Michael R. Darnel

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σ-INTERPOLATION LATTICE-ORDERED GROUPS

Michael R. Darnel, South Bend

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Abstract. In [1], Jakubík showed that the class of σ-interpolation lattice-ordered groups forms a radical class, but left open the question of whether the class forms a torsion class. In this paper, we show that this class does indeed form a torsion class.

A radical class of lattice-ordered groups is a class $\mathcal{C}$ closed under two operations:

(i) If $G \in \mathcal{C}$ and A is a convex ℓ-subgroup of G, then $A \in \mathcal{C}$; and

(ii) If $\{A_\lambda\}_\Lambda$ is a set of convex ℓ-subgroups of an ℓ-group G such that for all $\lambda \in \Lambda$, $A_\lambda \in \mathcal{C}$, then $\bigvee\limits_{\Lambda} A_\lambda \in \mathcal{C}$.

A *torsion class* of lattice-ordered groups is a radical class that is also closed with respect to ℓ-homomorphic images.

A lattice-ordered group G has the *σ-interpolation property* if for any countable subsets $A = \{a_n\}$ and $B = \{b_n\} \subseteq G$ such that for any $a_m \in A$ and any $b_n \in B$, $a_m \leqslant b_n$, then there exists $h \in G$ such that for all positive integers n, $a_n \leqslant h \leqslant b_n$. In [J] Jakubík proved that the class of all σ-interpolation lattice-ordered group forms a radical class.

Theorem. *The class of σ-interpolation lattice-ordered groups forms a torsion class.*

P r o o f. Let K be an ℓ-ideal of an ℓ-group G such that G has the σ-interpolation property. Let $\{a_m\}$, $\{b_n\} \subset G/K$ be sequences such that for every a_m and every b_n, $a_m \leqslant b_n$. For each K-coset $[g]$ in $\{a_m\} \cup \{b_n\}$, choose a representative d. Since $\{a_m\} \cup \{b_n\}$ is countable, we can enumerate its elements as $\{Kd_1, Kd_2, \ldots\}$.

Let $d_1' = d_1$.

Now let n be a positive integer such that for all $1 \leqslant i \leqslant n$, d'_i has been chosen such that whenever $1 \leqslant i, j \leqslant n$ and $Kd_i \in \{a_m\}$ and $Kd_j \in \{b_n\}$, $d'_i \leqslant d'_j$. Let $A'_n = \{i\colon 1 \leqslant i \leqslant n \text{ and } Kd'_i \in \{a_m\}\}$; define B'_n similarly.

Now if $Kd_{n+1} \in \{a_m\}$, then for all $j \in B_n$, there exists $k_j \in K$ such that $k_j d_{n+1} \leqslant d'_j$. In this case, let $d'_{n+1} = \Big(\bigwedge_{j \in B_n} k_j\Big) d_{n+1}$. Then $Kd'_{n+1} = Kd_{n+1}$ and $d'_{n+1} \leqslant d'_j$ for all $j \in B_n$. If $Kd_{n+1} \in \{b_n\}$, we similarly find, for all $i \in A_n$, k_i such that $d'_i \leqslant k_i d_{n+1}$, and let $d'_{n+1} = \Big(\bigvee_{i \in A_n} k_i\Big) d_{n+1}$.

Continue in this way until we exhaust $\{a_m\} \cup \{b_n\}$. Let $A' = \{d'_n\colon Kd'_n \in \{a_m\}\}$ and $B' = \{d'_n\colon Kd'_n \in \{b_n\}\}$, and enumerate both by the induced enumeration from $\{a_m\} \cup \{b_n\}$: $Ka'_i = a_i$ for all i, and $Kb'_j = b_j$ for all j. We obtain that $\{a'_m\}$ and $\{b'_n\}$ are sequences of G such that for any a'_m and any b'_n, $a'_n \leqslant b'_n$. By the σ-interpolation property, there exists $h \in G$ such that $a'_n \leqslant h \leqslant b'_n$.

But then $Ka_n \leqslant Kh \leqslant Kb_n$. So ℓ-homomorphic images also have the σ-interpolation property. □

References

[1] *Jakubík, J.*: On some completeness properties for lattice-ordered groups. Czechoslovak Math. J. *45 (120)* (1995), 253–266.

Author's address: Department of Mathematics, Indiana University South Bend, e-mail: `mdarnel@mathcs.iusb.edu`.