t) = 0, \end{cases} \tag{23)-(24}$$

where α is a constant satisfying $1 \leqslant \alpha < 3/2$. We show that this problem has a positive solution with the aid of Theorem 4.

Put

$$\bar{\omega}(t) \equiv \alpha \quad \text{and} \quad \underline{\omega}(t) = \mathrm{e}^{1-t}, \quad t \geqslant 1,$$

then we know that they are, respectively, a supersolution and a subsolution of (23) satisfying $\underline{\omega}(t) \leqslant \bar{\omega}(t)$, $t \geqslant 1$. Since

$$\sup\left\{t^2 \cdot \frac{y}{3t^2} \colon\ t \geqslant 1, \mathrm{e}^{1-t} \leqslant y \leqslant \alpha\right\} = \frac{\alpha}{3},$$

the assumption of Theorem 4 is satisfied with $\delta = 1$. Therefore equation (23) has a solution $\widehat{y}$ satisfying

$$\widehat{y}(1) = \alpha \quad \text{and} \quad \mathrm{e}^{1-t} \leqslant \widehat{y}(t) \leqslant \alpha, \quad t \geqslant 1.$$

We will show that actually $\widehat{y}$ gives a solution of boundary value problem (23)-(24), that is, we will show that $\widehat{y}(\infty) = 0$ below.

The positivity of $\widehat{y}$ implies that $\psi(\widehat{y}')$ is increasing for $t \geqslant 1$, and so $\lim_{t\to\infty} \psi(\widehat{y}'(t))$ exists (possibly is equal to $+\infty$). This, in turn, implies that $\widehat{y}'(\infty)$ exists. Since $\widehat{y}$ is bounded, $\widehat{y}'(t) \uparrow 0$ as $t \uparrow \infty$, from which we conclude that $\widehat{y}$ is a nonicreasing function. Therefore we find that $\widehat{y}(t) \downarrow \widehat{y}(\infty) \in [0, \alpha)$ as $t \uparrow \infty$ because of the boundedness of $\widehat{y}$. We must prove that $\widehat{y}(\infty) = 0$. Suppose to the contrary that $\widehat{y}(\infty) > 0$. Then, by integrating (23) twice and noting that $\inf_{0<u<1} \psi^{-1}(u)/u > 0$, we have

$$-\widehat{y}(\infty) + \widehat{y}(1) = \int_1^\infty \psi^{-1}\Big(\int_t^\infty \frac{\widehat{y}(s)}{3s^2}\,\mathrm{d}s\Big)\,\mathrm{d}t \geqslant \int_1^\infty \psi^{-1}\Big(\widehat{y}(\infty)\int_t^\infty \frac{\mathrm{d}s}{3s^2}\Big)\,\mathrm{d}t = \infty,$$

a contradiction. Thus $\widehat{y}(\infty) = 0$.

3.2. The second example is the following problem:

$$\text{(25)-(26)} \qquad \begin{cases} \Big(\frac{y'}{\sqrt{1+(y')^2}}\Big)' + \frac{\lambda}{t^2}y = 0, \quad t \geqslant 1, \\ \lim_{t\to\infty} y(t) = \infty, \quad \lim_{t\to\infty} y'(t) = 0, \end{cases}$$

where λ is a constant satisfying $0 < \lambda < 1/4$. We will prove the existence of a solution of this problem. Putting $\bar{\omega}(t) = 4mt^{1/2}$, we know that $\bar{\omega}$ becomes a supersolution of (25) provided

$$0 < m \leqslant \frac{\{1 - (4\lambda)^{2/3}\}^{1/2}}{2(4\lambda)^{1/3}}.$$

On the other hand, obviously the function $\underline{\omega}(t) \equiv 4m$ is a subsolution of (25) satisfying $\underline{\omega}(t) \leqslant \bar{\omega}(t)$. For them, the condition (16) is fulfilled with $\delta = 1/2$ if $m\lambda < 1/16$. This is always possible by taking a sufficiently small $m > 0$. Hence we find from Theorem 4 that equation (25) has a solution $\widehat{y}$ satisfying

$$4m \leqslant \widehat{y}(t) \leqslant 4mt^{1/2}, \quad t \geqslant 1$$

for a suitable $m > 0$. Arguing as in §3.1, we can easily show that actually this $\widehat{y}$ solves problem (25)-(26).

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