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CONSTRUCTIONS FOR TYPE I TREES WITH NONISOMORPHIC PERRON BRANCHES

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Abstract. A tree is classified as being type I provided that there are two or more Perron branches at its characteristic vertex. The question arises as to how one might construct such a tree in which the Perron branches at the characteristic vertex are not isomorphic. Motivated by an example of Grone and Merris, we produce a large class of such trees, and show how to construct others from them. We also investigate some of the properties of a subclass of these trees. Throughout, we exploit connections between characteristic vertices, algebraic connectivity, and Perron values of certain positive matrices associated with the tree.

1. Introduction and preliminaries

A *weighted graph* G consists of an undirected graph, and a collection of positive numbers such that each edge of the graph is associated with one of those positive numbers; if e is an edge and is associated with the number $\theta > 0$, we refer to θ as the *weight of* e. In the case that all of the weights are equal to 1, G is called an *unweighted graph.* For a weighted graph G on vertices labelled $1, \ldots, n$, the *Laplacian matrix of* G is the $n \times n$ matrix L with

$$L_{ij} = \begin{cases} -\theta, \text{ if } i \neq j \text{ and } i - j \text{ is an edge of } G \text{ with weight } \theta, \\ 0, \text{ if } i \neq j \text{ and } i - j \text{ is not edge of } G, \\ \text{the sum of the weights of the edges incident with } i, \text{ if } i = j. \end{cases}$$

It is well-known that L is a symmetric positive semi-definite M-matrix, and that if G is connected (which we will henceforth take to be the case), then the nullity of L is 1, and the null space of L is spanned by the all ones vector, 1_n. The second smallest

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eigenvalue of L is known as the *algebraic connectivity of* G (see [1]), and it has been the object of a good deal of study over the last two decades (see, for example the survey of Merris [5] for a list of references). The study of algebraic connectivity of weighted trees has been especially fruitful, and in particular, Fiedler [2] provides the following result, which classifies trees according to whether there is an eigenvector corresponding to the algebraic connectivity which has a zero entry.

Proposition 1. (Fiedler [2]) *Let T be a weighted tree, and let v be an eigenvector corresponding to the algebraic connectivity of T. Then one of the following holds:*

(i) *Some entry of v is zero. In this case there is a unique vertex k with $v_k = 0$ such that k is adjacent to a vertex l with $v_l \neq 0$. Further, along any path starting at vertex k, the corresponding entries in v are either increasing, decreasing, or identically 0. In this case, T is called a Type* I *tree, and k is called the characteristic vertex of T.*

(ii) *No entry of v is zero. In this case there is a unique pair of adjacent vertices i and j such that $v_i > 0 > v_j$. Further, along any path starting at vertex i and not passing through vertex j, the corresponding entries in v are increasing, while along any path starting at vertex j and not passing through vertex i, the corresponding entries in v are decreasing. In this case, T is called a Type* II *tree, and the vertices i and j are called the characteristic vertices of T.*

Merris [6] has shown that in fact, the identification of both the tree type and its characteristic vertices is independent of the choice of the eigenvector v.

Another approach to trees and their characteristic vertices is given by Kirkland, Neumann and Shader [4] (indeed that is the approach which will be employed in this paper). In order to describe it, we need some notation and terminology. We will denote the $k \times k$ all ones matrix by J_k, suppressing the subscript whenever the order is clear from the context. Let G be a connected weighted graph with Laplacian matrix L. If C is a subset of the vertices of G, then $L(C)$ denotes the principal submatrix of L corresponding to the vertices of C. For a vertex v of a weighted tree T, a *branch at* v is one of the connected components of $L \setminus \{v\}$. Note that if B is a branch at v with vertex set C, then $L(C)^{-1}$ is a positive matrix, so it has a Perron eigenvalue, and we refer to that eigenvalue as the *Perron value of* B. For a rooted tree T, we will also refer to the *Perron value for a rooted branch* T, by which we mean the Perron value of the branch T at vertex $x \notin T$, where x is adjacent to the root vertex of T. A branch B at vertex v is called a *Perron branch at* v provided that its Perron value is maximum amongst the Perron values of all of the branches at v. The following result shows how both the type of a weighted tree and its characteristic vertices can be discussed in terms of Perron branches.

Proposition 2. (Kirkland, Neumann and Shader [4]) *Suppose that T is a weighted tree with Laplacian matrix L. T is a type* I *tree if and only if there is a unique vertex k at which there are two or more Perron branches. Moreover in that case, k is the characteristic vertex, and the algebraic connectivity of T is the reciprocal of the Perron value of any Perron branch at k. T is a type* II *tree if and only if there are adjacent vertices i and j such that the unique Perron branch at vertex i is the branch containing j and the unique Perron branch at vertex j is the branch containing i. Moreover, in that case, vertices i and j are the characteristic vertices of T. Let C_i be the vertex set of the Perron branch at i, let C_j be the vertex set of the Perron branch at j, and let the weight of the edge $i-j$ be θ. There is a $\gamma \in (0,1)$ such that the Perron values of $L(C_i)^{-1} - \gamma/\theta J$ and $L(C_j)^{-1} - (1-\gamma)/\theta J$ are the same, and their common value is the reciprocal of the algebraic connectivity.*

In order to apply Proposition 2, it is necessary to compute the Perron value of a branch B at vertex v. Thus, we need to find $L(C)^{-1}$, where C is the vertex set of B (in [4], $L(C)^{-1}$ is called the *bottleneck matrix for the branch B at vertex v*). Fortunately, the following result shows how that can be done graph-theoretically.

Proposition 3. (Kirkland, Neumann and Shader [4]) *Suppose that T is a weighted tree, and for each edge e in T, denote its weight by $w(e)$. Let B be a branch of T at vertex v, and label the vertices of B from $1, \ldots, k$, say. Then the (i,j) entry of the bottleneck matrix for B is equal to $\sum_{e \in P_{i,j}} 1/w(e)$, where $P_{i,j}$ denotes the collection of edges in T on both the path from i to v and the path from j to v.*

From Proposition 2, it is easy to see that the following construction will yield a type I tree: Take two copies of a weighted tree which is rooted at a pendant vertex, and form a new tree by identifying the two copies of the root vertex into a single vertex, v. The resulting tree is type I with characteristic vertex v, since there are just two branches at v, which, since they are isomorphic, must necessarily have the same Perron value. This creates a type I weighted tree with two isomorphic Perron branches at the characteristic vertex.

The question naturally arises then: can we construct type I trees having *non*isomorphic Perron branches at the characteristic vertex? This question is perhaps too easy in the weighted case, since we can take any two rooted trees, identify their root vertices into a single vertex v, and then by adjusting the weights on the edges, ensure that both branches at v have the same Perron value. So we might revise our question and ask whether there are *unweighted* type I trees having nonisomorphic Perron branches at the characteristic vertex. The answer to this question is “yes”, as following example of Grone and Merris [3] shows; indeed much of the work in this paper is motivated by this example.

Example 1. (Grone and Merris [3]). Let T be the unweighted tree pictured in Figure 1. Then T is a type I tree with algebraic connectivity 0.139194 and characteristic vertex 6. Evidently the two (Perron) branches at vertex 6 are not isomorphic.

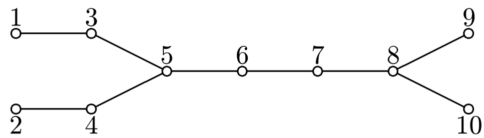

Figure 1

In this paper, we show how Example 1 fits into an entire class of unweighted type I trees having nonisomorphic Perron branches at the characteristic vertex. Further, we give a construction which enables us to take one such tree and produce another one. Finally, we investigate properties possessed by some of these special type I trees.

2. A Construction for Unweighted Type I Trees

We begin with a useful preliminary result. Recall that for an $m \times n$ matrix A and any matrix B, their Kronecker product, $A \otimes B$ is given by

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix}.$$

Lemma 1. *Let G be a connected weighted graph with Laplacian matrix L, and let C be a proper subset of vertices of G. Suppose that $C_1 \subseteq C$ and that the vertices of G are numbered so that those in C_1 come those before those in $C \setminus C_1$. Partition $L(C)^{-1}$ as* $L(C)^{-1} = \begin{array}{c|c} & \begin{array}{cc} C_1 & C \backslash C_1 \end{array} \\ \begin{array}{c} C_1 \\ C \backslash C_1 \end{array} & \left[\begin{array}{c|c} L_1 & L_2 \\ \hline L_2^T & L_3 \end{array}\right] \end{array}$*. Now form a new graph as follows: for each vertex v of C_1, add j new pendant vertices adjacent to v, giving each new edge a weight of 1. Let A denote the set of new vertices, and let $\widehat{L}$ be the Laplacian matrix of the new graph. Then $\widehat{L}(A \cup C)^{-1}$ is permutationally similar to*

$$\begin{array}{cc} & \begin{array}{ccc} A & C_1 & C \backslash C_1 \end{array} \\ \begin{array}{c} A \\ C_1 \\ C \backslash C_1 \end{array} & \left[\begin{array}{c|c|c} I + L_1 \otimes J_j & L_1 \otimes 1_j & L_2 \otimes 1_j \\ \hline L_1 \otimes 1_j^T & L_1 & L_2 \\ \hline L_2^T \otimes 1_j^T & L_2^T & L_3 \end{array}\right] \end{array}.$$

P r o o f. We have $L(C) = \left[\begin{array}{c|c} U & V \\ \hline V^T & W \end{array}\right]$ say, and by suitably labelling the vertices of A, we can suppose that

$$\widehat{L}(A\cup C) = \begin{array}{c} \\ {\scriptstyle A} \\ {\scriptstyle C_1} \\ {\scriptstyle C\backslash C_1} \end{array} \begin{array}{c} \begin{array}{ccc} \quad{\scriptstyle A}\quad & \quad{\scriptstyle C_1}\quad & {\scriptstyle C\backslash C_1} \end{array} \\ \left[\begin{array}{c|c|c} I & -I\otimes 1_j & 0 \\ \hline -I\otimes 1_j^T & U+jI & V \\ \hline 0 & V^T & W \end{array}\right] \end{array}.$$

Using the fact that

$$\left[\begin{array}{c|c} U & V \\ \hline V^T & W \end{array}\right] \left[\begin{array}{c|c} L_1 & L_2 \\ \hline L_2^T & L_3 \end{array}\right] = I,$$

it is now straightforward to verify that

$$\left[\begin{array}{c|c|c} I & -I\otimes 1_j & 0 \\ \hline -I\otimes 1_j^T & U+jI & V \\ \hline 0 & V^T & W \end{array}\right] \left[\begin{array}{c|c|c} I+L_1\otimes J_j & L_1\otimes 1_j & L_2\otimes 1_j \\ \hline L_1\otimes 1_j^T & L_1 & L_2 \\ \hline L_2^T\otimes 1_j^T & L_2^T & L_3 \end{array}\right] = I.$$

□

Given positive integers $k_1,\dots,k_m$, let $T(k_1,\dots,k_m)$ be the unweighted rooted tree formed by the following inductive procedure: Start with a root vertex v, say; then $T(k_1)$ is just the star on k_1+1 vertices with center vertex v. To get $T(k_1,\dots,k_{j+1})$ from $T(k_1,\dots,k_j)$, take each pendant vertex $p\neq v$ of $T(k_1,\dots,k_j)$, and add in k_{j+1} new pendant vertices, each adjacent to p. Figure 2 illustrates the construction. Notice that in Figure 1, the branch at vertex 6 containing vertex 5 is $T(2,1)$, while the branch at vertex 6 containing vertex 7 is $T(1,2)$.

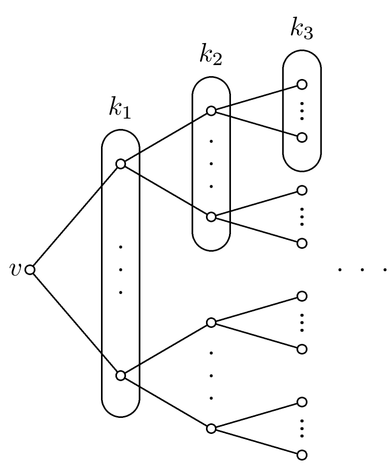

Figure 2

We now discuss how adding $T(k_1,\dots,k_m)$ to each vertex of a branch affects the bottleneck matrix.

Theorem 1. *Let M be the bottleneck matrix of a branch B at vertex a in some weighted tree. Modify B as follows: for each vertex x of B, take a distinct copy of $T(k_1,\dots,k_m)$, and identify its root vertex with x. Then the bottleneck matrix of the modified branch at a can be described as an $(m+1)\times(m+1)$ symmetric block matrix, where the (i,i) block is $I+\{(\dots((I+M\otimes J_{k_1})\otimes J_{k_2}+I)\otimes J_{k_3}\dots+I)\}\otimes J_{k_i}$, $1\leqslant i\leqslant m$, the (i,j) block is $I+\{(\dots((I+M\otimes J_{k_1})\otimes J_{k_2}+I)\otimes J_{k_3}\dots+I)\}\otimes J_{k_i}\otimes 1_{k_{i+1}}\otimes\dots\otimes 1_{k_j}$, $1\leqslant i<j\leqslant m$, and where the $(i,m+1)$ block is $M\otimes 1_{k_1}\otimes 1_{k_2}\otimes\dots\otimes 1_{k_i}$.*

P r o o f. We use induction on m, and note that when $m=1$, the modified bottleneck matrix is permutationally similar to $\left[\begin{array}{c|c} I+M\otimes J_{k_1} & M\otimes 1_{k_1} \\ \hline M\otimes 1_k^T & M \end{array}\right]$ by Lemma 1. Now suppose that the result holds for $m_0\geqslant 1$. Note that carrying through the construction with a copy of $T(k_1,\dots,k_{m_0},k_{m_0+1})$ at every vertex of B is the same as first using the construction with a copy of $T(k_1,\dots,k_{m_0})$ at every vertex of B, then adding k_{m_0+1} new pendant vertices adjacent to every pendant vertex of each of the new copies of $T(k_1,\dots,k_{m_0})$. Appealing to the induction step and Lemma 1 now yields the desired block form for the modified bottleneck matrix. $\square$

Corollary 1.1. *Let B be a branch at vertex a in a weighted tree, and let the Perron value of B be ϱ. Modify B as described in the statement of Theorem 1. Let $f(k_1,\dots,k_i)=\varrho k_i\dots k_1+k_i\dots k_2+\dots+k_i+1$ and form the matrix A of order $m+1$ whose entries in the i-th column on and above the diagonal are $f(k_1,\dots,k_{m+1-i})$, $1\leqslant i\leqslant m$, whose entry in the (i,j) position where $1\leqslant j<i\leqslant m$ is $k_{m+1-j}\dots k_{m+2-i}f(k_1,\dots,k_{m+1-i})$, and whose entries in the $(m+1,i)$ and $(i,m+1)$ positions are $\varrho k_{m+1-i}\dots k_1$ and ϱ, respectively. Then the Perron value of the modified branch at a is the same as the Perron value of A.*

P r o o f. Let v be the Perron vector for the bottleneck matrix for B, and let $[a_1\dots a_{m+1}]^T$ be a Perron vector for A. From the block formula given in Theorem 1, we find that the vector $\begin{bmatrix} a_1v\otimes 1_{k_1\dots k_m} \\ \vdots \\ a_mv\otimes 1_{k_1} \\ a_{m+1}v \end{bmatrix}$ is a Perron vector for the bottleneck matrix of the modified branch, and that its Perron value is the same as that of A. $\square$

Our next result establishes an intriguing connection between $T(k_1,\dots,k_m)$ and $T(k_m,\dots,k_1)$.

Theorem 2. *The Perron value of the rooted branch $T(k_1,\dots,k_m)$ is equal to the Perron value of the rooted branch $T(k_m,\dots,k_1)$.*

Proof. From Corollary 1.1, the Perron value of the rooted branch $T(k_1,\dots,k_m)$ is the same as that of the matrix A whose entries are described in that corollary, and where the value of ϱ is 1. It now follows that A can be factored as XY_1, where

$$X=\begin{bmatrix}1&1&\dots&1\\0&1&\dots&1\\\vdots&&\ddots&\vdots\\0&0&\dots&1\end{bmatrix}\quad\text{and}\quad Y_1=\begin{bmatrix}1&0&\dots0&0\\k_m&1&&\\k_mk_{m-1}&k_{m-1}&\ddots&\vdots\\\vdots&\vdots&1&0\\k_m\dots k_1&k_{m-1}\dots k_1&k_1&1\end{bmatrix}.$$

Similarly, the Perron value of the rooted branch $T(k_1,\dots,k_m)$ is the same as that of XY_2, where

$$Y_2=\begin{bmatrix}1&0&\dots0&0\\k_1&1&&\\k_1k_2&k_2&\ddots&\vdots\\\vdots&\vdots&1&0\\k_1\dots k_m&k_2\dots k_m&k_m&1\end{bmatrix}.$$

But note that each of XY_1, $Y_1^TX^T$ and $X^TY_1^T$ has the same Perron value. Further, if P is the permutation matrix with 1's on the back diagonal, then that common Perron value coincides with the Perron value of $PX^TP^TPY_1^TP^T$. But $PX^TP^T=X$ and $PY_1^TP^T=Y_2$, so we see that XY_1 and XY_2 have the same Perron value. □

The following result shows that Example 1 is part of a larger class of type I trees with nonisomorphic Perron branches at the characteristic vertex.

Corollary 2.1. *Suppose that $k_1,\dots,k_m\in\mathbb{N}$. Form an unweighted tree T by taking a vertex x and making it adjacent to the root vertices of both $T(k_1,\dots,k_m)$ and $T(k_m,\dots,k_1)$. Then T is a type I tree with characteristic vertex x, and the Perron branches at x are nonisomorphic if and only if $k_i\neq k_{m-i+1}$ for some $1\leqslant i\leqslant m$.*

Proof. The result follows directly from Proposition 2 and Theorem 2. □

Remark. Consider the tree T constructed in Corollary 2.1. The argument given in the proof of Theorem 2 shows that in fact, the bottleneck matrices for the two branches at x share $m+1$ eigenvalues, namely the eigenvalues of the matrix XY_1. It now follows that the reciprocal of any eigenvalue of XY_1 is necessarily an eigenvalue of the Laplacian matrix of T.

We now look at the effect of adding copies of a weighted tree S at every vertex of a branch B.

Theorem 3. *Let B be a branch of a weighted tree at vertex v having bottleneck matrix M. Suppose we have another rooted weighted tree S with root r, and let A be the direct sum of bottleneck matrices for the branches of S at r. Form a new branch B' from B as follows: for each vertex x of B, take a distinct copy of S and identify its root vertex with x. Label and partition the vertices of B' by putting all vertices corresponding to the same vertex of $S \setminus \{r\}$ into the same subset of the partition, and all of the vertices of B into the same subset. Then the bottleneck matrix for B' at v is $\left[\begin{array}{c|c} A & 0 \\ \hline 0^T & 0 \end{array}\right] \otimes I + J \otimes M$ (here the identity matrix is the same order as M, and the order of J is one more than that of A).*

P r o o f. Suppose that vertices i_1 and j_1 of B' are on branches at vertices i_0 and j_0 (respectively) in B, and that neither branch contains v. If $i_0 \neq j_0$, then from Proposition 3 the (i_1, j_1) entry of the bottleneck matrix for B' is just $M_{i_0 j_0}$. On the other hand, if $i_0 = j_0$ then by Proposition 3 the (i_1, j_1) entry of the bottleneck matrix for B' is $A_{ij} + M_{i_0 i_0}$, where i_1 and j_1 correspond to vertices i and j of S, respectively. The formula now follows. $\square$

For a square positive matrix M, we let $r(M)$ denote its Perron value.

Corollary 3.1. *Suppose that we have a weighted tree T with algebraic connectivity μ and let S be another rooted weighted tree with root r. Form a new tree T' as follows: for each vertex x of T, take a distinct copy of S and identify its root vertex with x. Then the type of tree and characteristic vertices of T' are the same as those of T. Further, if the A is the direct sum of bottleneck matrices for the branches of S at r, then the algebraic connectivity of T' is $1/r\left(\left[\begin{array}{c|c} A & 0 \\ \hline 0^T & 0 \end{array}\right] + (1/\mu)J\right)$.*

P r o o f. By Theorem 3, for a branch B of T with bottleneck matrix M, the corresponding branch of T' is $\left[\begin{array}{c|c} A & 0 \\ \hline 0^T & 0 \end{array}\right] \otimes I + J \otimes M$. In particular, if the Perron value of B is ϱ, then the Perron value of the corresponding branch in T' is easily seen to be $r\left(\left[\begin{array}{c|c} A & 0 \\ \hline 0^T & 0 \end{array}\right] + \varrho J\right)$. Hence if there is a vertex of T with two or more Perron branches (so that T is type I), that same vertex of T' also has two or more Perron branches, so that T' is also type I, with the same characteristic vertex. The formula for the algebraic connectivity now follows from Proposition 2 upon observing that $1/\mu = \varrho$.

Similarly, if T is type II, there are adjacent (characteristic) vertices u and v of T such that for some $\gamma \in (0,1)$, $r(M_1 - \gamma/\theta J) = \varrho = 1/\mu = r(M_2 - (1-\gamma)/\theta J)$, where M_1 is the bottleneck matrix for the branch at u containing v, M_2 is the bottleneck matrix for the branch at v containing u, and θ is the weight of the edge $u - v$. By Theorem 3, in T' the bottleneck matrix for the branch at u containing v is $\left[\begin{array}{c|c} A & 0 \\ \hline 0^T & 0 \end{array}\right] \otimes I + J \otimes M_1$ while in T' the bottleneck matrix for the branch at v containing u is $\left[\begin{array}{c|c} A & 0 \\ \hline 0^T & 0 \end{array}\right] \otimes I + J \otimes M_2$. Since $r\left(\left[\begin{array}{c|c} A & 0 \\ \hline 0^T & 0 \end{array}\right] \otimes I + J \otimes M_1 - \gamma/\theta J\right) = r\left(\left[\begin{array}{c|c} A & 0 \\ \hline 0^T & 0 \end{array}\right] \otimes I + J \otimes (M_1 - \gamma/\theta J)\right) = r\left(\left[\begin{array}{c|c} A & 0 \\ \hline 0^T & 0 \end{array}\right] + (1/\mu)J\right) = r\left(\left[\begin{array}{c|c} A & 0 \\ \hline 0^T & 0 \end{array}\right] \otimes I + J \otimes (M_2 - (1-\gamma)/\theta J)\right) = r\left(\left[\begin{array}{c|c} A & 0 \\ \hline 0^T & 0 \end{array}\right] \otimes I + J \otimes M_1 - (1-\gamma)/\theta J\right)$, we see that u and v are the characteristic vertices of the (type II) tree T', and that the algebraic connectivity of T' is the reciprocal of $r\left(\left[\begin{array}{c|c} A & 0 \\ \hline 0^T & 0 \end{array}\right] + (1/\mu)J\right)$. □

Corollary 3.2. *Let T be a weighted tree with algebraic connectivity μ, and modify it as follows: at each vertex x of T add in j new pendant vertices, each adjacent to x, with weight 1 for each of the new pendant edges. Then the algebraic connectivity of the new tree is $2\mu/\{j+1+\mu+\sqrt{(j-1)^2+2\mu(j+1)+\mu^2}\}$.*

P r o o f. From the hypotheses, we see that we are in the situation of Corollary 3.1, where S is the unweighted star on $j+1$ vertices, rooted at the center vertex. Thus A is the identity matrix of order j, and so by Corollary 3.1, the algebraic connectivity of the new tree is the reciprocal of the Perron value of $\left[\begin{array}{c|c} I & 0 \\ \hline 0^T & 0 \end{array}\right] + (1/\mu)J$. This last partitioned matrix has constant row sums in each block of the partitioning, and so it follows that the Perron value of $\left[\begin{array}{c|c} I & 0 \\ \hline 0^T & 0 \end{array}\right] + (1/\mu)J$ is the same as the Perron value of $\begin{bmatrix} j/\mu+1 & 1/\mu \\ j/\mu & 1/\mu \end{bmatrix}$. The result now follows by direct computation. □

Remark. Corollaries 2.1 and 3.1 together give us a way to construct many type I unweighted trees with nonisomorphic Perron branches. Start with a tree T having a vertex v at which there are just two branches: $T(k_1,\ldots,k_m)$ and $T(k_m,\ldots,k_1)$, with v adjacent to each of the appropriate root vertices. Now at each vertex of T, identify that vertex with the root vertex of a distinct copy of some rooted unweighted tree S. By Corollary 2.1, T is type I with characteristic vertex v, and hence by Corollary 3.1, the modified tree is also type I with characteristic vertex v. However, the Perron branches at v in the modified tree are not isomorphic provided that $k_i \neq k_{m-i+1}$ for some i. Note also that the construction of Corollary 3.2 can be iterated with a number of trees $S_1, \ldots, S_n$, yielding even more type I trees with nonisomorphic Perron branches at the characteristic vertex.

3. Perron Properties of $T(l, l, \ldots, l, k, l, \ldots, l)$

In this section we compare Perron values for certain type of branches in unweighted trees.

Lemma 2. *Let ϱ be the Perron root of the rooted branch $T(k_1, \ldots, k_m)$. Then*

$$\det \begin{bmatrix} 1-\varrho & \varrho k_1 & 0 & 0 & 0\ldots 0 \\ 1 & 1-\varrho & \varrho k_2 & 0 & 0\ldots 0 \\ 1 & 1 & 1-\varrho & \varrho k_3 & 0\ldots 0 \\ \vdots & & & \ddots & \\ 1 & 1\ \ldots & 1 & 1-\varrho & \varrho k_m \\ 1 & 1\ \ldots & 1 & 1 & 1-\varrho \end{bmatrix} = 0.$$

Proof. It follows from Theorem 2 that ϱ satisfies $\det(XY - \varrho I) = 0$, where

$$X = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 0 & 1 & \ldots & 1 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 1 & 0 & \ldots 0 & 0 \\ k_m & 1 & & \\ k_m k_{m-1} & k_{m-1} & \ddots & \vdots \\ \vdots & \vdots & 1 & 0 \\ k_m \ldots k_1 & k_{m-1} \ldots k_1 & k_1 & 1 \end{bmatrix}.$$

Consequently, $\det(Y - \varrho X^{-1}) = 0$, and a straightforward computation shows that

$$\begin{bmatrix} 1-\varrho & \varrho k_1 & 0 & 0 & 0\ldots 0 \\ 1 & 1-\varrho & \varrho k_2 & 0 & 0\ldots 0 \\ 1 & 1 & 1-\varrho & \varrho k_3 & 0\ldots 0 \\ \vdots & & & \ddots & \\ 1 & 1\ \ldots & 1 & 1-\varrho & \varrho k_m \\ 1 & 1\ \ldots & 1 & 1 & 1-\varrho \end{bmatrix} = D^{-1}(Y - \varrho X^{-1})D,$$

where D is the diagonal matrix whose first diagonal entry is 1 and whose i-th diagonal entry is $k_1, \ldots, k_{i-1}$, $1 \leqslant i \leqslant m+1$. □

Fix $1 \leqslant i \leqslant m$, and let

$$f_{m+1,i}(\varrho) = \det \begin{bmatrix} 1-\varrho & \varrho k_1 & 0 & 0 & 0\dots0 \\ 1 & 1-\varrho & \varrho k_2 & 0 & 0\dots0 \\ 1 & 1 & 1-\varrho & \varrho k_3 & 0\dots0 \\ \vdots & & & \ddots & \\ 1 & 1\ \dots & 1 & 1-\varrho & \varrho k_m \\ 1 & 1\ \dots & 1 & 1 & 1-\varrho \end{bmatrix}$$

where $k_i = k$ and $k_j = l$ for all $j \neq i$. Similarly let

$$g_{m+1,i}(\varrho) = \det \begin{bmatrix} 1 & \varrho k_1 & 0 & 0 & 0\dots0 \\ 1 & 1-\varrho & \varrho k_2 & 0 & 0\dots0 \\ 1 & 1 & 1-\varrho & \varrho k_3 & 0\dots0 \\ \vdots & & & \ddots & \\ 1 & 1\ \dots & 1 & 1-\varrho & \varrho k_m \\ 1 & 1\ \dots & 1 & 1 & 1-\varrho \end{bmatrix}$$

where $k_i = k$ and $k_j = l$ for all $j \neq i$. Let

$$D_{m+1}(\varrho) = \det \begin{bmatrix} 1-\varrho & \varrho l & 0 & 0 & 0\dots0 \\ 1 & 1-\varrho & \varrho l & 0 & 0\dots0 \\ 1 & 1 & 1-\varrho & \varrho l & 0\dots0 \\ \vdots & & & \ddots & \\ 1 & 1\ \dots & 1 & 1-\varrho & \varrho l \\ 1 & 1\ \dots & 1 & 1 & 1-\varrho \end{bmatrix}$$

and let

$$A_{m+1}(\varrho) = \det \begin{bmatrix} 1 & \varrho l & 0 & 0 & 0\dots0 \\ 1 & 1-\varrho & \varrho l & 0 & 0\dots0 \\ 1 & 1 & 1-\varrho & \varrho l & 0\dots0 \\ \vdots & & & \ddots & \\ 1 & 1\ \dots & 1 & 1-\varrho & \varrho l \\ 1 & 1\ \dots & 1 & 1 & 1-\varrho \end{bmatrix},$$

where the order of each matrix is $m+1$.

Lemma 3. *Suppose that $m \geqslant 3$ and that $1 \leqslant i \leqslant (m-1)/2$. Then $f_{m+1,i}(\varrho) - f_{m+1,i+1}(\varrho) = \varrho^{2i+1} l^i (k-l) A_{m-2i}$ and $g_{m+1,i}(\varrho) - g_{m+1,i+1}(\varrho) = \varrho^{2i} l^{i-1} (k-l) D_{m-2i}$.*

Proof. We will proceed by induction on i. So suppose that $i = 1$. Expanding $f_{m+1,1}$ and $f_{m+1,2}$ along the first row, we have $f_{m+1,1} - f_{m+1,2} = (1-\varrho)D_m -$

$\varrho k A_m - (1-\varrho) f_{m,1} + \varrho l g_{m,1}$. Since the matrices corresponding to D_m and $f_{m,1}$ differ only in the first row, it follows that

$$
\begin{aligned}
&D_m - f_{m,1} \\
&= \det\left\{\begin{bmatrix} 1-\varrho & \varrho l & 0 & 0 & 0\dots 0 \\ 1 & 1-\varrho & \varrho l & 0 & 0\dots 0 \\ 1 & 1 & 1-\varrho & \varrho l & 0\dots 0 \\ \vdots & & & \ddots & \\ 1 & 1 & \dots & 1 & 1-\varrho & \varrho l \\ 1 & 1 & \dots & 1 & 1 & 1-\varrho \end{bmatrix} - \begin{bmatrix} 1-\varrho & \varrho k & 0 & 0 & 0\dots 0 \\ 1 & 1-\varrho & \varrho l & 0 & 0\dots 0 \\ 1 & 1 & 1-\varrho & \varrho l & 0\dots 0 \\ \vdots & & & \ddots & \\ 1 & 1 & \dots & 1 & 1-\varrho & \varrho l \\ 1 & 1 & \dots & 1 & 1 & 1-\varrho \end{bmatrix}\right\} \\
&= \det\begin{bmatrix} 0 & \varrho(l-k) & 0 & 0 & 0\dots 0 \\ 1 & 1-\varrho & \varrho l & 0 & 0\dots 0 \\ 1 & 1 & 1-\varrho & \varrho l & 0\dots 0 \\ \vdots & & & \ddots & \\ 1 & 1 & \dots & 1 & 1-\varrho & \varrho l \\ 1 & 1 & \dots & 1 & 1 & 1-\varrho \end{bmatrix} \\
&= (k-l)\varrho A_{m-1}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&-kA_m + l g_{m,1} \\
&= \det\begin{bmatrix} l & \varrho lk & 0 & 0 & 0\dots 0 \\ 1 & 1-\varrho & \varrho l & 0 & 0\dots 0 \\ 1 & 1 & 1-\varrho & \varrho l & 0\dots 0 \\ \vdots & & & \ddots & \\ 1 & 1 & \dots & 1 & 1-\varrho & \varrho l \\ 1 & 1 & \dots & 1 & 1 & 1-\varrho \end{bmatrix} - \det\begin{bmatrix} k & \varrho lk & 0 & 0 & 0\dots 0 \\ 1 & 1-\varrho & \varrho l & 0 & 0\dots 0 \\ 1 & 1 & 1-\varrho & \varrho l & 0\dots 0 \\ \vdots & & & \ddots & \\ 1 & 1 & \dots & 1 & 1-\varrho & \varrho l \\ 1 & 1 & \dots & 1 & 1 & 1-\varrho \end{bmatrix} \\
&= \det\begin{bmatrix} l-k & 0 & 0 & 0 & 0\dots 0 \\ 1 & 1-\varrho & \varrho l & 0 & 0\dots 0 \\ 1 & 1 & 1-\varrho & \varrho l & 0\dots 0 \\ \vdots & & & \ddots & \\ 1 & 1 & \dots & 1 & 1-\varrho & \varrho l \\ 1 & 1 & \dots & 1 & 1 & 1-\varrho \end{bmatrix} \\
&= (l-k)D_{m-1}.
\end{aligned}
$$

Thus we see that

$$f_{m+1,1} - f_{m+1,2} = (k-l)\varrho((1-\varrho)A_{m-1} - D_{m-1}) = (k-l)\varrho$$

$$\times \left\{ \det \begin{bmatrix} 1-\varrho & (1-\varrho)\varrho l & 0 & 0 & 0\dots0 \\ 1 & 1-\varrho & \varrho l & 0 & 0\dots0 \\ 1 & 1 & 1-\varrho & \varrho l & 0\dots0 \\ \vdots & & & \ddots & \\ 1 & 1\ \dots & 1 & 1-\varrho & \varrho l \\ 1 & 1\ \dots & 1 & 1 & 1-\varrho \end{bmatrix} - \det \begin{bmatrix} 1-\varrho & \varrho l & 0 & 0 & 0\dots0 \\ 1 & 1-\varrho & \varrho l & 0 & 0\dots0 \\ 1 & 1 & 1-\varrho & \varrho l & 0\dots0 \\ \vdots & & & \ddots & \\ 1 & 1\ \dots & 1 & 1-\varrho & \varrho l \\ 1 & 1\ \dots & 1 & 1 & 1-\varrho \end{bmatrix} \right\}$$

$$= (k-l)\varrho \det \begin{bmatrix} 0 & -\varrho^2 l & 0 & 0 & 0\dots0 \\ 1 & 1-\varrho & \varrho l & 0 & 0\dots0 \\ 1 & 1 & 1-\varrho & \varrho l & 0\dots0 \\ \vdots & & & \ddots & \\ 1 & 1\ \dots & 1 & 1-\varrho & \varrho l \\ 1 & 1\ \dots & 1 & 1 & 1-\varrho \end{bmatrix} = \varrho^3 l(k-l)A_{m-1}.$$

Now expanding $g_{m+1,i}$ and $g_{m+1,i+1}$ along the first row, we find that

$$g_{m+1,i}(\varrho) - g_{m+1,i+1}(\varrho) = D_m - f_{m,1} + \varrho l g_{m,1} - \varrho k A_m$$
$$= \varrho(k-l)A_{m-1} - \varrho(k-l)D_{m-1} = (k-l)\varrho$$

$$\times \left\{ \det \begin{bmatrix} 1 & \varrho l & 0 & 0 & 0\dots0 \\ 1 & 1-\varrho & \varrho l & 0 & 0\dots0 \\ 1 & 1 & 1-\varrho & \varrho l & 0\dots0 \\ \vdots & & & \ddots & \\ 1 & 1\ \dots & 1 & 1-\varrho & \varrho l \\ 1 & 1\ \dots & 1 & 1 & 1-\varrho \end{bmatrix} - \det \begin{bmatrix} 1-\varrho & \varrho l & 0 & 0 & 0\dots0 \\ 1 & 1-\varrho & \varrho l & 0 & 0\dots0 \\ 1 & 1 & 1-\varrho & \varrho l & 0\dots0 \\ \vdots & & & \ddots & \\ 1 & 1\ \dots & 1 & 1-\varrho & \varrho l \\ 1 & 1\ \dots & 1 & 1 & 1-\varrho \end{bmatrix} \right\}$$

$$= (k-l)\varrho \det \begin{bmatrix} \varrho & 0 & 0 & 0 & 0\dots0 \\ 1 & 1-\varrho & \varrho l & 0 & 0\dots0 \\ 1 & 1 & 1-\varrho & \varrho l & 0\dots0 \\ \vdots & & & \ddots & \\ 1 & 1\ \dots & 1 & 1-\varrho & \varrho l \\ 1 & 1\ \dots & 1 & 1 & 1-\varrho \end{bmatrix} = (k-l)\varrho^2 D_{m-1}.$$

This establishes the basis for the induction.

Next we suppose that $i+1 \leqslant (m-1)/2$, and that the induction hypothesis holds for i. Expanding along first rows as above, and then applying the induction hypothesis, we find that

$$\begin{aligned}
f_{m+1,i+1} - f_{m+1,i+2} &= (1-\varrho)\{f_{m,i} - f_{m+i-1}\} - \varrho l\{g_{m,1} - g_{m,i+1}\} \\
&= (1-\varrho)\{\varrho^{2i+1} l^i (k-l) A_{m-1-2i}\} - \varrho l\{\varrho^{2i} l^{i-1}(k-l) D_{m-1-2i}\} \\
&= \varrho^{2i+1} l^i (k-l)\{(1-\varrho)A_{m-1-2i} - D_{m-1-2i}\} \\
&= \varrho^{2i+3} l^{i+1}(k-l) A_{m-2-2i}.
\end{aligned}$$

Proceeding similarly, we have

$$\begin{aligned}
g_{m+1,i+1} - g_{m+1,i+2} &= \{f_{m,i} - f_{m,i+1}\} - \varrho l(g_{m,i} - g_{m,i+1}\} \\
&= \{\varrho^{2i+1} l^i (k-l) A_{m-1-2i}\} - \varrho l(\varrho^{2i} l^{i-1}(k-l) D_{m-1-2i}\} \\
&= \varrho^{2i+1} l^i (k-l)\{A_{m-1-2i} - D_{m-1-2i}\} = \varrho^{2i+2} l^i (k-l) D_{m-2-2i}.
\end{aligned}$$

This completes the induction step, and the proof. □

Suppose that $m \geqslant 3$, $i \leqslant (m-1)/2$ and that $k, l \in \mathbb{N}$ with $l \neq k$. Let $r(m,i,l,k)$ be the Perron root of the rooted branch $T(k_1,\dots,k_m)$, where $k_i = k$ and $k_j = l$ for all $1 \leqslant j \leqslant m$ with $j \neq i$. Our final result describes the behaviour of $r(m,i,l,k)$ for different values of i.

Theorem 4. *Suppose that $m \geqslant 3$. If $k > l$, then $r(m,i,l,k) < r(m,i+1,l,k)$ for all i such that $i \leqslant (m-1)/2$. If $k < l$, then $r(m,i,l,k) > r(m,i+1,l,k)$ for all i such that $i \leqslant (m-1)/2$.*

Proof. Note that by Lemma 3, $r(m,i,l,k)$ is the maximum positive solution to the equation $f_{m+1,i}(\varrho) = 0$. From Lemma 4, we find that for $i \leqslant (m-1)/2$, $f_{m+1,i}(\varrho) - f_{m+1,i+1}(\varrho) = \varrho^{2i+1} l^i (k-l) A_{m-2i}$. We next claim that $\operatorname{sgn}(A_p) = (-1)^{p-1}$ whenever ϱ exceeds the maximum positive root of $D_p(\varrho) = 0$. Note that in fact the maximum positive root of $D_p(\varrho) = 0$ is, by Lemma 2, the same as $r(XY)$, where X and Y are the $p \times p$ matrices

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & 1 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ l & 1 & & & \\ l & l & \ddots & & \vdots \\ \vdots & \vdots & & 1 & 0 \\ l & l & & l & 1 \end{bmatrix},$$

respectively. Consequently, we find that the maximum positive root of $D_p(\varrho) = 0$ is increasing in p.

We prove our claim by induction on p, and note that when $p = 1$, $A_p = 1 > 0$. Suppose now that the induction hypothesis holds for $p \geqslant 1$ and that ϱ exceeds the maximum positive root of $D_{p+1}(\varrho) = 0$. Then certainly ϱ exceeds the maximum positive root of $D_p(\varrho) = 0$. Expanding A_{p+1} along the top row we have $A_{p+1} = D_p - \varrho l A_p$. Since D_p is a polynomial of degree p in ϱ with leading coefficient $(-1)^p$, and since ϱ is larger than the maximum positive root of D_p, it follows that $\operatorname{sgn}(D_p) = (-1)^p$. Further, by the induction hypothesis, $\operatorname{sgn}(A_p) = (-1)^{p-1}$, so we find that $\operatorname{sgn}(A_{p+1}) = (-1)^p$, completing the proof of the claim.

Suppose now that $k > l$, and note that $r(m, i, l, k)$ exceeds the maximum positive root of D_{m-2i}. Evaluating $f_{m+1,i}(\varrho) - f_{m+1,i+1}(\varrho)$ at $\varrho = r(m, i, l, k)$, we have that $\operatorname{sgn}(f_{m+1,i}(\varrho) - f_{m+1,i+1}(\varrho)) = -\operatorname{sgn}(f_{m+1,i+1}(\varrho)) = \operatorname{sgn}(\varrho^{2i+1} l^i (k - l) A_{m-2i}) = (-1)^{m-1}$, so that $\operatorname{sgn}(f_{m+1,i+1}(\varrho)) = (-1)^m$. But the leading coefficient in the polynomial $f_{m+1,i+1}(\varrho)$ is $(-1)^{m+1}$, so by the Intermediate Value Theorem, $f_{m+1,i+1}$ must have a root larger than $r(m, i, l, k)$. Thus $r(m, i, l, k) < r(m, i + 1, l, k)$. The proof of the statement for $k < l$ is analogous. $\square$

Remark. Theorem 2 and 4 together give us complete information on the ordering of the values $r(m, i, l, k)$ as i runs from 1 to m. From Theorem 2 we find that $r(m, i, l, k) = r(m, m + 1 - i, l, k)$, and so applying Theorem 4, we have for $k > l$, $r(m, 1, l, k) = r(m, m, l, k) < r(m, 2, l, k) = r(m, m-1, l, k) < \ldots < r(m, m/2, l, k) = r(m, (m + 2)/2, l, k)$ if m is even, and $r(m, 1, l, k) = r(m, m, l, k) < r(m, 2, l, k) = r(m, m - 1, l, k) < \ldots < r(m, (m - 1)/2, l, k) = r(m, (m + 3)/2, l, k) < r(m, (m + 1)/2, l, k)$ if m is odd. Similarly, if $k < l$, $r(m, 1, l, k) = r(m, m, l, k) > r(m, 2, l, k) = r(m, m - 1, l, k) > \ldots > r(m, m/2, l, k) = r(m, (m + 2)/2, l, k)$ if m is even, and $r(m, 1, l, k) = r(m, m, l, k) > r(m, 2, l, k) = r(m, m - 1, l, k) > \ldots > r(m, (m - 1)/2, l, k) = r(m, (m + 3)/2, l, k) > r(m, (m + 1)/2, l, k)$ if m is odd.

Suppose that we are given natural numbers $k_1, \ldots, k_m$, and are asked to extremize the Perron value of $T(k_{\pi(1)}, \ldots, k_{\pi(m)})$ over all permutations π of $\{1, \ldots, m\}$. We suspect that the maximizing permutation will have the property that $k_{\pi(i)}$ is nondecreasing in i for values of i which are less than or equal to some j, then nonincreasing in i for values of i beyond j. Similarly, we also suspect that the minimizing permutation will have $k_{\pi(i)}$ nonincreasing in i for values of i which are less than or equal to some h, then nondecreasing in i for values of i beyond h. Theorem 4 supports these suspicions under restrictive hypotheses on the k_i's but we are unable to say much about the general case at present.

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