Czechoslovak Mathematical Journal, 50 (125) (2000), 59–65

ON SETS WITH BAIRE PROPERTY IN TOPOLOGICAL SPACES

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(Received February 6, 1997)

Abstract. Steinhaus [9] prove that if a set A has a positive Lebesgue measure in the line then its distance set contains an interval. He obtained even stronger forms of this result in [9], which are concerned with mutual distances between points in an infinite sequence of sets. Similar theorems in the case we replace distance by mutual ratio were established by Bose-Majumdar [1]. In the present paper, we endeavour to obtain some results related to sets with Baire property in locally compact topological spaces, particular cases of which yield the Baire category analogues of the above results of Steinhaus [9] and their corresponding form for ratios by Bose-Majumdar [1].

Keywords: Baire property, first category, second category

MSC 2000: 28A05

Introduction

Let $A, B \subseteq \mathbb{R}$ (the real line). The distance set of A and B written as $D(A, B)$ is the set of all distances $|x - y|$ between points x and y, where $x \in A$, $y \in B$. If A, B ($\subseteq \mathbb{R} \setminus \{0\}$), we define in an analogous way their ratio set $R(A, B)$ as the set of all possible ratios $\frac{x}{y}$ or $\frac{y}{x}$ where $x \in A$, $y \in B$.

Steinhaus [9] showed that $D(A, B)$ contains an interval if A and B are both Lebesgue measurable with measures $m(A)$, $m(B)$ (> 0). Exactly analogous formulations related to $R(A, B)$ were produced by Bose-Majumdar [1].

In the same paper [9] (dealing only with subsets of the real line $\mathbb{R}$), Steinhaus proved even stronger theorems. They are:

Theorem X. *If $\{A_n\}_{n=1}^{\infty}$ is any infinite sequence of Lebesgue measurable sets with positive measures, then there exists an infinite sequence $\{a_n\}_{n=1}^{\infty}$ of distinct*

The research is supported by the research grant of C.S.I.R.

points such that $a_n \in A_n$ $(n = 1, 2, 3)$ and their mutual distances are all rational numbers.

Theorem XI. *E being any infinite Lebesgue measurable set, there exists an enumerable set P composed of points whose mutual distances are rational numbers, and a set Z of measure zero such that $P \subseteq A \subseteq P' \cup Z$, where P' represents the derived set of P.*

The fact that Theorems X and XI in the form as stated above hold equally well when sets are taken with non-zero abscissae and mutual distance between points is replaced by mutual ratio was established by Bose Majumdar (Theorems X and XI, [1]).

A set $A \subseteq \mathbb{R}$ is said to possess the Baire property [6] if it can be represented as the symmetric difference $G \triangle P$ of an open set G and a set P of first category in $\mathbb{R}$. Equivalently, A has the Baire property if $A = (G \setminus P) \cup Q$, where G is open and P, Q are first category sets in $\mathbb{R}$. However, the above definition could be translated unequivocally to any topological space.

Steinhaus theorem on the distance set has an exact analogue in the realm of Baire category (that is, with sets having the Baire property) and is due to Piccard [7]. He showed that if A and B are second category subsets of $\mathbb{R}$ with Baire property, then their distance set $D(A, B)$ contains an interval. If A and B are taken with non-zero abscissae, the corresponding form for ratio sets (with greater generality) has also been obtained [3]. Piccard's theorem has been generalized by K.P.S. and M. Bhaskara Rao [2] for sets in a topological group, and in topological vector spaces by Z. Kominek [4]. In [8], Sander extended Piccard's result to sets in an arbitrary topological space with reference to the classes of globally solvable mappings.

The above two theorems of Steinhaus (Theorems X and XI) have also been obtained in more general forms in the categorical setting. Alongside with the measure-theoretic results these were set forth by Miller, Xenikakis and Polychronis [5] for sets with the Baire property in the real line in the light of certain specified classes of mappings $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ each of which have continuous first order partial derivatives f_x and f_y non-vanishing on an open set containing $A \times A$ where A is the union of the sets in the sequence $\{A_n\}_{n=1}^{\infty}$. Particular cases of these results were also proved in the corresponding form of Theorems X and XI for ratios by Bose-Majumdar [1] Even in spite of these facts, Miller's theorem has some essential drawbacks. Although it proves that whenever C is given to be a dense subset of $\mathbb{R}$, there exist points $a_i \in A_i$ such that $f(a_i, a_{i+1}) \in C$ $(i = 1, 2, 3, \ldots)$, it fails to ensure that $f(ai, aj) \in C$ for all i, j. In this paper, we establish that if X is an arbitrary locally compact Hausdorff topological space and f is chosen arbitrarily from the class of globally solvable mappings $f\colon X \times X \to X$, then preferably better-organized extensions of Steinhaus theorems (Theorem X and Theorem XI) for sets with the Baire property can be found, particular cases of which also yield the category analogues of their corresponding form (with ratios) by Bose-Majumdar [1].

Let X be any topological space. Let $f\colon X \times X \to X$ and define $f_x\colon X \to X$ and $f^y\colon X \to X$ by $f_x(y) = f(x,y)$ and $f^y(x) = f(x,y)$ for all $x, y \in X$. The functions f_x and f^y are respectively the x and y sections of the function f. Then f is said to be globally solvable [8], if there exist two continuous functions ψ, $\varphi\colon X \times X \to X$ such that $f(x,y) = z$ is equivalent to $x = \psi(y,z)$ and $y = \varphi(x,y)$ for all x, y, z $(\in X)$. It follows that f_x, f^y, ψ^z, φ^z are homeomorphisms. Form now on, we will consider our space X to be a locally compact Hausdorff topological space, and denote by the symbols $A \setminus B$, $A \triangle B$ and $\overline{A}$ the difference, symmetric difference of two sets and the closure of any set in X. A set A with the Baire property will be defined likewise as above if $A = G \triangle P$, where G is open and P is of the first category in X. Equivalently, A has the Baire property if it can be expressed as $(G \setminus P) \cup Q$ where G is open and P, Q are first category sets in X.

Now, given D to be any dense subset of X, we have the following result.

Theorem 1. *Let $\{A_n\}_{n=1}^{\infty}$ be any sequence of second category sets with the Baire property in X and let $f\colon X \times X \to X$ be a globally solvable mapping. Then there exist infinite sequences $\{a_n\}_{n=1}^{\infty}$ and $\{\eta_n\}_{n=1}^{\infty}$ of distinct points in X such that $a_n \in A_n$, $\eta_n \in D$ $(n = 1,2,3,\ldots)$ and the relation $f(a_1,\eta_1) = f(a_2,\eta_2) = \ldots = f(a_n,\eta_n) = \ldots$ is satisfied.*

Lemma. *If A and B are non-empty open subsets of X, then there exists $\eta \in D$ such that $A_n \cap f^{\eta}(B) \neq \emptyset$.*

P r o o f. Let $x \in B$ and $z \in A$. Since f is globally solvable, there exists $y \in X$ such that $f(x,y) = z$ (evidently $y = \varphi(x,z)$). Consequently, $f(x,y) \in A$. Now as D is dense in X, A is open and f is globally solvable, there exists $\eta \in D$ such that $f(x,\eta) \in A$. Hence $A \cap f^{\eta}(B) \neq \emptyset$. □

Note. The above lemma does not require X to be locally compact Hausdorff.

P r o o f of the Theorem. We choose a compact subset C_0 of X with non-empty interior G_0 (this choice is justified since X is locally compact) and let P_0 be an arbitrary set of the first category in X. We set $A_0 = G_0 \setminus P_0$. Since A_n are second category sets with the Baire property, we can write $A_n = (G_n \setminus P_n) \cup Q_n$ where for each n, G_n is a non-empty open set and P_n, Q_n are first category sets in X $(n = 1,2,3,\ldots)$. Again by the definition of first category sets, we may write $P_n = \bigcup\limits_{j=1}^{\infty} F_j^{(n)}$ $(n = 0,1,2,3,\ldots)$ where $F_j^{(n)}$ are nowhere dense subsets of X.

We consider the set $G_0 \setminus \overline{F_1^{(0)}}$ and choose and fix $\eta_0 \in D$. Since the closure of any nowhere dense set is again nowhere dense, the set $G_0 \setminus \overline{F_1^{(0)}}$ is non-empty open and therefore as f is globally solvable, $f^{\eta_0}(G_0 \setminus \overline{F_1^{(0)}})$ is non-empty open. Also by regularity of X (X being locally compact Hausdorff, it is regular) there exists a non-empty open set H_{00} such that $H_{00} \subseteq \overline{H_{00}} \subseteq f^{\eta_0}(G_0 \setminus \overline{F_1^{(0)}})$.

Now again, as $\overline{F_2^{(0)}}$ is closed and nowhere dense and f is globally solvable, $H_{00} \cap f^{\eta_0}(G_0 \setminus \overline{F_2^{(0)}})$ is a non-empty open set. We set $D_1 = D \setminus \{\eta_0\}$. Since X is Hausdorff, D_1 is again dense in X. Since $\overline{F_1^{(1)}}$ is closed and nowhere dense, by the above lemma there exists $\eta_1 \in D_1$ such that $H_{00} \cap f^{\eta_0}(G_0 \setminus \overline{F_2^{(0)}}) \cap f^{\eta_1}(G_1 \setminus \overline{F_1^{(1)}})$ is a non-empty open set. Consequently, by regularity of X there exists a non-empty open set H_{11} such that $H_{11} \subseteq \overline{H_{11}} \subseteq H_{00} \cap f^{\eta_0}(G_0 \setminus \overline{F_2^{(0)}}) \cap f^{\eta_1}(G \setminus \overline{F_1^{(1)}})$.

Now again, as $\overline{F_3^{(0)}}$ and $\overline{F_2^{(1)}}$ are closed and nowhere dense sets and f is globally solvable, $H_{11} \cap f^{\eta_0}(G_0 \setminus \overline{F_3^{(0)}}) \cap f^{\eta_1}(G_1 \setminus \overline{F_2^{(1)}})$ is a non-empty open set. We set $D_2 = D_1 \setminus \{\eta_1\}$. Since X is Hausdorff, D_2 is again dense in X. Since $\overline{F_1^{(2)}}$ is closed and nowhere dense, by the above lemma there exists $\eta_2 \in D_2$ such that $H_{11} \cap f^{\eta_0}(G_0 \setminus \overline{F_3^{(0)}}) \cap f^{\eta_1}(G_1 \setminus \overline{F_2^{(1)}}) \cap f^{\eta_2}(G_2 \setminus \overline{F_1^{(2)}})$ is a non-empty open set. Consequently, by regularity of X, there exists a non-empty open set H_{22} such that $H_{22} \subseteq \overline{H_{22}} \subseteq H_{11} \cap f^{\eta_0}(G_0 \setminus \overline{F_3^{(0)}}) \cap f^{\eta_1}(G_1 \setminus \overline{F_2^{(1)}}) \cap f^{\eta_2}(G_2 \setminus \overline{F_1^{(2)}})$.

Proceeding likewise on the same arguments as stated above, at the n-th stage we get a non-empty open set $H_{n-1,n-1}$ such that $H_{n-1,n-1} \subseteq \overline{H_{n-1,n-1}} \subseteq H_{n-2,n-2} \cap f^{\eta_0}(G_0 \setminus \overline{F_n^{(0)}}) \cap f^{\eta_1}(G_1 \setminus \overline{F_{n-1}^{(1)}}) \cap f^{\eta_2}(G_2 \setminus \overline{F_{n-2}^{(2)}}) \cap \ldots \cap f^{\eta_{n-1}}(G_{n-1} \setminus \overline{F_1^{(n-1)}})$ where $\eta_0, \eta_1, \eta_2, \ldots, \eta_{n-1}$ are distinct elements in D. The following description may be helpful for a better understanding of the above process. Thus if we continue indefinitely, we get a decreasing sequence $\overline{H_{00}}, \overline{H_{11}}, \overline{H_{22}}, \ldots, \overline{H_{n-1,n-1}}, \overline{H_{nn}}$, of non-empty closed sets each of which is contained in the compact set C_0 we started with, which has the finite intersection property. Consequently, $\bigcap_{n=1}^{\infty} \overline{H_{nn}} \neq \emptyset$ and hence after a rearrangement we have $\{f^{\eta_0}(G_0 \setminus \overline{F_1^{(0)}}) \cap f^{\eta_0}(G_0 \setminus \overline{F_2^{(0)}}) \cap f^{\eta_0}(G_0 \setminus \overline{F_3^{(0)}}) \cap f^{\eta_0}(G_0 \setminus \overline{F_n^{(0)}}) \ldots\} \cap \{f^{\eta_1}(G_1 \setminus \overline{F_1^{(1)}}) \cap f^{\eta_1}(G_1 \setminus \overline{F_2^{(1)}}) \cap f^{\eta_1}(G_1 \setminus \overline{F_3^{(1)}}) \cap f^{\eta_1}(G_1 \setminus \overline{F_n^{(1)}}) \cap \ldots\} \cap \{f^{\eta_n}(G_n \setminus \overline{F_1^{(n)}}) \cap f^{\eta_n}(G_n \setminus \overline{F_2^{(n)}}) \cap \ldots \cap f^{\eta_n}(G_n \setminus \overline{F_n^{(n)}}) \ldots\} \cap \ldots \neq \emptyset$. Since f is globally solvable, the above arrangement may be written in a more compact form, viz. $f^{\eta_0}(G_0 \setminus \bigcup_{j=1}^{\infty} \overline{F_j^{(0)}}) \cap f^{\eta_1}(G_1 \setminus \bigcup_{j=1}^{\infty} \overline{F_j^{(1)}}) \cap \ldots \cap f^{\eta_n}(G_n \setminus \bigcup_{j=1}^{\infty} \overline{F_j^{(n)}}) \cap \ldots \neq \emptyset$. Therefore $f^{\eta_0}(G_0 \setminus P_0) \cap f^{\eta_1}(G_1 \setminus P_1) \cap \ldots \cap f^{\eta_n}(G_n \setminus P_n) \cap \ldots \neq \emptyset$. Hence $f^{\eta_1}(A_1) \cap f^{\eta_2}(A_2) \cap f^{\eta_n}(A_n) \cap \ldots \neq \emptyset$.

Now f being globally solvable, there exists a sequence $\{a_n\}_{n=1}^{\infty}$ of distinct points in X such that $a_n \in A_n$ $(n = 1, 2, 3, \ldots)$ and relation

$$f(a_1, \eta_1) = f(a_2, \eta_2) = f(a_3, \eta_3) = \ldots,$$

is satisfied where $\{\eta_n\}_{n-1}^{\infty}$ is a sequence of distinct points in D. □

In particular, if we set (i) $X = \mathbb{R}$ (with its usual topology), $D = Q$ (the set of rational numbers) and $f(x, y) = x + y (x, y \in \mathbb{R})$ and (ii) $X = \mathbb{R} \setminus \{0\}$ (with the induced topology), $D = Q \setminus \{0\}$ and $f(x, y) = xy$ $(x, y \in \mathbb{R} \setminus \{0\})$, we get as particular cases of the above theorem the following corollaries.

Corollary 1.1. *If $\{A_n\}_{n-1}^{\infty}$ is any infinite sequence of second category sets in $\mathbb{R}$ with the Baire property, then there exists an infinite sequence $\{a_n\}_{n-1}^{\infty}$ of distinct points such that $a_n \in A_n$ $(n = 1, 2, 3, \ldots)$ and their mutual distances are all elements in Q.*

Corollary 1.2. *If $\{A_n\}_{n=1}^{\infty}$ is any infinite sequence of second category sets in $\mathbb{R}$ with non-zero abscissae and the Baire property, then there exists an infinite sequence $\{a_n\}_{n=1}^{\infty}$ of distinct points such that $a_n \in A_n$ $(n = 1, 2, 3,)$ and their mutual ratios are all elements in Q.*

The above two results are the Baire category analogues of Theorem X of Steinhaus (stated above) and of its corresponding form (for ratios) by Bose-Majumdar (Theorem X, [1]).

In our next theorem, in addition to the local compactness and Hausdorff property, we assume that the underlying space X is also second countable without having any isolated points. As before, here D is again any dense subset of X.

Theorem 2. *Let A be any non-empty set with the Baire property and let $f\colon X \times X \to X$ be globally solvable. Then there exists an enumerable set $P = \{a_1, a_2, a_3, \ldots\}$ and a set H of the first category in X such that $P \subseteq A \subseteq P' \cup H$, where P' is the derived set of P and $\{\eta_n\}_{n=1}^{\infty}$ is a sequence of points in D for which the relation*

$$f(a_1, \eta_1) = f(a_2, \eta_2) = \ldots = f(a_n, \eta_n) = \ldots$$

is satisfied.

P r o o f of the Theorem. If A is of the first category, the choice of any enumerable set for which the above relation is satisfied shall meet our purpose (the reader may note that in this case we may choose $\{a_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ as constant sequences). Now let us suppose that A is of the second category. Let $\{U_n\}_{n=1}^{\infty}$ denote a countable base in X. From the sets $U_n \cap A$ $(n = 1, 2, 3,)$, we suppress those terms that represent sets of the first category, being left thereby with a subsequence $\{U_{n_k} \cap A\}_{k=1}^{\infty}$ whose terms represent sets of the second category. They having also the Baire property we may choose by virtue of Theorem 1 sequences $\{a_k\}_{k=1}^{\infty}$ and $\{\eta_k\}_{k=1}^{\infty}$ of distinct points in X such that $a_k \in A \cap U_{n_k}$, $\eta_k \in D$ $(k = 1, 2, 3,)$, for which the relation

$$f(a_1, \eta_1) = f(a_2, \eta_2) = \ldots = f(a_k, \eta_k) = \ldots$$

is satisfied.

Now let H be the union of those sets in the sequence $\{U_n \cap A\}_{n=1}^{\infty}$ that have been suppressed and set $P = \{a_1, a_2, \ldots, a_k, \ldots\}$. Clearly H is of the first category, P is countable and $P \subseteq A$. To complete the proof we need only to show that $A \subseteq P' \cup H$. Let $\xi \in A \setminus H$. Since none of the sets from the sequence $\{A \cap U_n\}_{n=1}^{\infty}$ that contains ξ has been suppressed there exists a subsequence $\{U_{n_{k_j}}\}_{j=1}^{\infty}$ of $\{U_{n_k}\}_{k=1}^{\infty}$ such that $\xi \in A \cap U_{n_{k_j}}$ $(j = 1, 2, 3, \ldots)$. Hence $\xi \in P'$ and therefore $A \subseteq P' \cup H$. Hence the theorem. □

Note. One may note from the proof of the above theorem that if A is taken to be a set of the second category, then the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{\eta_n\}_{n=1}^{\infty}$ can be chosen so as to consist of distinct points.

Corollary 2.1. *If E is any non-empty set in $\mathbb{R}$ with the Baire property, then there exists an enumerable set P composed of points whose mutual distances are all elements in Q, and a set H of the first category such that $P \subseteq A \subseteq P'UH$, where P' is the derived set of P.*

Corollary 2.2. *If A is any non-empty set in $\mathbb{R}$ with non-zero abscissae and the Baire property, then there exists an enumerable set P composed of points whose mutual ratios are all elements in Q, and a set H of the first category such that $P \subseteq A \subseteq P'UH$, where P' is the derived set of P.*

The above two results are the Baire category analogues of Theorem XI of Steinhaus (stated above) and of its corresponding form (for ratios) by Bose-Majumdar (Theorem XI, [1]).

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