Imrich Fabrici

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A NOTE ON PRINCIPAL IDEALS AND $\mathcal{J}$-CLASSES IN THE DIRECT PRODUCT OF TWO SEMIGROUPS

Imrich Fabrici, Bratislava

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Let S_1, S_2 be two semigroups, $a \in S_1$, $b \in S_2$, $S_1 \times S_2$ the direct product of these semigroups. $J(a)$ is a principal two-sided ideal in S_1, $J(b)$ is a principal two-sided ideal in S_2, $J(a,b)$ is a principal two-sided ideal in $S_1 \times S_2$. J_a is a $\mathcal{J}$-class containing the element a in S_1, J_b is a $\mathcal{J}$-class containing the element b in S_2, $J_{(a,b)}$ is a $\mathcal{J}$-class containing the element (a,b) in $S_1 \times S_2$.

In the papers [2], [3] among other problems, conditions under which the equalities

$$J(a,b) = J(a) \times J(b); \tag{1}$$

$$J_{(a,b)} = J_a \times J_b \tag{2}$$

hold in $S_1 \times S_2$ have been studied. The following question arises: Does the validity of (1) imply the validity of (2) and vice versa?

The aim of this note is to show that if (1) holds, then also (2) holds. However, if (2) holds, then (1) need not hold.

The investigation of conditions under which the equality (1) holds is divided into two cases:

I. $a \in (S_1a \cup aS_1 \cup S_1aS_1) \wedge b \in (S_2b \cup bS_2 \cup S_2bS_2)$, but $(a,b) \notin [(S_1a \times S_2b) \cup (aS_1 \times bS_2) \cup (S_1aS_1 \times S_2bS_2)]$;

II. $(a,b) \in [(S_1a \times S_2b) \cup (aS_1 \times bS_2) \cup (S_1aS_1 \times S_2bS_2)]$.

Case I may occur in the following ten cases, which are given in Lemma 3 [2]:

1. $[a \in S_1a \wedge a \notin (aS_1 \cup S_1aS_1)] \wedge [b \in bS_2 \wedge b \notin (S_2b \cup S_2bS_2)]$;
2. $[a \in aS_1 \wedge a \notin (S_1a \cup S_1aS_2)] \wedge [b \in S_2b \wedge b \notin (bS_2 \cup S_2bS_2)]$;

3. $[a \in aS_1 \wedge a \notin (S_1a \cup S_1aS_1)] \wedge [b \in S_2bS_2 \wedge b \notin (S_2b \cup bS_2)]$;
4. $[a \in S_1a \wedge a \notin (aS_1 \cup S_1aS_1)] \wedge [b \in S_2bS_2 \wedge b \notin (S_2b \cup bS_2)]$;
5. $[a \in S_1aS_1 \wedge a \notin (S_1a \cup aS_1)] \wedge [b \in S_2b \wedge b \notin (bS_2 \cup S_2bS_2)]$;
6. $[a \in S_1aS_1 \wedge a \notin (S_1a \cup aS_1)] \wedge [b \in bS_2 \wedge b \notin (S_2b \cup S_2bS_2)]$;
7. $[a \in (S_1a \cap S_1aS_1) \wedge a \notin aS_1] \wedge [b \in bS_2 \wedge b \notin (S_2b \cup S_2bS_2)]$;
8. $[a \in (aS_1 \cap S_1aS_1) \wedge a \notin S_1a] \wedge [b \in S_2b \wedge b \notin (bS_2 \cup S_2bS_2)]$;
9. $[a \in aS_1 \wedge a \notin (S_1a \cup S_1aS_1)] \wedge [b \in (S_2b \cap S_2bS_2) \wedge b \notin bS_2]$;
10. $[a \in S_1a \wedge a \notin (aS_1 \cup S_1aS_1)] \wedge [b \in (bS_2 \cup S_2bS_2) \wedge b \notin S_2b]$.

In [2] it has been proved that the equality (1) may occur only in cases 1 and 2 under certain additional conditions. In all the remaining cases $J(a,b) \subset J(a) \times J(b)$ holds. As we want to show that the equality (1) implies the equality (2), we consider from I only cases 1 and 2 and the corresponding additional conditions for the equality (1).

Lemma 1. (See Theorem 1 and Theorem 2 in [2].) (a) *Let*

$$[a \in S_1a \wedge a \notin (aS_1 \cup S_1aS_1)] \wedge [b \in bS_2 \wedge b \notin (S_2b \cup S_2bS_2)].$$

Then $J(a,b) = J(a) \times J(b)$ *iff*

$$[aS_1 = S_1aS_1 \wedge S_1a = P_1 \cup \{a\}] \wedge [S_2b = S_2bS_2 \wedge bS_2 = P_2 \cup \{b\}].$$

(b) *Let*

$$[a \in aS_1 \wedge a \notin (S_1a \cup S_1aS_1)] \wedge [b \in S_2b \wedge b \notin (bS_2 \cup S_2bS_2)].$$

Then $J(a,b) = J(a) \times J(b)$ *iff*

$$[S_1a = S_1aS_1 \wedge aS_1 = P_1 \cup \{a\}] \wedge [bS_2 = S_2bS_2 \wedge S_2b = P_2 \cup \{b\}],$$

where $P_1 = S_1a \cap aS_1$, $P_2 = S_2b \cap bS_2$.

Lemma 2. (Theorem 2 of [3]) *Let*

$$(a,b) \notin [(S_1a \times S_2b) \cup (aS_1 \times bS_2) \cup (S_1aS_1 \times S_2bS_2)].$$

Then

$$J_{(a,b)} = \{(a,b)\}.$$

Lemma 3. *Let any one of cases* (a) *and* (b) *of Lemma* 1 *hold. Then* $J_a = \{a\}$ *in* S_1 *and* $J_b = \{b\}$ *in* S_2.

P r o o f. (a) Suppose that $J(a,b) = J(a) \times J(b)$. Then

$$[aS_1 = S_1aS_1 \wedge S_1a = P_1 \cup \{a\}] \wedge [S_2b = S_2bS_2 \wedge bS_2 = P_2 \cup \{b\}].$$

As $a \in S_1a$, we have $aS_1 \subseteq S_1aS_1$ and $J(a) = S_1a \cup aS_1$, $a \in J_a$. $S_1a = P_1 \cup \{a\}$. If J_a contained more than one element, e.g. if $c \in J_a$, $c \neq a$, then we would have $c \in aS_1$. The relation $a \in S_1$ a implies $c \in S_1aS_1$. Hence we get $J(c) \subseteq S_1aS_1$ and since $c \in J_a$, we have $J(c) = J(a) \subseteq S_1aS_1 = aS_1$. And because $J_a \subseteq J(a) \subseteq aS_1$, it implies $a \in aS_1$, which contradicts the fact $a \notin (aS_1 \cup S_1aS_1)$. Consequently, $J_a = \{a\}$. In a similar way we could show that $J_b = \{b\}$.

(b) The proof of this part is similar to that of part (a).

In Case II, if $(a,b) \in [(S_1a \times S_2b) \cup (aS_1 \times bS_2) \cup (S_1aS_1 \times S_2bS_2)]$, it is necessary to consider several cases:

(i) (a,b) belongs to each component;

(ii) (a,b) belongs to two components only;

(iii) (a,b) belongs to just one component.

In (i) and (ii) and in (iii) provided $(a,b) \in (S_1aS_1 \times S_2bS_2)$ we get $(a,b) \in (S_1aS_2 \times S_2bS_2)$ and by Theorem 5 in [2] we have $J(a,b) = J(a) \times J(b)$ while by Theorem 3 in [3], $J_{(a,b)} = J_a \times J_b$. Hence, from (iii) the following two possibilities remain:

1. $(a,b) \in (S_1a \times S_2b) \wedge (a,b) \notin [(aS_1 \times bS_2) \cup (S_1aS_1 \times S_2bS_2)]$;
2. $(a,b) \in (aS_1 \times bS_2) \wedge (a,b) \notin [(S_1a \times S_2b) \cup (S_1aS_1 \times S_2bS_2)]$.

The relation $(a,b) \notin [(aS_1 \times bS_2) \cup (S_1aS_1 \times S_2bS_2)]$ includes the following possibilities:

1. $[a \notin (aS_1 \cup S_1aS_1)] \wedge [b \in (bS_2 \cap S_2bS_2)]$;
2. $[a \in (aS_1 \cap S_1aS_1)] \wedge [b \notin (bS_2 \cup S_2bS_2)]$;
3. $[a \notin (aS_1 \cup S_1aS_1)] \wedge [d \notin (bS_2 \cup S_2bS_2)]$;
4. $[a \in aS_1 \wedge a \notin S_1aS_1] \wedge [b \notin bS_2 \wedge b \in S_2bS_2]$;
5. $[a \notin aS_1 \wedge a \in S_1aS_1] \wedge [b \in bS_2 \wedge b \notin S_2bS_2]$.

However, 4 and 5 cannot occur, as $(a,b) \in (S_1a \times S_2b)$ and 4, $a \in aS_1$ imply $S_1a \subseteq S_1aS_1$. Since $a \in S_1a \subseteq S_1aS_1$ implies $a \in S_1aS_1$, we arrive at a contradiction with $a \notin S_1aS_1$. Case 5 can be verified similarly. Combining $(a,b) \in (S_1a \times S_2b)$ with each of 1, 2, 3, we get the following three possibilities:

(α) $[a \in S_1a \wedge a \notin (aS_1 \cup S_1aS_1] \wedge [b \in (S_2b \cap bS_2 \cap S_2bS_2)]$;

(β) $[a \in (S_1a \cap aS_1 \cap S_1aS_1)] \wedge [b \in S_2b \wedge b \notin (bS_2 \cup S_2bS_2)]$;

(γ) $[a \in S_1a \wedge a \notin (aS_1 \cup S_1aS_1)] \wedge [b \in S_2b \wedge b \notin (bS_2 \cup S_2bS_2)]$. □

Lemma 4. (See Theorems 6, 7, 8 in [2].) (a) *Let* $[a \in S_1a \wedge a \notin (aS_1 \cup S_1aS_1)] \wedge [b \in (S_2b \cap bS_2 \cap S_2bS_2)]$. *Then* $J(a,b) = J(a) \times J(b)$ *iff* $S_2b = S_2bS_2$.

(b) *Let* $[a \in (S_1 a \cap aS_1 \cap S_1 aS_1)] \wedge [b \in S_2 b \wedge b \notin (bS_2 \cup S_2 bS_2)]$. *Then* $J(a,b) = J(a) \times J(b)$ *iff* $S_1 a = S_1 aS_1$.

(c) *Let* $[a \in S_1 a \wedge a \notin (aS_1 \cup S_1 aS_1)] \wedge [b \in S_2 b \wedge b \notin (bS_2 \cup S_2 bS_2)]$. *Then* $J(a,b) = J(a) \times J(b)$ *iff* $(aS_1 \subseteq S_1 aS_1 \subset S_1 a) \wedge (bS_2 \subseteq S_2 bS_2 \subset S_2 b)$.

Lemma 5. *Let any one of* (a), (b), (c) *from Lemma* 4 *hold. Then*

$$J_{(a,b)} = J_a \times J_b.$$

P r o o f. (a) Let $[a \in S_1 a \wedge a \notin (aS_1 \cup S_1 aS_1)] \wedge [b \in (S_2 b \cap bS_2 \cap S_2 bS_2)]$ and $S_2 b = S_2 bS_2$. From the relations $a \in S_1 a$, $b \in S_2 bS_2$ we get $J(a) = S_1 a \cup S_1 aS_1$ in S_1, $J(b) = S_2 bS_2$ in S_2. Further, $J_a \subseteq J(a) = S_1 a \cup S_1 aS_1$, $J_b \subseteq J(b) = S_2 bS_2$. Let $c \in J_a$. If $c = a$, then $S_1 a = S_1 c$. If $c \# a$, then $J(a) = S_1 a \cup S_1 aS_1 = S_1 c \cup S_1 cS_1 = J(c)$. This implies $c \in (S_1 a \cup S_1 aS_1) \wedge a \in (S_1 c \cup S_1 cS_1)$. $c \notin S_1 aS_1$, since if $c \in S_1 aS_1$, then $S_1 c \subseteq S_1 aS_1$ and $S_1 cS_1 \subseteq S_1 aS_1$. So $J(c) = S_1 c \cup S_1 cS_1 \subseteq S_1 aS_1$. However, $J(a) = J(c)$, so $a \in S_1 aS_1$, which is a contradiction. Similarly, $a \notin S_1 cS_1$. It remains a single possibility: $a \in S_1 c \wedge c \in S_1 a$. From it we have $S_1 a \subseteq S_1 c \wedge S_1 c \subseteq S_1 a$. This implies $S_1 c = S_1 a$. Then $J_a \subseteq S_1 a$ and $J_b \subseteq S_2 bS_2$. Let $(c,d) \in J_a \times J_b$. If we want to show that $J_{(a,b)} = J_a \times J_b$, it is sufficient to show that $J_{(a,b)} = J_{(c,d)}$. Since $(c,d) \in J_a \times J_b = J_c \times J_d$, then $(a,b) \in J_c \times J_d$. $(c,d) \in J_a \times J_b \subseteq (S_1 a \times S_2 bS_2)$. This implies $(S_1 c \times S_2 d) \subseteq (S_1 a \times S_2 bS_2)$, $(cS_1 \times dS_2) \subseteq (S_1 aS_1 \times S_2 bS_2)$, $(S_1 cS_1 \times S_2 dS_2) \subseteq (S_1 aS_1 \times S_2 bS_2)$. Then

$$J(c,d) \subseteq (S_1 a \times S_2 bS_2) \cup (S_1 aS_1 \times S_2 bS_2) = (S_1 a \times S_2 b) \cup (S_1 aS_1 \times S_2 bS_2) = J(a,b).$$

From the relation $(a,b) \in J_c \times J_d \subseteq (S_1 c \times S_2 dS_2)$ we get in a similar way that $J(a,b) \subseteq J(c,d)$. The last relation together with the relation $J(c,d) \subseteq J(a,b)$ gives $J(c,d) = J(a,b)$ and $J_{(a,b)} = J_{(c,d)}$, hence $J_a \times J_b \subseteq J_{(a,b)}$. And because we generally have $J_{(a,b)} \subseteq J_a \times J_b$ (Theorem 1 in [3]), we conclude that

$$J_{(a,b)} = J_a \times J_b.$$

(b) The proof is analogous to that of case (a).

(c) Let $[a \in S_1 a \wedge a \notin (aS_1 \cup S_1 aS_1)] \wedge [b \in S_2 b \wedge b \notin (bS_2 \cup S_2 bS_2)]$ and moreover $(aS_1 \subseteq S_1 aS_1 \subset S_1 a) \wedge (bS_2 \subseteq S_2 bS_2 \subset S_2 b)$. Then $J(a) = S_1 a$ in S_1, $J(b) = S_2 b$ in S_2, $J_a \subseteq J(a) = S_1 a$, $J_b \subseteq J(b) = S_2 b$. Let $(c,d) \in J_a \times J_b$, then $(a,b) \in J_c \times J_d \subseteq J(c) \times J(d)$ and $J(a) = J(c)$ in S_1, $J(b) = J(d)$ in S_2. If $(c,d) \in J_a \times J_b \subseteq J(a) \times J(b) = (S_1 a \times S_2 b)$, then $(S_1 c \times S_2 d) \subseteq (S_1 a \times S_2 b)$, $(cS_1 \times dS_2) \subseteq (S_1 aS_1 \times S_2 bS_2)$, $(S_1 cS_1 \times S_2 bS_2) \subseteq (S_1 aS_1 \times S_2 bS_2) \subseteq (S_1 a \times S_2 b)$, so $J(c,d) \subseteq (S_1a \times S_2b) = J(a,b)$, and hence $J(c,d) \subseteq J(a,b)$. From the relation $(a,b) \in J_c \times J_d \subseteq (S_1c \times S_2d)$ we can obtain in a similar way that $J(a,b) \subseteq J(c,d)$. Both these relations imply $J(a,b) = J(c,d)$, so $J_a \times J_b \subseteq J_{(a,b)}$. And because the inclusion $J_{(a,b)} \subseteq J_a \times J_b$ holds in general, we have

$$J_{(a,b)} = J_a \times J_b.$$

From the possibility 2,

$$[(a,b) \in (aS_1 \times bS_2) \wedge (a,b) \notin (S_1a \times S_2b) \cup (S_1aS_1 \times S_2bS_2)],$$

and from the relation $(a,b) \notin [(S_1a \times S_2b) \cup (S_1aS_1 \times S_2bS_2)]$ we can obtain in a similar way the following three similar cases:

(α') $[a \in aS_1 \wedge a \notin (S_1a \cup S_1aS_1)] \wedge [b \in (S_2b \cap bS_2 \cap S_2bS_2)]$;

(β') $[a \in (S_1a \cap aS_1 \cap S_1aS_1)] \wedge [b \in bS_2 \wedge b \notin (S_2b \cup S_2bS_2)]$;

(γ') $[a \in aS_1 \wedge a \notin (S_1a \cup S_1aS_1)] \wedge [b \in bS_2 \wedge b \notin (S_2b \cup S_2bS_2)]$. □

Lemma 6. (See Theorems 9, 10, 11 in [2].) *Let*

(a) $[a \in aS_1 \wedge a \notin (S_1a \cup S_1aS_1)] \wedge [b \in (S_2b \cap bS_2 \cap S_2bS_2)]$.

Then $J(a,b) = J(a) \times J(b)$ *iff* $bS_2 = S_2bS_2$.

(b) $[a \in (S_1a \cap aS_1 \cap S_1aS_1)] \wedge [b \in bS_2 \wedge b \notin (S_2b \cup S_2bS_2)]$.

Then $J(a,b) = J(a) \times J(b)$ *iff* $aS_1 = S_1aS_1$.

(c) $[a \in aS_1 \wedge a \notin (S_1a \cup S_1aS_1)] \wedge [b \in bS_2 \wedge b \notin (S_2b \cup S_2bS_2)]$.

Then $J(a,b) = J(a) \times J(b)$ *iff* $(S_1a \subseteq S_1aS_1 \subset aS_1) \wedge (S_2b \subseteq S_2bS_2 \subset bS_2)$.

Lemma 7. *Let any one of* (a), (b), (c) *from Lemma* 6 *hold. Then*

$$J_{(a,b)} = J_a \times J_b.$$

The proof is analogous to that of Lemma 5.

From Lemmas 3, 5 and 7 we obtain

Theorem 1. *Let* $(a,b) \in S_1 \times S_2$. *If* $J(a,b) = J(a) \times J(b)$, *then*

$$J_{(a,b)} = J_a \times J_b.$$

Now we are going to show that the equality (2) *does not imply the validity of the equality* (1).

Example 1. Let $S_1 = \{a_1, a_2, a_3, a_4\}$, $S_2 = \{b_1, b_2, b_3, b_4\}$ be two semigroups and let binary associative operations be given by the following tables:

	a_1	a_2	a_3	a_4
a_1	a_1	a_2	a_1	a_2
a_2	a_2	a_1	a_2	a_1
a_3	a_1	a_2	a_1	a_2
a_4	a_2	a_1	a_2	a_3

	b_1	b_2	b_3	b_4
b_1	b_1	b_1	b_1	b_1
b_2	b_1	b_1	b_1	b_1
b_3	b_1	b_1	b_1	b_1
b_4	b_1	b_1	b_1	b_2

$J(a_4) = \{a_1, a_2, a_3, a_4\}$ in S_1, $J_{a_4} = \{a_4\}$, $J(b_4) = \{b_1, b_2, b_4\}$ in S_2, $J_{b_4} = \{b_4\}$. For the element (a_4, b_4) we have $(a_4, b_4) \notin [(S_1a_4 \times S_2b_4) \cup (a_4S_1 \times b_4S_2) \cup (S_1a_4S_1 \times S_2b_4S_2)]$, hence $J_{(a_4,b_4)} = \{(a_4, b_4)\}$ and $J_{a_4} \times J_{b_4} = \{(a_4, b_4)\}$, therefore $J_{(a_4,b_4)} = J_{a_4} \times J_{b_4}$. But $J(a_4, b_4) = (a_4, b_4) \cup (S_1a_4 \times S_2b_4) \cup (a_4S_1 \times b_4S_2) \cup (S_1a_4S_1 \times S_2b_4S_2) = \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2), (a_3, b_1), (a_3, b_2), (a_4, b_4)\}$ and $J(a_4) \times J(b_4) = \{a_1, a_2, a_3, a_4\} \times \{b_1, b_2, b_4\}$, so $J(a_4, b_4) \subset J(a_4) \times J(b_4)$. In this example $J_{a_4} \times J_{b_4} = \{(a_4, b_4)\}$.

Let us suppose that $|J_a \times J_b| > 1$. The question if at least in this case the validity of the equality (2) does not imply the validity of the equality (1). We will show that even in this case it need not be so.

Example 2. Let $S_1 = \{a_1, a_2, a_3, a_4\}$, $S_2 = \{b_1, b_2, b_3, b_4\}$ be two semigroups and let binary associative operations be given by the following tables:

	a_1	a_2	a_3	a_4
a_1	a_1	a_1	a_1	a_1
a_2	a_1	a_1	a_1	a_2
a_3	a_1	a_1	a_1	a_1
a_4	a_1	a_1	a_3	a_4

	b_1	b_2	b_3	b_4
b_1	b_1	b_2	b_2	b_2
b_2	b_1	b_2	b_2	b_2
b_3	b_1	b_2	b_3	b_4
b_4	b_1	b_2	b_4	b_3

$a_3 \in S_1a_3$, $J(a_3) = \{a_1, a_3\}$, $J_{a_3} = \{a_3\}$, $b_4 \in S_2b_4$, $J(b_4) = \{b_1, b_2, b_3, b_4\}$, $J_{b_4} = \{b_3, b_4\}$, so $|J_{a_3} \times J_{b_4}| > 1$ and $J_{(a_3,b_4)} = J_{a_3} \times J_{b_4}$. However,

$$\begin{aligned} J(a_3, b_4) &= \{a_1, a_3\} \times \{b_2, b_3, b_4\} \cup \{a_1\} \times \{b_1, b_2, b_3, b_4\} = \\ &= \{a_1, a_3\} \times \{b_2, b_3, b_4\} \cup \{(a_1, b_1)\}, \\ J(a_3) \times J(b_4) &= \{a_1, a_3\} \times \{b_1, b_2, b_3, b_4\}. \end{aligned}$$

$(a_3, b_1) \in J(a_3) \times J(b_4)$, but $(a_3, b_1) \notin J(a_3, b_4)$, hence

$$J(a_3, b_4) \subset J(a_3) \times J(b_4).$$

Finally, we would like to show what are the answers to the above questions in the case of one-sided principal ideals and the corresponding classes. We will consider left principal ideals and $\mathscr{L}$-classes.

Lemma 8. (Theorem 1 of [4]) *Let* $(a,b) \in S_1 \times S_2$. *Then* $L(a,b) = L(a) \times L(b)$ *iff at least one of the following conditions is satisfied:*

1. $S_1 a = \{a\}$;
2. $S_2 b = \{b\}$;
3. $a \in S_1 a \wedge b \in S_2 b$.

Lemma 9. (Theorem 3 of [4]) *Let* $(a,b) \in S_1 \times S_2$. *Then* $L_{(a,b)} = L_a \times L_b$ *iff at least one of the following conditions is satisfied:*

1. $L_a = \{a\}$ *in* S_1, $L_b\{b\}$ *in* S_2;
2. $a \in S_1 a \wedge b \in S_2 b$.

Theorem 2. *Let* $(a,b) \in S_1 \times S_2$. *If* $L(a,b) = L(a) \times L(b)$, *then* $L_{(a,b)} = L_a \times L_b$.

Proof. It is sufficient to show that if any one of the conditions of Lemma 8 holds, then at least one condition of Lemma 9 is satisfied. Let 1 of Lemma 8 hold, so $S_1 a = \{a\}$. For $b \in S_2$ there are only two possibilities: (i) $b \in S_2 b$, (ii) $b \notin S_2 b$. If $b \in S_2 b$ and $S_1 a = \{a\}$, then $a \in S_1 a \wedge b \in S_2 b$, so 2 of Lemma 9 holds. If $b \notin S_2 b$, then by Lemma 2 [4] $L_b = \{b\}$. $S_1 a = \{a\}$ implies that $L_a = \{a\}$. So $L_a \times L_b = \{(a,b)\}$ and since $L_{(a,b)} \subseteq L_a \times L_b$, we get

$$L_{(a,b)} = L_a \times L_b.$$

If 2 of Lemma 8 holds, we can proceed analogously. If 3 of Lemma 8 holds, then 2 of Lemma 9 holds, as well.

However, if $L_{(a,b)} = L_a \times L_b$ in $S_1 \times S_2$, then in general the equality $L(a,b) = L(a) \times L(b)$ need not hold. □

Example 3. Let $S_1 = \{a_1, a_2, a_3, a_4\}$, $S_2 = \{b_1, b_2, b_3, b_4\}$ be two semigroups and let binary associative operations be given by the following tables:

	a_1	a_2	a_3	a_4
a_1	a_1	a_1	a_1	a_1
a_2	a_1	a_1	a_1	a_2
a_3	a_1	a_1	a_1	a_1
a_4	a_2	a_1	a_3	a_4

	b_1	b_2	b_3	b_4
b_1	b_1	b_1	b_1	b_1
b_2	b_1	b_1	b_1	b_2
b_3	b_1	b_1	b_1	b_3
b_4	b_1	b_1	b_1	b_4

$L(a_2) = \{a_1, a_2\}$, $L_{a_2} = \{a_2\}$. $L(b_2) = \{b_1, b_2\}$, $L_{b_2} = \{b_2\}$. $L_{(a_2,b_2)} = \{(a_2,b_2)\} = L_{a_2} \times L_{b_2}$. However, $L(a_2,b_2) = (a_2,b_2) \cup (S_1 a_2 \times S_2 b_2) = \{(a_1,b_1), (a_2,b_2)\}$ and

$$L(a_2) \times L(b_2) = \{a_1, a_2\} \times \{b_1, b_2\}, \text{ so}$$
$$L(a_2,b_2) \subset L(a_2) \times L(b_2).$$

However, unlike in the case of $\mathcal{J}$-classes, for $\mathcal{L}$-classes the following holds:

Theorem 3. *Let $L_{(a,b)} = L_a \times L_b$ in $S_1 \times S_2$ and let $|L_a \times L_b| > 1$. Then*

$$L(a,b) = L(a) \times L(b).$$

P r o o f. Let $L_{(a,b)} = L_a \times L_b$ and $|L_a \times L_b| > 1$. We shall consider three cases:
1. $|L_a| > 1, \wedge |L_b| > 1$;
2. $|L_a| > 1, \wedge |L_b| = \{b\}$;
3. $|L_a| = \{a\} \wedge |L_b| > 1$.

If 1 holds then Theorem 4 [4] implies $a \in S_1 a \wedge b \in S_2 b$ and by Lemma 8 we have $L(a,b) = L(a) \times L(b)$.

If 2 holds, then $b \notin S_2 b$ cannot occur, since in this case $L_a \times L_b$ is the union of at least two mutually different $\mathcal{L}$-classes (Theorem 5 [4]), which contradicts our hypothesis. So $b \in S_2 b$ must hold and together with $a \in S_1 a$ this that 3 of Lemma 8 is satisfied, so

$$L(a,b) = L(a) \times L(b).$$

If 3 holds, we proceed analogously. □

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Author's address: Department of Mathematics, Slovak Technical University, Radlinského 9, 812 37 Bratislava, Slovakia.