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N. Parhi

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OSCILLATIONS OF HIGHER ORDER DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE

N. Parhi, Berhampur

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Abstract. In this paper, sufficient conditions have been obtained for oscillation of solutions of a class of nth order linear neutral delay-differential equations. Some of these results have been used to study oscillatory behaviour of solutions of a class of boundary value problems for neutral hyperbolic partial differential equations.

Keywords: Oscillation, nonoscillation, boundary value problem, neutral equations, hyperbolic equations.

MSC 2000: 34C10, 34K15, 34K40, 35B05, 35L20

1. During the last few years many authors have obtained sufficient conditions for oscillation of solutions of neutral differential equations of higher orders (see [1, 2, 6, 8]). The conditions assumed differ from authors to authors due to the different techniques they use and different type of equations they consider. It is interesting to note that the conditions assumed by different authors for a similar type of equations are often not comparable. In a recent paper [6]. P. K. Mohanty and the author have considered oscillation of solutions of a class of linear homogeneous neutral differential equations of order n. In the present work we consider equations of the form

$$(y(t) - py(t-\tau))^{(n)} + \sum_{i=1}^{m} q_i(t)y(t-\tau_i(t)) = 0, \tag{1}$$

where $0 \leqslant p < 1$, $\tau > 0$ and τ_i, $q_i \in C([0,\infty),\mathbb{R})$, $1 \leqslant i \leqslant m$, such that $\tau_i(t) \geqslant 0$. These equations and the conditions assumed here are different from those in earlier works.

By a solution of (1) we mean a real-valued continuous function y on $[T_y,\infty)$ for some $T_y > 0$ such that $(y(t) - py(t-\tau))$ is n-times continuously differentiable and (1) is satisfied for $t \in [T_y, \infty)$. Such a solution is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory. Eq. (1) is oscillatory if all its solutions are oscillatory.

In Section 2 sufficient conditions are obtained for oscillation of solutions of (1). Some of the results of this section are used to predict oscillation of some Neumann and Dirichlet boundary value problems for neutral hyperbolic partial differential equations in Section 3.

We need the following lemmas for our work:

Lemma 1.1. [7] *If*

$$(\mathrm{H}_1) \qquad 0 \leqslant q_i(t) \leqslant q_0,\ 0 \leqslant \tau_i(t) \leqslant \tau_0, \quad t \in [0,\infty),\ 1 \leqslant i \leqslant m,$$

where q_0 *and* τ_0 *are positive constants, and*

$$(\mathrm{H}_2) \qquad \lim_{t\to\infty} \inf \inf_{\lambda>0} \left[\lambda^{-1} \sum_{i=1}^{m} q_i(t) \exp(\lambda \tau_i(t))\right] > 1,$$

then (2) *is oscillatory, where*

$$(2) \qquad x'(t) + \sum_{i=1}^{m} q_i(t) x(t - \tau_i(t)) = 0.$$

Lemma 1.2. [7] *If*

(H_3) $\quad \tau_i(t) = \tau_i, \quad t \in [0,\infty),$ *is a positive constant and* $q_i(t) \geqslant 0,\ 1 \leqslant i \leqslant m,$

and (H_2) *is satisfied, then* (2) *is oscillatory.*

Lemma 1.3. ([3], p. 67). *If* $q_i, \tau_i \in C([0,\infty),[0,\infty))$, $1 \leqslant i \leqslant m$, *then the differential inequality*

$$(3) \qquad u'(t) + \sum_{i=1}^{m} q_i(t) u(t - \tau_i(t)) \leqslant 0$$

has an eventually positive solution if and only if (2) *has an eventually positive solution.*

Lemma 1.4. ([4], [5], p. 193). *Let* $u \in C^n([0,\infty),\mathbb{R})$ *be of constant sign, let* $u^{(n)}(t)$ *be of constant sign and not identically equal to zero in any interval* $[t_0,\infty)$, $t_0 \geqslant 0$, *and* $u(t)u^{(n)}(t) \leqslant 0$. *Then*

(i) *there exists a* $t_1 > 0$ *such that* $u^{(k)}(t)$, $k = 1, \dots, n-1$, *is of constant sign on* $[t_1, \infty)$,

(ii) *there exists an integer* r, $0 \leqslant r \leqslant n-1$, *which is even if* n *is odd and is odd if* n *is even, such that*

$$u(t)u^{(k)}(t) > 0, \;\; k = 0, 1, \dots, r, \quad t \geqslant t_1,$$
$$(-1)^{n+k-1}u(t)u^{(k)}(t) > 0, \quad k = r+1, \dots, n-1, \;\; t \geqslant t_1,$$

and

(iii)

$$|u(t)| \geqslant \frac{(t-t_1)^{n-1}}{(n-1)\dots(n-r)}|u^{(n-1)}(2^{n-r-1}t)|, \quad t \geqslant t_1.$$

Lemma 1.5. *Let* $n \geqslant 3$ *be an odd integer,* $\alpha \in C([0,\infty),[0,\infty))$, $0 < \alpha(t) \leqslant \alpha_0$, *and* $u \in C^n([0,\infty),\mathbb{R})$ *such that* $(-1)^i u^{(i)}(t) > 0$, $0 \leqslant i \leqslant n-1$, *and* $u^{(n)}(t) \leqslant 0$. *Then*

$$u(t-\alpha(t)) \geqslant \frac{(\alpha(t))^{n-1}}{(n-1)!}u^{(n-1)}(t)$$

for $t \geqslant \alpha_0$.

P r o o f. By Taylor's expansion we have

$$\begin{aligned} u(t-\alpha(t)) = u(t) + (-\alpha(t))u'(t) + \frac{(-\alpha(t))^2}{2!}u''(t) + \dots \\ + \frac{(-\alpha(t))^{n-1}}{(n-1)!}u^{(n-1)}(t) + \frac{(-\alpha(t))^n}{n!}u^{(n)}(t-\theta\alpha(t)), \end{aligned}$$

where $0 \leqslant \theta \leqslant 1$. Thus

$$u(t-\alpha(t)) \geqslant \frac{(\alpha(t))^{n-1}}{(n-1)!}u^{(n-1)}(t)$$

for $t \geqslant \alpha_0$, since n is an odd integer. Hence the lemma is proved. □

2. In this section we obtain sufficient conditions for oscillation of (1).

Remark. If $f\colon (0,\infty) \to (0,\infty)$ is given by $f(\lambda) = \lambda^{-1}e^{\lambda\sigma}$, where $\sigma > 0$ is a constant, then $\lim\limits_{\lambda\to\infty} f(\lambda) = \infty$, $\lim\limits_{\lambda\to 0+} f(\lambda) = \infty$ and $f'(\lambda) = \lambda^{-2}(\lambda\sigma - 1)e^{\lambda\sigma}$. Thus $f'(\lambda) > 0$ for $\lambda > \sigma^{-1}$ and $f'(\lambda) < 0$ for $\lambda < \sigma^{-1}$. Hence the least value is obtained at $\lambda = \sigma^{-1}$. Consequently, we have

$$\inf_{\lambda>0} f(\lambda) = f(\sigma^{-1}) = \sigma e.$$

Theorem 2.1. *Let $n \geqslant 1$ be an odd integer and let* (H_1) *hold. If*

$$(\mathrm{H}_4) \qquad \liminf_{t\to\infty} \inf_{\lambda>0} \left[\lambda^{-1} \sum_{i=1}^{m} q_i(t)(\tau_i(t))^{n-1} \exp(\lambda n^{-1} \tau_i(t))\right] > n^{n-1},$$

then (1) *is oscillatory.*

P r o o f. Let $y(t)$ be a nonoscillatory solution of (1) on $[T_y, \infty)$, $T_y > 0$. Without any loss of generality, we may assume that $y(t) > 0$ for $t \geqslant t_0 \geqslant T_y$. We set, for $t \geqslant t_1 \geqslant t_0 + \max\{\tau, \tau_0\}$,

$$z(t) = y(t) - py(t-\tau). \tag{4}$$

Thus

$$\begin{gathered} z(t) \leqslant y(t),\ z(t) > -py(t-\tau) \quad \text{and} \\ z^{(n)}(t) + \sum_{i=1}^{m} q_i(t) y(t-\tau_i(t)) = 0 \end{gathered} \tag{5}$$

for $t \geqslant t_1$. It follows from (H_4) that

$$\begin{aligned} n^{n-1} &< \liminf_{t\to\infty} \inf_{\lambda>0} \left[\lambda^{-1} \tau_0^{n-1} \exp(\lambda n^{-1}\tau_0) \sum_{i=1}^{m} q_i(t)\right] \\ &< n^{-1} \tau_0^{n-1} \tau_0 e \liminf_{t\to\infty} \left(\sum_{i=1}^{m} q_i(t)\right), \end{aligned}$$

that is,

$$\liminf_{t\to\infty} \sum_{i=1}^{m} q_i(t) > \mathrm{e}^{-1} \tau_0^{-n}\ n^n > 0.$$

Thus

$$\int_0^\infty \left(\sum_{i=1}^{m} q_i(t)\right) \mathrm{d}t = \infty.$$

Hence there exists a $k \in \{1, \ldots, m\}$ such that

$$\int_0^\infty q_k(t)\,\mathrm{d}t = \infty. \tag{6}$$

Clearly, (5) yields

$$z^{(n)}(t) \leqslant -q_k(t) y(t-\tau_k(t)) \leqslant 0 \tag{7}$$

for $t \geqslant t_1$. Since $q_k(t) \not\equiv 0$, then (7) implies that $z^{(i)}(t)$, $0 \leqslant i \leqslant n-1$, is of constant sign for $t \geqslant t_2 \geqslant t_1$.

Suppose that $z(t) < 0$ for $t \geqslant t_2$. We may note that this case does not arise if $p = 0$. Since n is odd, then $z'(t) \leqslant 0$ if $n = 1$ and $z'(t) < 0$ if $n \geqslant 3$ for $t \geqslant t_2$. If $\lim_{t\to\infty} z(t) = \mu_0$, then $-\infty \leqslant \mu_0 < 0$. Thus $z(t) < -\mu$, where $0 < \mu < \infty$, for $t \geqslant t_3 \geqslant t_2$. From (5) we obtain $-py(t-\tau) < -\mu$, $t \geqslant t_3$, that is, $\mu < py(t-\tau) < y(t-\tau)$, $t \geqslant t_3$, that is, $\mu < y(t)$, $t \geqslant t_3$. Hence (7) yields, for $t \geqslant t_3 + \tau_0$, that

$$z^{(n)}(t) \leqslant -\mu q_k(t).$$

Then $\lim_{t\to\infty} z(t) = -\infty$ in view of (6). This in turn implies that $\lim_{t\to\infty} y(t) = \infty$ by (5). Hence there exists a sequence $\langle t_j \rangle$ such that $\lim_{j\to\infty} t_j = \infty$, $\lim_{j\to\infty} y(t_j) = \infty$ and $y(t_j) = \max\{y(t)\colon\, t_3 + \tau_0 \leqslant t \leqslant t_i\}$. We may choose j large enough such that $t_j - \tau > t_3 + \tau_0$. Thus

$$z(t_j) = y(t_j) - py(t_j - \tau) \geqslant (1-p)y(t_j)$$

and hence $\lim_{j\to\infty} z(t_j) = \infty$, a contradiction. Then $z(t) > 0$ for $t \geqslant t_2$. This implies that $z^{(n-1)}(t) > 0$, $t \geqslant t_2$. If $n = 1$, then (5) yields that

$$z'(t) + \sum_{i=1}^{m} q_i(t) z(t - \tau_i(t)) \leqslant 0$$

for $t \geqslant t_2 + \tau_0$, that is, $z(t)$ is an eventually positive solution of (2), a contradiction in view of (H_4) and Lemma 1.1. If $n \geqslant 3$, then Lemma 1.4 implies that there exists an even integer r, $0 \leqslant r \leqslant n-1$, such that

$$\begin{aligned} z^{(\ell)}(t) &> 0, \quad 0 \leqslant \ell \leqslant r,\\ (-1)^{n+\ell-1} z^{(\ell)}(t) &> 0, \quad r+1 \leqslant \ell \leqslant n-1, \end{aligned}$$

for $t \geqslant t_4 > t_2$. If $r = 0$, then $z'(t) < 0$ for $t \geqslant t_4$. If $r \geqslant 2$, then $z(t) > 0$, $z'(t) > 0$ and $z''(t) > 0$ for $t \geqslant t_4$ and hence $\lim_{t\to\infty} z(t) = \infty$. Consequently, $\lim_{t\to\infty} y(t) = \infty$ and (7) yields that $z^{(n)}(t) < -Lq_k(t)$ for large t, where $L > 0$ is a constant. Thus $\lim_{t\to\infty} z(t) = -\infty$, a contradiction. Then $r = 0$, that is, $(-1)^i z^{(i)}(t) > 0$, $0 \leqslant i \leqslant n-1$, for $t \geqslant t_4$. From (5) we obtain, for $t \geqslant t_4 + \tau_0$,

$$z^{(n)}(t) + \sum_{i=1}^{m} q_i(t) z(t - \tau_i(t)) \leqslant 0. \tag{8}$$

By Lemma 1.5,

$$\begin{aligned} z(t-\tau_i(t)) &= z\Big(t - \frac{\tau_i(t)}{n} - \frac{n-1}{n}\tau_i(t)\Big) \\ &\geqslant \frac{1}{(n-1)!}\Big(\frac{n-1}{n}\Big)^{n-1}(\tau_i(t))^{n-1} z^{(n-1)}\Big(t - \frac{\tau_i(t)}{n}\Big) \end{aligned}$$

for $t \geqslant t_4 + \tau_0$. Hence (8) yields

$$z^{(n)}(t) + \frac{1}{(n-1)!}\Big(\frac{n-1}{n}\Big)^{n-1}\sum_{i=1}^{m} q_i(t)(\tau_i(t))^{n-1} z^{(n-1)}\Big(t - \frac{\tau_i(t)}{n}\Big) \leqslant 0,$$

that is, $z^{(n-1)}(t)$ is an eventually positive solution of

$$u'(t) + \frac{1}{(n-1)!}\Big(\frac{n-1}{n}\Big)^{n-1}\sum_{i=1}^{m} q_i(t)(\tau_i(t))^{n-1} u\Big(t - \frac{\tau_i(t)}{n}\Big) \leqslant 0,$$

a contradiction due to (H_4) and Lemmas 1.1 and 1.3. Hence the theorem is proved. $\square$

Remark. If $n = 1$ and $p = 0$, then (1) reduces to (2) and (H_4) reduces to (H_2). Thus Theorem 2.1 may be viewed as a generalization of Lemma 1.1.

Remark. If $n \geqslant 3$ is an odd integer, then we may prove Theorem 2.1 with an assumption weaker than (H_4).

Theorem 2.2. *Let $n \geqslant 3$ be an odd integer and let (H_1) hold. If*

$$(\mathrm{H}_5) \qquad \liminf_{t\to\infty} \inf_{\lambda>0} \Big[\lambda^{-1}\sum_{i=1}^{m} q_i(t)(\tau_i(t))^{n-1}\exp(\lambda n^{-1}\tau_i(t))\Big] > (n-1)!\Big(\frac{n}{n-1}\Big)^{n-1},$$

then (1) is oscillatory.

The proof is similar to that of Theorem 2.1 and hence is omitted.

Remark. Theorems 2.1 and 2.2 remain true for $p = 1$. Indeed, $p = 1$ implies that $z(t) = y(t) - y(t-\tau)$ (see (4)). If $z(t) < 0$ for $t \geqslant t_2$, then $y(t) < y(t-\tau)$, $t \geqslant t_2$, and hence $y(t)$ is bounded. On the other hand, proceeding as in the proof of Theorem 2.1 one obtains in this case $\lim\limits_{t\to\infty} z(t) = -\infty$, which implies that $\lim\limits_{t\to\infty} y(t) = \infty$, a contradiction. The rest of the proof is the same as that of Theorem 2.1.

Remark. We may notice that the assumptions (H_4) and (H_5) are independent of p.

Theorem 2.3. *Suppose that* $0 < p < 1$, $n \geqslant 3$, *is an odd integer and* (H_1) *holds. If*

$$(\mathrm{H}_6) \qquad \lim_{t\to\infty} \inf \inf_{\lambda>0} \left[\lambda^{-1} \sum_{i=1}^{m} q_i(t)(\tau_i(t))^{n-1} \exp(\lambda n^{-1}\tau_i(t))\right] > n^{n-1}(1-p),$$

then (1) *is oscillatory.*

Proof. One may proceed as in the proof of Theorem 2.1 to obtain $z(t) > 0$ for $t \geqslant t_2$ and $(-1)^i z^{(i)}(t) > 0$, $0 \leqslant i \leqslant n-1$, for $t \geqslant t_4 \geqslant t_2$. From (4) we get, for $t \geqslant t_5 \geqslant \max\{t_4, t_1 + \ell\tau\}$,

$$\begin{aligned} y(t) &= z(t) + py(t-\tau) \\ &= z(t) + pz(t-\tau) + p^2 y(t-2\tau) \\ &= z(t) + pz(t-\tau) + p^2 z(t-2\tau) + p^3 y(t-3\tau) \\ &\vdots \\ &= z(t) + pz(t-\tau) + \ldots + p^\ell z(t-\ell\tau) + p^{\ell+1} y(t-(\ell+1)\tau). \end{aligned}$$

Since $z'(t) < 0$ for $t \geqslant t_4$, we have

$$\begin{aligned} y(t) &\geqslant (1 + p + \ldots + p^\ell) z(t) \\ &\geqslant \frac{(1-p^{\ell+1})}{1-p} z(t) \end{aligned}$$

for $t \geqslant t_5$. For $0 < \varepsilon < \frac{1}{2}$ we may choose ℓ sufficiently large such that $p^{\ell+1} < \varepsilon$. Thus, for $t \geqslant t_5$,

$$y(t) > \frac{1-\varepsilon}{1-p} z(t).$$

Then (5) yields, for $t \geqslant t_5 + \tau_0$,

$$z^{(n)}(t) + \frac{1-\varepsilon}{1-p} \sum_{i=1}^{m} q_i(t) z(t-\tau_i(t)) \leqslant 0.$$

As in the proof of Theorem 2.1,

$$z(t-\tau_i(t)) > \frac{1}{(n-1)!} \Big(\frac{n-1}{n}\Big)^{n-1} (\tau_i(t))^{n-1} z^{(n-1)}\Big(t - \frac{\tau_i(t)}{n}\Big)$$

for $t \geqslant t_5 + \tau_0$. Hence $z^{(n-1)}(t)$ is an eventually positive solution of

$$u'(t) + \frac{1-\varepsilon}{1-p}\frac{1}{(n-1)!}\Big(\frac{n-1}{n}\Big)^{n-1}\sum_{i=1}^{m} q_i(t)(\tau_i(t))^{n-1}u\Big(t - \frac{\tau_i(t)}{n}\Big) \leqslant 0,$$

a contradiction due to the assumption (H_6) and Lemmas 1.1. and 1.3. This completes the proof of the theorem. □

Remark. For $0 < p < 1$, (H_4) $\Longrightarrow$ (H_6). Further, (H_6) $\Longrightarrow$ (H_5) if $0 < p \leqslant \frac{1}{2}$ but these assumptions are not comparable if $p > \frac{1}{2}$. We may note that Theorem 2.3 does not hold if (H_6) is replaced by the weaker condition

$$\liminf_{t\to\infty} \inf_{\lambda>0} \Big[\lambda^{-1}\sum_{i=1}^{m} q_i(t)(\tau_i(t))^{n-1}\exp(\lambda n^{-1}\tau_i(t))\Big] > (n-1)!\Big(\frac{n}{n-1}\Big)^{n-1}(1-p).$$

Corresponding to Lemma 1.2 we have three similar results.

Theorem 2.4. *Let $n \geqslant 1$ be an odd integer and let* (H_3) *hold. If*

$$(\mathrm{H}_4') \qquad \liminf_{t\to\infty} \inf_{\lambda>0} \Big[\lambda^{-1}\sum_{i=1}^{m} q_i(t)\tau_i^{n-1}\exp(\lambda n^{-1}\tau_i)\Big] > n^{n-1},$$

then (1) *with $\tau_i(t) = \tau_i$, $1 \leqslant i \leqslant m$, is oscillatory.*

Theorem 2.5. *Let $n \geqslant 3$ be an odd integer and let* (H_3) *hold. If*

$$(\mathrm{H}_5') \qquad \liminf_{t\to\infty} \inf_{\lambda>0} \Big[\lambda^{-1}\sum_{i=1}^{m} q_i(t)\tau_i^{n-1}\exp(\lambda n^{-1}\tau_i)\Big] > (n-1)!\Big(\frac{n}{n-1}\Big)^{n-1},$$

then (1) *with $\tau_i(t) = \tau_i$, $1 \leqslant i \leqslant m$, is oscillatory.*

Theorem 2.6. *Let $0 < p < 1$, $n \geqslant 3$, be an odd integer and let* (H_3) *hold. If*

$$(\mathrm{H}_6') \qquad \liminf_{t\to\infty} \inf_{\lambda>0} \Big[\lambda^{-1}\sum_{i=1}^{m} q_i(t)\tau_i^{n-1}\exp(\lambda n^{-1}\tau_i)\Big] > n^{n-1}(1-p),$$

then (1) *with $\tau_i(t) = \tau_i$, $1 \leqslant i \leqslant m$, is oscillatory.*

Theorem 2.7. *Suppose that $n \geqslant 2$ is an even integer and* (H_1) *holds. If*

$$(\mathrm{H}_7) \qquad \liminf_{t\to\infty} \inf_{\lambda>0} \Big[\lambda^{-1}\sum_{i=1}^{m} q_i(t)(\tau_i(t))^{n-1}\exp(\lambda\tau_i(t))\Big] > (n-1)!2^{(n-1)(2n-1)},$$

then every solution of (1) *is oscillatory or tends to zero as $t \to \infty$.*

P r o o f. Let $y(t)$ be a solution of (1) on $[T_y,\infty)$, $T_y>0$. If $y(t)$ is oscillatory, then we have nothing to prove. Suppose that $y(t)$ is non-oscillatory. Hence we may assume, without any loss of generality, that $y(t)>0$ for $t\geqslant t_0>T_y$. Then setting $z(t)$ as in (4) for $t\geqslant t_1\geqslant t_0+\max[\tau,\tau_0)$ we get (5) for $t\geqslant t_1$. Since $z^{(n)}(t)\leqslant 0$ for $t\geqslant t_1$, then $z^{(i)}(t)$, $0\leqslant i\leqslant n-1$, is of constant sign for $t\geqslant t_2\geqslant t_1$. Let $z(t)>0$ for $t\geqslant t_2$. It follows from Lemma 1.4 that there exists an odd integer r, $1\leqslant r\leqslant n-1$, such that

$$\begin{aligned}
&z^{(\ell)}(t)>0,\quad 0\leqslant \ell\leqslant r,\\
&(-1)^{n+\ell-1}z^{(\ell)}(t)>0,\quad r+1\leqslant \ell\leqslant n-1,\\
&|z(t)|\geqslant \frac{(t-t_2)^{n-1}}{(n-1)\dots(n-r)}|z^{(n-1)}(2^{n-r-1}t)|
\end{aligned}$$

for $t\geqslant t_2$. As $z^{(n-1)}(t)<0$ implies that $z(t)<0$ for large t, we have $z^{(n-1)}(t)>0$ for $t\geqslant t_2$. Moreover, $z'(t)>0$ for $t\geqslant t_2$ since $r\geqslant 1$ is an odd integer. Hence

$$z(t)\geqslant \frac{(t-t_2)^{n-1}}{(n-1)\dots(n-r)}z^{(n-1)}(2^{n-r-1}t),$$

$t\geqslant t_2$. Thus, for $t\geqslant t_3\geqslant t_2 2^{n-2}(1-2^{-n})^{-1}$,

$$\begin{aligned}
z(t)\geqslant z(2^{r+1-n}t)&\geqslant \frac{(2^{r+1-n}t-t_2)^{n-1}}{(n-1)\dots(n-r)}z^{(n-1)}(t)\\
&\geqslant \frac{(n-r-1)!(t-t_2 2^{n-r-1})^{n-1}}{(n-1)!2^{(n-r-1)(n-1)}}z^{(n-1)}(t)\\
&> \frac{(t-t_2 2^{n-r-1})^{n-1}}{(n-1)!2^{(n-1)^2}}z^{(n-1)}(t)\\
&> \frac{t^{n-1}z^{(n-1)}(t)}{(n-1)!\,2^{(n-1)(2n-1)}}.
\end{aligned}$$

Then (5) yields

$$z^{(n)}(t)+\frac{1}{(n-1)!2^{(n-1)(2n-1)}}\sum_{i=1}^{m}q_i(t)(t-\tau_i(t))^{n-1}z^{(n-1)}(t-\tau_i(t))\leqslant 0$$

for $t\geqslant t_3$. Hence, for $t\geqslant t_3+2\tau_0$, we obtain

$$z^{(n)}(t)+\frac{1}{(n-1)!2^{(n-1)(2n-1)}}\sum_{i=1}^{m}q_i(t)(\tau_i(t))^{n-1}z^{(n-1)}(t-\tau_i(t))\leqslant 0,$$

that is, $z^{(n-1)}(t)$ is an eventually positive solution of

$$u'(t) + \frac{1}{(n-1)!\,2^{(n-1)(2n-1)}} \sum_{i=1}^{m} q_i(t)(\tau_i(t))^{n-1} u(t-\tau_i(t)) \leqslant 0,$$

a contradiction in view of the assumption (H_7) and Lemmas 1.1 and 1.3. Thus $z(t) < 0$ for $t \geqslant t_2$. We may note that this case does not arise if $p = 0$. Clearly, (6) follows from (H_7) for some $k \in \{1, \ldots, m\}$. If $z'(t) < 0$ for $t \geqslant t_2$, then proceeding as in Theorem 2.1 we arrive at a contradiction. Suppose that $z'(t) > 0$ for $t \geqslant t_2$. Thus $-\infty < \lambda_0 \leqslant 0$, where $\lambda_0 = \lim\limits_{t\to\infty} z(t)$. If $\lambda_0 < 0$, then we obtain a contradiction as in the case $z'(t) < 0$ for $t \geqslant t_2$. Hence $\lambda_0 = 0$. We claim that $y(t)$ is bounded. If not, then there exists a sequence $\langle t_j \rangle$ such that $\lim\limits_{j\to\infty} t_j = \infty$, $\lim\limits_{j\to\infty} y(t_j) = \infty$ and $y(t_j) = \max\{y(t)\colon\ t_2 \leqslant t \leqslant t_j\}$. It is possible to choose j sufficiently large such that $t_j - \tau > t_2$. Hence

$$z(t_j) = y(t_j) - py(t_j - \tau) \geqslant (1-p)y(t_j).$$

Thus $\lim\limits_{j\to\infty} z(t_j) = \infty$, a contradiction. Hence our claim holds. From (4) we obtain

$$\begin{aligned} \lim_{t\to\infty} \sup z(t) &= \lim_{t\to\infty} \sup[y(t) - py(t-\tau)] \\ &\geqslant \lim_{t\to\infty} \sup y(t) + \lim_{t\to\infty} \inf(-py(t-\tau)) \\ &\geqslant (1-p) \lim_{t\to\infty} \sup y(t), \end{aligned}$$

that is, $\lim\limits_{t\to\infty} \sup y(t) \leqslant 0$. Hence $\lim\limits_{t\to\infty} y(t) = 0$ and the proof of the theorem is complete. □

Corollary 2.8. *If all conditions of Theorem* 2.7 *are satisfied, then every unbounded solution of* (1) *oscillates.*

Corollary 2.9. *Suppose that the conditions of Theorem* 2.7 *are satisfied. Then the equation*

$$y^{(n)}(t) + \sum_{i=1}^{m} q_i(t) y(t - \tau_i(t)) = 0 \tag{9}$$

is oscillatory.

Corollary 2.10. *Let $n > 0$ be an integer and let* (H_1) *hold. If*

$$\lim_{t\to\infty} \inf \inf_{\lambda>0} \left[\lambda^{-1} \sum_{i=1}^{m} q_i(t)(\tau_i(t))^{n-1} \exp(\lambda n^{-1} \tau_i(t))\right] > (n-1)!\,2^{(n-1)(2n-1)} \tag{H_8}$$

then (9) *is oscillatory.*

This follows from Theorems 2.1 and 2.7 since (H_8) implies (H_4) and (H_7).

Remark. We may notice that (H_8) reduces to (H_2) for $n = 1$.

Theorem 2.11. *Suppose that $n \geqslant 2$ is an even integer, $0 \leqslant p \leqslant 1$, $\tau < \sigma_0 \leqslant \tau_i(t) \leqslant \tau_0$ and $0 \leqslant q_i(t) \leqslant q_0$, $1 \leqslant i \leqslant m$, where σ_0, τ_0, q_0 are constants. If* (H_7) *holds and*

$$(\mathrm{H}_9) \qquad \limsup_{t\to\infty} \int_{t-(\sigma_0-\tau)}^{t} (t-s)^{n-1}\Bigl(\sum_{i=1}^{m} q_i(s)\Bigr)\,\mathrm{d}s > (n-1)!$$

then every solution of (1) *oscillates.*

P r o o f. We proceed as in the proof of Theorem 2.7 to obtain a contradiction in the case $z(t) > 0$ for $t \geqslant t_2$. Thus $z(t) < 0$ for $t \geqslant t_2$. We may note that this case does not arise if $p = 0$. Hence $0 < p \leqslant 1$ for this case. If $z'(t) < 0$ for $t \geqslant t_2$, then a contradiction is obtained as in the proof of Theorem 2.7. Thus $z'(t) > 0$ for $t \geqslant t_2$. Consequently, $z(t)$ is bounded and $(-1)^{k+1}z^{(k)}(t) > 0$, $1 \leqslant k \leqslant n-1$, for $t \geqslant t_3 \geqslant t_2$. From (5) we obtain $z(t) > -y(t-\tau)$ and hence

$$\begin{aligned} 0 &\geqslant z^{(n)}(t) - \sum_{i=1}^{m} q_i(t)z(t-\tau_i(t)+\tau) \\ &\geqslant z^{(n)}(t) - \Bigl(\sum_{i=1}^{m} q_i(t)\Bigr)z(t-\sigma_0+\tau) \end{aligned}$$

for $t \geqslant t_3 + \tau_0$. By Taylor's expansion, for $t_3 + \tau_0 + \sigma_0 < s < t$,

$$\begin{aligned} z(s-(\sigma_0-\tau)) = z(t-(\sigma_0-\tau)) + (s-t)z'(t-(\sigma_0-\tau)) + \frac{(s-t)^2}{2!}z''(t-(\sigma_0-\tau)) \\ + \ldots + \frac{(s-t)^{n-1}}{(n-1)!}z^{(n-1)}(t-(\sigma_0-\tau)) + \frac{(s-t)^n}{n!}z^{(n)}(\xi), \end{aligned}$$

where ξ lies between $s-(\sigma_0-\tau)$ and $t-(\sigma_0-\tau)$. Thus

$$z(s-(\sigma_0-\tau)) \leqslant \frac{(s-t)^{n-1}}{(n-1)!}z^{(n-1)}(t-(\sigma_0-\tau))$$

and hence

$$0 \geqslant z^{(n)}(s) + \frac{(t-s)^{n-1}}{(n-1)!}\Bigl(\sum_{i=1}^{m} q_i(s)\Bigr)z^{(n-1)}(t-(\sigma_0-\tau)).$$

Integrating from $t-(\sigma_0-\tau)$ to t, for $t > t_3+\tau_0+2\sigma_0$, we obtain

$$\begin{aligned}\frac{z^{(n-1)}(t-(\sigma_0-\tau))}{(n-1)!}\int_{t-(\sigma_0-\tau)}^{t}(t-s)^{n-1}\Big(\sum_{i=1}^{m}q_i(s)\Big)\,\mathrm{d}s &\leqslant z^{(n-1)}(t-(\sigma_0-\tau))-z^{(n-1)}(t)\\ &< z^{(n-1)}(t-(\sigma_0-\tau)),\end{aligned}$$

that is,

$$\int_{t-(\sigma_0-\tau)}^{t}(t-s)^{n-1}\Big(\sum_{i=1}^{m}q_i(s)\Big)\,\mathrm{d}s < (n-1)!,$$

a contradiction to (H_9), which completes the proof of the theorem. □

Remark. It seems that (H_7) and (H_9) are not comparable in general. However, for $m=1$, $\tau_1(t)=\sigma_0$, $q_1(t)=q_0$ and $n=1$, (H_9) implies (H_7) because (H_7) reduces to $eq_0\sigma_0>1$ and (H_9) reduces to $q_0(\sigma_0-\tau)>1$.

Theorem 2.12. *Let $n \geqslant 2$ be an even integer and let (H_3) hold. If*

$$(\mathrm{H}_7')\qquad \liminf_{t\to\infty}\inf_{\lambda>0}\Big[\lambda^{-1}\sum_{i=1}^{m}q_i(t)\tau_i^{n-1}\exp(\lambda\tau_i)\Big] > (n-1)!\,2^{(n-1)(2n-1)},$$

then (1) *with $\tau_i(t)=\tau_i$, $1\leqslant i\leqslant m$, is oscillatory.*

Remark. From the proof of Theorems 2.1, 2.2, 2.3 and 2.7 it is clear that the following results hold for the equation

$$(10)\qquad (y(t)-p(t)y(t-\tau))^{(n)}+\sum_{i=1}^{m}q_i(t)y(t-\tau_i(t))=0,$$

where $p\in C([0,\infty),\mathbb{R})$ and $\tau, q_i, \tau_i, 1\leqslant i\leqslant m$, are the same as in (1).

Theorem 2.13. (i) *Suppose that the conditions of Theorem* 2.1 *are satisfied. If $0\leqslant p(t)\leqslant p_2<1$, where p_2 is a constant, then* (10) *is oscillatory.*

(ii) *If the conditions of Theorem* 2.2 *are satisfied and $0\leqslant p(t)\leqslant p_2<1$, then* (10) *is oscillatory.*

(iii) *Let $0<p_1\leqslant p(t)\leqslant p_2<1$, let $p(t)$ be periodic of a period τ, let $n\geqslant 3$ be an odd integer and let (H_1) hold. If (H_6) holds with p replaced by p_1, then* (10) *is oscillatory.*

(iv) *If the conditions of Theorem* 2.7 *are satisfied and $0\leqslant p(t)\leqslant p_2<1$, then every solution of* (10) *oscillates or tends to zero as $t\to\infty$.*

Theorem 2.14. *Let $n \geqslant 1$ be an odd integer, $0 \leqslant p \leqslant 1$, $q_i(t) \leqslant 0$ and $\tau < \sigma_0 \leqslant \tau_i(t) \leqslant \tau_0$, $1 \leqslant i \leqslant m$, where σ_0 and τ_0 are constants. If*

$$(\mathrm{H}_{10}) \qquad \limsup_{t\to\infty} \int\limits_{t-(\sigma_0-\tau)}^{t} (t-s)^{n-1}\Big(-\sum_{i=1}^{m} q_i(s)\Big)\,\mathrm{d}s > (n-1)!$$

then the bounded solutions of (1) *oscillate.*

P r o o f. Let $y(t)$ be a bounded solution of (1) on $[T_y, \infty)$, $T_y > 0$. If possible, let $y(t)$ be nonoscillatory. We may assume, without any loss of generality, that $y(t) > 0$ for $t \geqslant t_0 > T_y$. Setting $z(t)$ as in (4) for $t \geqslant t_1 \geqslant t_0 + \max\{\tau, \tau_0\}$, we get

$$(11) \qquad \begin{gathered} z(t) \leqslant y(t),\ z(t) > -py(t-\tau) \geqslant -y(t-\tau) \quad \text{and} \\ z^{(n)}(t) = -\sum_{i=1}^{m} q_i(t) y(t-\tau_i(t)) \geqslant 0. \end{gathered}$$

Since $q_i(t) \not\equiv 0$, $1 \leqslant i \leqslant m$, then $z^{(k)}(t)$, $0 \leqslant k \leqslant n-1$, is of constant sign for $t \geqslant t_2 \geqslant t_1$. Further, $y(t)$ being bounded implies that $z(t)$ is bounded. Clearly, it follows from (H_{10}) that

$$(12) \qquad \int_0^\infty \Big(\sum_{i=1}^{m} q_i(t)\Big)\,\mathrm{d}t = -\infty.$$

Indeed, if

$$\int_0^\infty \Big(\sum_{i=1}^{m} q_i(t)\Big)\,\mathrm{d}t > -\infty,$$

then, for $t \geqslant 2(\sigma_0 - \tau)$,

$$\begin{aligned} \int_{t-(\sigma_0-\tau)}^{t} (t-s)^{n-1}\Big(\sum_{i=1}^{m} q_i(s)\Big)\,\mathrm{d}s &\geqslant (\sigma_0-\tau)^{n-1}\int_{t-(\sigma_0-\tau)}^{t}\Big(\sum_{i=1}^{m} q_i(s)\Big)\,\mathrm{d}s \\ &= (\sigma_0-\tau)^{n-1}\Big[\int_{\sigma_0-\tau}^{t}\Big(\sum_{i=1}^{m} q_i(s)\Big)\,\mathrm{d}s - \int_{\sigma_0-\tau}^{t-(\sigma_0-\tau)}\Big(\sum_{i=1}^{m} q_i(s)\Big)\,\mathrm{d}s\Big] \end{aligned}$$

implies that

$$\liminf_{t\to\infty} \int_{t-(\sigma_0-\tau)}^{t} (t-s)^{n-1}\Bigl(\sum_{i=1}^{m} q_i(s)\Bigr)\,\mathrm{d}s$$

$$\geqslant (\sigma_0-\tau)^{n-1}\liminf_{t\to\infty}\Bigl[\int_{\sigma_0-\tau}^{t}\Bigl(\sum_{i=1}^{m} q_i(s)\Bigr)\,\mathrm{d}s - \int_{\sigma_0-\tau}^{t-(\sigma_0-\tau)}\Bigl(\sum_{i=1}^{m} q_i(s)\Bigr)\,\mathrm{d}s\Bigr]$$

$$\geqslant (\sigma_0-\tau)^{n-1}\Bigl[\liminf_{t\to\infty}\int_{\sigma_0-\tau}^{t}\Bigl(\sum_{i=1}^{m} q_i(s)\Bigr)\,\mathrm{d}s - \limsup_{t\to\infty}\int_{\sigma_0-\tau}^{t-(\sigma_0-\tau)}\Bigl(\sum_{i=1}^{m} q_i(s)\Bigr)\,\mathrm{d}s\Bigr] = 0,$$

a contradiction to (H_{10}).

If $n=1$, then $z'(t)\geqslant 0$ for $t\geqslant t_2$. If $n\geqslant 3$, then the boundedness of $z(t)$ implies that $(-1)^{k+1}z^{(k)}(t)>0$, $1\leqslant k\leqslant n-1$, for $t\geqslant t_2$. Let $z(t)>0$ for $t\geqslant t_2$. Then $0<\lim\limits_{t\to\infty} z(t)<\infty$ and hence by (11), $\liminf\limits_{t\to\infty} y(t)\geqslant \lim\limits_{t\to\infty} z(t)>0$. Thus $y(t)>\lambda>0$ for $t\geqslant t_3\geqslant t_2$. Consequently, for $t\geqslant t_3+\tau_0$, we obtain

$$\begin{aligned}
&\int_{t_3}^{t}\Bigl(\sum_{i=1}^{m} q_i(s)y(s-\tau_i(s))\Bigr)\,\mathrm{d}s\\
&\quad=\int_{t_3}^{t_3+\tau_0}\Bigl(\sum_{i=1}^{m} q_i(s)y(s-\tau_i(s))\Bigr)\,\mathrm{d}s+\int_{t_3+\tau_0}^{t}\Bigl(\sum_{i=1}^{m} q_i(s)y(s-\tau_i(s))\Bigr)\,\mathrm{d}s\\
&\quad<\lambda\int_{t_3+\tau_0}^{t}\Bigl(\sum_{i=1}^{m} q_i(s)\Bigr)\,\mathrm{d}s,
\end{aligned}$$

that is,

$$\lim_{t\to\infty}\int_{t_3}^{t}\Bigl(\sum_{i=1}^{m} q_i(s)y(s-\tau_i(s))\Bigr)\,\mathrm{d}s=-\infty.$$

On the other hand, integrating (11) yields

$$\int_{t_3}^{t}\Bigl(\sum_{i=1}^{m} q_i(s)y(s-\tau_i(s))\Bigr)\,\mathrm{d}s = z^{(n-1)}(t_3)-z^{(n-1)}(t)>z^{(n-1)}(t_3),$$

a contradiction. Hence $z(t)<0$ for $t\geqslant t_2$. By (11), we have for $t\geqslant t_2+\tau_0$

$$\begin{aligned}
0&=z^{(n)}(t)+\sum_{i=1}^{m} q_i(t)y(t-\tau_i(t))\\
&\leqslant z^{(n)}(t)-\sum_{i=1}^{m} q_i(t)z(t-\tau_i(t)+\tau)\\
&\leqslant z^{(n)}(t)-\Bigl(\sum_{i=1}^{m} q_i(t)\Bigr)z(t-(\sigma_0-\tau))
\end{aligned}$$

because $z'(t) \geqslant 0$ for $t \geqslant t_2$. By Taylor's expansion, for $t_2 + \sigma_0 < s < t$,

$$
\begin{aligned}
z(s-(\sigma_0-\tau)) = z(t-(\sigma_0-\tau)) + (s-t)z'(t-(\sigma_0-\tau)) + \frac{(s-t)^2}{2!}z''(t-(\sigma_0-\tau)) \\
+ \ldots + \frac{(s-t)^{n-1}}{(n-1)!}z^{(n-1)}(t-(\sigma_0-\tau)) + \frac{(s-t)^n}{n!}z^{(n)}(\xi),
\end{aligned}
$$

where ξ lies between $s-(\sigma_0-\tau)$ and $t-(\sigma_0-\tau)$. Hence

$$
z(s-(\sigma_0-\tau)) \leqslant \frac{(s-t)^{n-1}}{(n-1)!}z^{(n-1)}(t-(\sigma_0-\tau)).
$$

Thus, for $s \geqslant t_2 + \tau_0 + \sigma_0$,

$$
\begin{aligned}
0 &\leqslant z^{(n)}(s) - \Big(\sum_{i=1}^{m} q_i(s)\Big) z(s-(\sigma_0-\tau)) \\
&\leqslant z^{(n)}(s) - \Big(\sum_{i=1}^{m} q_i(s)\Big) \frac{(s-t)^{n-1}}{(n-1)!} z^{(n-1)}(t-(\sigma_0-\tau)).
\end{aligned}
$$

Integrating from $t-(\sigma_0-\tau)$ to t, for $t \geqslant t_2 + 2\sigma_0 + \tau_0$, yields

$$
\begin{aligned}
&\frac{z^{(n-1)}(t-(\sigma_0-\tau))}{(n-1)!} \int\limits_{t-(\sigma_0-\tau)}^{t} (t-s)^{n-1}\Big(-\sum_{i=1}^{m} q_i(s)\Big)\,\mathrm{d}s \\
&\qquad \geqslant z^{(n-1)}(t-(\sigma_0-\tau)) - z^{(n-1)}(t) \\
&\qquad > z^{(n-1)}(t-(\sigma_0-\tau)),
\end{aligned}
$$

that is,

$$
\int\limits_{t-(\sigma_0-\tau)}^{t} (t-s)^{n-1}\Big(-\sum_{i=1}^{m} q_i(s)\Big)\,\mathrm{d}s < (n-1)!,
$$

a contradiction to (H_{10}). Hence the theorem is proved. □

Theorem 2.15. *Let $n \geqslant 2$ be an even integer, $0 \leqslant p \leqslant 1$, $q_i(t) \leqslant 0$ and $0 < \sigma_0 \leqslant \tau_i(t) \leqslant \tau_0$, $1 \leqslant i \leqslant m$, where σ_0 and τ_0 are constants. If*

$$
(\mathrm{H}_{11}) \qquad \limsup_{t\to\infty} \int_{t-\sigma_0}^{t} (t-s)^{n-1}\Big(-\sum_{i=1}^{m} q_i(s)\Big)\,\mathrm{d}s > (n-1)!,
$$

then the bounded solutions of (1) *oscillate.*

P r o o f. Let $y(t)$ be a bounded nonoscillatory solution of (1) such that $y(t) > 0$ for $t \geqslant t_0 > 0$. Setting $z(t)$ as in (4), we get (11) for $t \geqslant t_1 \geqslant t_0 + \max\{\tau, \tau_0\}$. Further, ($\mathrm{H}_{11}$) implies (12). Since boundedness of $y(t)$ implies that $z(t)$ is bounded, then $(-1)^k z^{(k)}(t) > 0$ for $1 \leqslant k \leqslant n-1$ and $t \geqslant t_2 \geqslant t_1$. Let $z(t) < 0$ for $t \geqslant t_2$. Thus there exists $0 < \mu < \infty$ such that $z(t) < -\mu$ for $t \geqslant t_3 \geqslant t_2$. Then by (11), $y(t) > \mu$ for $t \geqslant t_3$. Proceeding as in the proof of Theorem 2.14, we obtain a contradiction. Hence $z(t) > 0$ for $t \geqslant t_2$. By Taylor's expansion, for $t_2 + \sigma_0 < s < t$,

$$\begin{aligned} z(s-\sigma_0) = z(t-\sigma_0) + (s-t)z'(t-\sigma_0) + \frac{(s-t)^2}{2!} z''(t-\sigma_0) + \ldots \\ + \frac{(s-t)^{n-1}}{(n-1)!} z^{(n-1)}(t-\sigma_0) + \frac{(s-t)^n}{n!} z^{(n)}(\xi), \end{aligned}$$

where ξ lies between $s-\sigma_0$ and $t-\sigma_0$. Hence

$$z(s-\sigma_0) > \frac{(s-t)^{n-1}}{(n-1)!} z^{(n-1)}(t-\sigma_0).$$

Consequently, (11) implies that, for $t_2 + \sigma_0 + \tau_0 < s$,

$$\begin{aligned} 0 &= z^{(n)}(s) + \sum_{i=1}^m q_i(s) y(s-\tau_i(s)) \\ &\leqslant z^{(n)}(s) + \sum_{i=1}^m q_i(s) z(s-\tau_i(s)) \\ &\leqslant z^{(n)}(s) + \Bigl(\sum_{i=1}^m q_i(s)\Bigr) z(s-\sigma_0) \\ &\leqslant z^{(n)}(s) + \Bigl(-\sum_{i=1}^m q_i(s)\Bigr) \frac{(t-s)^{n-1}}{(n-1)!.} z^{(n-1)}(t-\sigma_0) \end{aligned}$$

since $z'(t) < 0$. Integrating from $t-\sigma_0$ to t, for $t \geqslant t_2 + 2\sigma_0 + \tau_0$, we obtain

$$\begin{aligned} &\frac{z^{(n-1)}(t-\sigma_0)}{(n-1)!} \int_{t-\sigma_0}^t (t-s)^{n-1} \Bigl(-\sum_{i=1}^m q_i(s)\Bigr) \, \mathrm{d}s \\ &\geqslant z^{(n-1)}(t-\sigma_0) - z^{(n-1)}(t) > z^{(n-1)}(t-\sigma_0), \end{aligned}$$

that is,

$$\int_{t-\sigma_0}^t (t-s)^{n-1} \Bigl(-\sum_{i=1}^m q_i(s)\Bigr) \, \mathrm{d}s < (n-1)!,$$

a contradiction to (H_{11}), which completes the proof of the theorem. □

Remark. We may note that $(\mathrm{H}_{10}) \implies (\mathrm{H}_{11})$. Further, theorems similar to Theorems 2.14 and 2.15 hold for (10) if we assume $0 \leqslant p(t) \leqslant 1$.

3. In this section we use some of the results of the previous section to obtain sufficient conditions for the oscillation of solutions of Dirichlet and Neumann boundary value problems for a class of neutral hyperbolic partial differential equations. We consider

$$(12)\qquad u_{tt}(x,t) - \beta u_{tt}(x,t-\tau) - \Big[b(t)\Delta u(x,t) + \sum_{j=1}^{\ell} b_j(t)\Delta u(x,t-\sigma_j)\Big] + \sum_{i=1}^{m} q_i(t)u(x,t-\tau_i(t)) = 0$$

$(x,t) \in \Omega X(0,\infty)$, where Ω is a bounded domain in $\mathbb{R}^n$ with piece-wise smooth boundary $\Gamma \equiv \partial\Omega$ and Δ is the Laplacian in $\mathbb{R}^n$, with the boundary condition

$$(\mathrm{NBC})\qquad \frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \Gamma X(0,\infty)$$

or

$$(\mathrm{DBC})\qquad u = 0 \qquad \text{on } \Gamma X(0,\infty),$$

where ν denotes the unit exterior normal vector to Γ. We assume that $0 \leqslant \beta \leqslant 1$, $\tau > 0$, $q_i, \tau_i, b, b_j \in C([0,\infty,\mathbb{R})$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant \ell$, such that $0 \leqslant \tau_i(t) \leqslant \tau_0$ and $b(t) > 0$, where τ_0 is a constant. Let $T_0 = \max\{\tau, \sigma_j, \tau_0 \colon 1 \leqslant j \leqslant \ell\}$. By a solution of the problem (12), (NBC) we mean a real-valued continuous function $u(x,t)$ on $\Omega X(-T_0,\infty)$ such that $u_{tt}(x,t)$ and $\Delta u(x,t)$ exist, (12) is satisfied identically on $\Omega X(0,\infty)$ and (NBC) holds. A solution $u(x,t)$ of the problem (12), (NBC) is said to be oscillatory if $u(x,t)$ has a zero in $\Omega X(t_0,\infty)$ for every $t_0 \geqslant 0$. It is known that the first eigenvalue λ_1 of the eigenvalue problem

$$-\Delta w = \lambda w \quad \text{in} \quad \Omega,\ w = 0 \quad \text{on} \quad \Gamma$$

is positive and the associated eigenfunction $\varphi(x)$ is of one a sign and hence may be chosen positive in Ω. For a sufficiently smooth function $u(x,t)$ we denote

$$U(t) = \int_\Omega u(x,t)\,\mathrm{d}x \quad \text{and} \quad \widetilde{U}(t) = \int_\Omega u(x,t)\varphi(x)\,\mathrm{d}x, \quad t > 0.$$

Theorem 3.1. *Suppose that $\tau \leqslant \sigma_0 \leqslant \tau_i(t)$ and $0 \leqslant q_i(t) \leqslant q_0$, $1 \leqslant i \leqslant m$, where σ_0 and q_0 are constants. If* (H_7) *and* (H_9) *hold, then every solution of the problem* (12), (NBC) *oscillates in* $\Omega X(0,\infty)$.

P r o o f. Let $u(x,t)$ be a solution of the problem (12), (NBC) which does not oscillate in $\Omega X(0,\infty)$. Then there exists a $t_0>0$ such that $u(x,t)\neq 0$ in $\Omega X(t_0,\infty)$. We may take $u(x,t)>0$ in $\Omega X(t_0,\infty)$. For $t>t_0+T_0$ we integrate (12) with respect to x over the domain Ω to obtain

$$U''(t)-\beta U''(t-\tau)+\sum_{i=1}^{m} q_i(t)U(t-\tau_i(t))=0,$$

that is, $U(t)$ is a positive solution of (1) with $n=2$ and $p=\beta$, a contradiction due to Theorem 2.11. Hence the theorem is proved. □

Theorem 3.2. *Let the conditions of Theorem* 3.1 *hold. If* $\sigma_0\leqslant\min\{\sigma_j\colon 1\leqslant j\leqslant\ell\}$ *and* $0<b(t)$, $b_j(t)\leqslant q_0/\lambda_1$, $1\leqslant j\leqslant\ell$, *then every solution of the problem* (12), (DBC) *oscillates in* $\Omega X(0,\infty)$.

P r o o f. If $u(x,t)$ is a solution of the problem (12), (DBC) which does not oscillate in $\Omega X(0,\infty)$, then we may take $u(x,t)>0$ in $\Omega X(t_0,\infty)$ for some $t_0\geqslant 0$. Since

$$\begin{aligned}\int_\Omega \Delta u(x,t)\varphi(x)\,\mathrm{d}x&=\int_\Omega u(x,t)\Delta\varphi\,\mathrm{d}x+\int_\Gamma\frac{\partial u}{\partial\nu}\varphi\,\mathrm{d}s-\int_\Gamma u\frac{\partial\varphi}{\partial\nu}\,\mathrm{d}s\\&=-\lambda_1\int_\Omega u(x,t)\varphi\,\mathrm{d}x=-\lambda_1 U(t),\end{aligned}$$

then multiplying (12) through by $\Phi(\kappa)$ and integrating the resulting identity with respect to x over the domain Ω we get

$$\begin{aligned}\widetilde{U}''(t)-\beta\widetilde{U}''(t-\tau)+\lambda_1\Bigl(b(t)\widetilde{U}(t)+\sum_{j=1}^{\ell} b_j(t)\widetilde{U}(t-\sigma_j)\Bigr)&\\+\sum_{i=1}^{m} q_i(t)\widetilde{U}(t-\tau_i(t))&=0.\end{aligned}$$

A contradiction is obtained due to Theorem 2.11 since $\widetilde{U}(t)>0$ for $t\geqslant t_0+T_0$. This completes the proof of the theorem. □

Theorem 3.3. *Suppose that* $q_i(t)\leqslant 0$ *and* $0<\sigma_0\leqslant\tau_i(t)$, $1\leqslant i\leqslant m$, *where* σ_0 *is a constant. If* (H_{11}) *holds, then every bounded solution of the problem* (12), (NBC) *oscillates in* $\Omega X(0,\infty)$.

In view of Theorem 2.15, the proof is similar to that of Theorem 3.1 and hence is omitted.

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References

[1] *K. Gopalsamy, B.S. Lalli and B.G. Zhang*: Oscillation of odd order neutral differential equations. Czechoslovak Math. J. *42(117)* (1992), 313–323.

[2] *K. Gopalsamy, S.R. Grace and B.S. Lalli*: Oscillation of even order neutral differential equations. Indian J. Math. *35* (1993), 9–25.

[3] *I. Györi and G. Ladas*: "Oscillation Theory of Delay-Differential Equations". Clarendon Press, Oxford, 1991.

[4] *I.T. Kiguradze*: On the question of variability of solutions of nonlinear differential equations. Differentsial'nye Uravneniya *1* (1965), 995–1006; Translation: Differential Equations *1* (1965), 773–782.

[5] *G.S. Ladde, V. Lakshmikantham and B.G. Zhang*: "Oscillation Theory of Differential Equations with Deviating Arguments". Marcel Dekker, INC, New York, 1987.

[6] *N. Parhi and P.K. Mohanty*: Oscillations of neutral differential equations of higher order. Bull. Inst. Math. Acad. Sinica *24* (1996).

[7] *C. Yuanji*: Oscillation in nonautonomous scalar differential equations with deviating arguments. Proc. Amer. Math. Soc. *110* (1990), 711–719.

[8] *B.G. Zhang and K. Gopalsamy*: Oscillations and nonoscillations in higher order neutral equations. J. Math. Phys. Sci. *25* (1991), 152–165.

Author's address: Department of Mathematics, Berhampur University, Berhampur-760007, INDIA.