Czechoslovak Mathematical Journal

Kamil John

U-ideals of factorable operators

Czechoslovak Mathematical Journal, Vol. 49 (1999), No. 3, 607–616

Persistent URL: http://dml.cz/dmlcz/127513

Czechoslovak Mathematical Journal, 49 (124) (1999), 607–616

U-IDEALS OF FACTORABLE OPERATORS

Kamil John,* Praha

(Received November 13, 1996)

Abstract. We suggest a method of renorming of spaces of operators which are suitably approximable by sequences of operators from a given class. Further we generalize J. Johnsons's construction of ideals of compact operators in the space of bounded operators and observe e.g. that under our renormings compact operators are u-ideals in the: space of 2-absolutely summing operators or in the space of operators factorable through a Hilbert space.

MSC 2000: 46A32, 46B20, 46H10, 46B99, 46B25

Keywords: factorization of linear operators, u-ideal, approximation properties, unconditional basis

Let K a be subspace of a Banach space L. Following [G, K, S] we say that K is an ideal in L if K° is the kern of a contractive projection P in L^*. Moreover, K is a u-ideal in $(L, \|\cdot\|)$ if $\|\operatorname{Id}_{L^*} - 2P\| \leqslant 1$.

In [J2] it was observed that Johnson's argument that $\mathscr{K}(X,Y)$ is an ideal in $\mathscr{L}(X,Y)$ can be carried out even when X and Y do not have the (compact) approximation property but when any $f \in \mathscr{L}(X,Y)$ is suitably approximated by a sequence $\{f_n\}$ of compact operators, $f_n \xrightarrow{w'} f$ (for the topology w′ see the definition below). In fact any $\varphi \in \mathscr{K}^*$ may be uniquely extended to the w′-sequential closure of $\mathscr{K}$. This attitude is particularly suited for the situation of factorable operators because of this unicity of extensions but it was used also in other situations [Li], [F]. Here we develop a slightly formal scheme which further generalizes the idea. We show that when we suitably renorm spaces of factorable operator then Johnson's construction even gives u-ideals. More precisely, we observe that $\mathscr{K}(X,Y) \cap \mathscr{A}(X,Y)$ is a u-ideal in $\mathscr{A}(X,Y)$ for operator ideals $\mathscr{A} = \mathscr{P}_2$ or $\mathscr{A} = \Gamma_2$ (cf. [Pie]). Here we

* The work was supported by the grants of AVČR No. A1019504 and of GAČR No. 201/94/0069

denote by $\mathscr{F}(X,Y), \mathscr{K}(X,Y), \Gamma_2(X,Y), \mathscr{P}_2(X,Y)$ and $\mathscr{L}(X,Y)$ the Banach spaces of linear operators from the Banach space X to the Banach space Y which are respectively finite-dimensional, compact, factorable through a Hilbert space, absolutely 2-summing or bounded.

To conclude the introduction we remark that we have not been able to get any reasonable corresponding results on M-ideals (except the trivial ones in [E, J]).

Let $\{f_n\} \subset \mathscr{L}(X,Y)$ be a sequence of operators and let $f \in \mathscr{L}(X,Y)$. We will denote by w the (locally convex) topology on $\mathscr{L}(Y^*, X^*)$ projectively generated by the linear forms of the form $x^{**} \otimes y^*$ for all $x^{**} \in X^{**}$ and all $y^* \in Y^*$. Following Kalton [Ka] we shall denote by w' the topology on $\mathscr{L}(X,Y)$ induced by the topology w from $\mathscr{L}(Y^*, X^*)$. Here we consider $\mathscr{L}(X,Y) \subset \mathscr{L}(Y^*, X^*)$ via the adjoint map. If $\mathscr{T}$ is a locally convex topology we will write $\mathscr{T}\text{-}\sum f_n = f$ if $\lim_n \sum_{i=1}^{n} f_i = f$ in the topology $\mathscr{T}$. Thus we will write $w'\text{-}\sum f_n = f$ if for all $x^{**} \in X^{**}$ and all $y^* \in Y^*$ we have $\lim_n \sum_{i=1}^{n} (x^{**} f_i^*) y^* = (x^{**} f^*) y^*$.

Definition. Let $\{f_n\}$ be a sequence of elements of a Banach space, let $\mathscr{T}$ be a topology on this Banach space and let $\mathscr{T}\text{-}\sum f_n = f$ exist in this Banach space. We will denote by $K_u(\{f_n\})$ the number (possibly also ∞)

$$\sup\left\{\max\left(\left\|\sum_{i=1}^{n} \varepsilon_i f_i\right\|, \left\|f - 2\sum_{i=1}^{n} \eta_i f_i\right\|\right);\ n \geqslant 1, |\varepsilon_i| \leqslant 1, 0 \leqslant \eta_i \leqslant 1\right\}.$$

Suppose further that $\|\cdot\|$ is a Banach space norm on a class of operators $\mathscr{X} \subset \mathscr{L}(X,Y)$ and that $\mathscr{T}$ means e.g. the w'-topology. We will say that $\mathscr{S}(X,Y)$ is an $(\mathscr{S})$-class of sequences $\{f_n\}_{n=1}^{\infty}$ of operators $f_n\colon\ X \to Y, f_n \in \mathscr{X}$ when the following holds:

(i) $\mathscr{S}(X,Y)$ forms a vector space (with the co-ordinatewise operations),

(ii) $\mathscr{S}(X,Y)$ contains with each $\{f_n\}$ also each $\{f_1, \ldots, f_m, 0, 0, \ldots\}$,

(iii) $\mathscr{S}(X,Y)$ is closed in the following sense:

If $\{f_{np}\}_{n=1}^{\infty} \in \mathscr{S}(X,Y)$ for all p, if $\sum_{p=1}^{\infty} K_u(\{f_{np}\}_n) \leqslant C$ for some constant C and if $n_i p_i$ is any ordering of the cartesian product of natural numbers then also $\{f_{n_i p_i}\} \in \mathscr{S}(X,Y)$.

Further we will say that an $(\mathscr{S})$-class $\mathscr{S}$ has the property $(\mathscr{U})$ if $\sum f_n$ is weakly unconditionally Cauchy (WUC), i.e. if $K_u(\{f_n\}) < \infty$ for every $\{f_n\} \in \mathscr{S}$.

The following proposition strengthens some results in [J2].

Proposition 1. *Let X, Y be Banach spaces and let $\mathscr{S}(X,Y)$ be an $(\mathscr{S})$-class of sequences $\{f_n\}$ of compact operators such that for every $f \in \mathscr{L}(X,Y)$ there is a* *sequence* $\{f_n\} \subset \mathscr{K}(X,Y)$, $\{f_n\} \in \mathscr{S}(X,Y)$ *with* w'-$\sum f_n = f$. *Let* $\|\cdot\|$ *be a norm on* $\mathscr{L}(X,Y)$ *equivalent to the sup norm on* $\mathscr{L}(X,Y)$ *and suppose that* $\mathscr{S}(X,Y)$ *has the property* $(\mathscr{U})$. *Then the norm* $|\!|\!|\cdot|\!|\!|$,

$$|\!|\!|f|\!|\!| = \inf\Big\{K_u(\{f_n\});\ w'\text{-}\sum f_n = f, \{f_n\} \in \mathscr{S}(X,Y), \{f_n\} \subset \mathscr{K}(X,Y)\Big\}$$

for $f \in \mathscr{L}(X,Y)$ *is an equivalent norm on* $\mathscr{L}(X,Y)$ *and the space* $\mathscr{K}(X,Y)$ *is a* u*-ideal in* $(\mathscr{L}(X,Y), |\!|\!|\cdot|\!|\!|)$.

P r o o f. We first show that the norm $|\!|\!|\cdot|\!|\!|$ is equivalent to the usual sup norm. We observe that $\|\cdot\| \leqslant |\!|\!|\cdot|\!|\!|$ on $\mathscr{L}$. In fact, by the definition of $K_u(\{f_n\})$, we have $\|f\| \leqslant K_u(\{f_n\})$ for any w'-$\sum f_n = f$. Passing to the infimum proves the claim. Evidently $|\!|\!|\cdot|\!|\!|$ is a norm on $\mathscr{L}$. Now we observe that $(\mathscr{L}, |\!|\!|\cdot|\!|\!|)$ is complete. To prove this it is sufficient to show that if $f_p \in \mathscr{L}$, $\sum\limits_{p=1}^{\infty} |\!|\!|f_p|\!|\!| < \infty$ then $\sum\limits_{p=1}^{\infty} f_p \in \mathscr{L}$ exists in $\mathscr{L}$ and $|\!|\!|\sum f_p|\!|\!| \leqslant \sum |\!|\!|f_p|\!|\!|$ (cf. Theorem 6.2.3 [Pie]). To see this let $\{f_{np}\}_n \in \mathscr{S}$, $f_{np} \in \mathscr{K}$ be such that for each p we have w'-$\sum\limits_n f_{np} = f_p$, $K_u(\{f_{np}\}_n) \leqslant |\!|\!|f_p|\!|\!| + \frac{\varepsilon}{2^p}$. If $|x^{**}| \leqslant 1$, $|y^*| \leqslant 1$ and if the sup norm $|\cdot|$ satisfies on $\mathscr{L}(X,Y)$ the inequality $|\cdot| \leqslant c\|\cdot\|$ then we have for suitable $\eta_i = \pm 1$

$$\sum_{i=1}^{n} \big|x^{**}(f_{ip}^* y^*)\big| = x^{**}\left(\sum_{i=1}^{n} \varepsilon_i f_{ip}^* y^*\right) \leqslant c\left\|\sum_{i=1}^{n} \varepsilon_i f_{ip}\right\| \tag{1}$$
$$\leqslant cK_u(\{f_{np}\}_n) \leqslant c|\!|\!|f_p|\!|\!| + \frac{c\varepsilon}{2^p} \quad \text{for all } n.$$

Let $\{g_i\} = \{f_{n_i p_i}\}$ be a reordering of $\{f_{np}\}$ into a sequence. Then we have

$$\sum x^{**}(g_i^* y^*) = \sum_{n,p} x^{**}(f_{np}^* y^*) = \sum_{p} x^{**}(f_p^* y^*) \tag{2}$$

because by (1) the convergence is absolute.

Observe now that $\sum\limits_p f_p \in \mathscr{L}$ converges in the norm $\|\cdot\|$ because $\|f_p\| \leqslant |\!|\!|f_p|\!|\!|$, and similarly also $\sum\limits_{p=1}^{\infty} f_{np} \in \mathscr{K}$ exists in the norm $\|\cdot\|$ because $\mathscr{K}$ is $\|\ \|$-complete. Indeed, we have

$$\sum_p \|f_{np}\| \leqslant \sum_p K_u(\{f_{np}\}_n) \leqslant \sum |\!|\!|f_p|\!|\!| + \varepsilon$$

for all n and thus by the assumption (iii) we have made on the class $\mathscr{S}$ it follows that $\{g_i\} \in \mathscr{S}(X,Y)$. Now (2) implies that

$$w'\text{-}\sum_i g_i = \sum_p f_p.$$

This implies that

(3)

$$\begin{aligned}
&\Big\|\!\Big| \sum f_p \Big\|\!\Big| \leqslant K_u(\{g_i\}_i)\\
&= \sup\Big\{\max\Big(\Big\|\sum_{j=1}^{n}\varepsilon_j g_j\Big\|, \Big\|\sum_p f_p - 2\sum_i \eta_i g_i\Big\|\Big);\ n\geqslant 1, |\varepsilon_j|\leqslant 1, 0\leqslant \eta_i\leqslant 1\Big\}\\
&\leqslant \sup\Big\{\max\Big(\Big\|\sum_{i,p=1}^{n}\varepsilon_{ip} f_{ip}\Big\|, \Big\|\sum_p f_p - 2\sum_{i,p} \eta_{ip} f_{ip}\Big\|\Big);\ n\geqslant 1, |\varepsilon_{ip}|\leqslant 1, 0\leqslant \eta_{ip}\leqslant 1\Big\}\\
&\leqslant \sup\Big\{\max\Big(\sum_{p=1}^{n}\Big\|\sum_{i=1}^{n}\varepsilon_{ip} f_{ip}\Big\|, \sum_p\Big\| f_p - 2\sum_{i} \eta_{ip} f_{ip}\Big\|\Big);\ n\geqslant 1, |\varepsilon_{ip}|\leqslant 1, 0\leqslant \eta_{ip}\leqslant 1\Big\}\\
&\leqslant \sum_p K_u(\{f_{np}\}_n)) \leqslant \varepsilon + \sum_p \|\!| f_p \|\!|,
\end{aligned}$$

showing that $K_u(\{g_i\}) < \infty$ and $\|\!|\sum_p f_p\|\!| \leqslant \sum_p \|\!| f_p\|\!|$. In (3) we have used that the sums $\sum f_p$ and $\sum_{i,p}\varepsilon_{ip}f_{ip}$ absolutely converge in the norm $\|\cdot\|$. Finally, the open mapping theorem yields that the norms $\|\!|\cdot\|\!|$ and $\|\cdot\|$ are equivalent.

To show that $\mathscr{K}(X,Y)$ is an ideal in $\mathscr{L} = (\mathscr{L}(X,Y), \|\!|\cdot\|\!|)$ we again follow [J2], namely we define the projection P in $\mathscr{L}^*$:

$$(P\varphi)f = \sum \varphi(f_n) \quad\text{for}\quad \varphi\in\mathscr{L}^* \quad\text{and for}\quad f\in\mathscr{L}, \tag{4}$$

where $w'\text{-}\sum f_n = f$, $\{f_n\}\in\mathscr{S}$ and $\{f_n\}\subset\mathscr{K}(X,Y)$.

Now we observe that the sum in (4) converges and does not depend on the sequence $\{f_n\}$ with $w'\text{-}\sum f_n = f$. Indeed, let $s_n = \sum_i^n f_i$. The uniform boundedness principle implies that $\{\varphi(s_n)\}$ is bounded and thus $\limsup_n \varphi(s_n) = \lim_k \varphi(s_{n_k})$ and $\liminf_n \varphi(s_n) = \lim_k \varphi(s_{m_k})$ for suitable subsequences $\{n_k\}$ and $\{m_k\}$ of natural numbers. Thus $\limsup\varphi(s_n) - \liminf\varphi(s_n) = \lim_k \varphi(s_{n_k} - s_{m_k}) = 0$, because $s_{n_k} - s_{m_k} \to 0$ weakly by (K). Similarly we show that if $\{g_n\}\subset\mathscr{K}$ then $\sum\varphi(f_n) = \sum\varphi(g_n)$ for any $\varphi\in\mathscr{K}^*$. Thus P is well defined.

Now it is not difficult to check that P is a bounded linear projection in $\mathscr{L}^*$ and that $\operatorname{Ker} P = \mathscr{K}^\circ$ (cf. [J2]).

Given $\varepsilon > 0$ we choose $\|\!|\varphi\|\!| = 1$, $\|\!| f\|\!| = 1$ so that $\|\!| P\|\!| \leqslant P(\varphi)(f) + \varepsilon$. Thus

$$\begin{aligned}
\|\!| P\|\!| &\leqslant \sum\varphi(f_n) + \varepsilon \leqslant \sup_m \sum_{i=1}^{\infty}\varphi(f_m) + \varepsilon \leqslant \|\!|\varphi\|\!| \sup_m \Big\|\!\Big|\sum_{i=1}^{m} f_i\Big\|\!\Big| + \varepsilon \leqslant K_u(\{\hat{f}_n\}_n) + \varepsilon\\
&\leqslant K_u(\{f_n\}_n) + \varepsilon \leqslant \|\!| f\|\!| + 2\varepsilon \leqslant 1 + 2\varepsilon
\end{aligned}$$

for suitable $w'\text{-}\sum f_n = f$, $K_u(\{f_n\}) \leqslant |||f||| + \varepsilon$. Here we have used that by (ii)

$$\{\hat{f}_n\} = \{f_1, \ldots, f_m, 0, 0, \ldots\} \in \mathscr{S}(X, Y)$$

and $w'\text{-}\sum \hat{f}_n = \sum\limits_{i=1}^{m} f_i$.

Finally, we suppose that the class $\mathscr{S}(X, Y)$ has the property $(\mathscr{U})$ and show that

$$|||\varphi - 2P\varphi||| \leqslant |||\varphi|||$$

for all $\varphi \in \mathscr{L}^*$. Indeed, let $f \in \mathscr{L}$. Then

$$\begin{aligned}(5)\qquad |||\varphi - 2P\varphi||| &= \sup\Big\{\lim_n \Big|\Big(f - 2\sum_{i=1}^{n} f_i\Big)\varphi\Big|;\ f \in \mathscr{L}, |||f||| \leqslant 1\Big\} \\ &\leqslant |||\varphi||| \cdot \sup_n\Big\{\Big|\Big|\Big|f - 2\sum_{i=1}^{n} f_i\Big|\Big|\Big|;\ f \in \mathscr{L}, |||f||| \leqslant 1\Big\}.\end{aligned}$$

Let $\varepsilon > 0$, $f \in \mathscr{L}$ and let $\{f_n\} \subset \mathscr{K}(X, Y)$ be the sequence from $\mathscr{S}$ such that $w'\text{-}\sum f_n = f$ and such that $\|K_u(\{f_n\})\| \leqslant |||f||| + \varepsilon$. Let m be fixed and let $\{\hat{f}_n\} = \{f_1, f_2, \ldots, f_m, 0, 0, 0, \ldots\} \in \mathscr{S}$. Then $\{g_n\} = \{f_n - 2\hat{f}_n\} \in \mathscr{S}$, $g_n \in \mathscr{K}(X, Y)$ and $w'\text{-}\sum g_n = f - 2\sum\limits_{i=1}^{m} f_i$. Thus

$$\Big|\Big|\Big|f - 2\sum_{i=1}^{m} f_i\Big|\Big|\Big| \leqslant K_u(\{g_n\}_n) \leqslant K_u(\{f_n\}) \leqslant |||f||| + \varepsilon.$$

The middle inequality follows by a simple calculation directly from the definition of $K_u(\{f_n\})$.

The last inequality together with (5) imply that $\mathscr{K}(X, Y)$ is u-ideal in $\mathscr{L}$. $\square$

Corollary 1. *Let every operator $f \in \mathscr{L}(X, Y)$ be factorable through a Banach space Z_f, Z_f having a shrinking unconditional basis (more generally an unconditional shrinking finite-dimensional decomposition). Then $\mathscr{K}(X, Y)$ is a u-ideal in $(\mathscr{L}(X, Y), |||\cdot|||)$ where $|||\cdot|||$ is a norm equivalent to the sup norm $|\cdot|$ on $\mathscr{L}(X, Y)$.*

P r o o f. Let the class $\mathscr{S}(X, Y)$ consist of all sequences $\{f_n\}$ of the form $f_n = Ak_nB$, where $f \in \mathscr{L}(X, Y)$ and $f = AB$ is the factorization of f through Z_f, Z_f having a shrinking 1-unconditional basis. (Note that Z_f may vary with f.) Let $\|\cdot\|$ be the factorization norm on $\mathscr{Z}(X, Y)$ defined by

$$\|f\| = \inf |A| \cdot |B| \cdot K_u(\{k_n\}_n) \quad \text{for} \quad f \in \mathscr{L}(X, Y).$$

Here the k_n's are the canonical projections onto the subspaces of Z_f which form the shrinking 1-unconditional decomposition of Z_f, so that $K_u(\{k_n\}_n) = 1$ and $K_u(\{k_n\}_n)$ is computed in the norm $|\cdot|$. The infimum is taken over all the above described factorizations $f = AB$.

Below we observe that $\|\cdot\|$ is a norm and similarly as in the proof of Proposition 1 we can show that $\|\cdot\|$ is equivalent to the sup norm on $\mathscr{L}(X,Y)$.

Notice also that $|\!|\!|\cdot|\!|\!| = \|\cdot\|$ *where* $|\!|\!|\cdot|\!|\!|$ *is the norm from Proposition* 1. (The norm $|\!|\!|\cdot|\!|\!|$ is built on the norm $\|\cdot\|$.) Indeed, $|\!|\!|f|\!|\!| \leqslant K_u(\{Ak_nB\}_n) \leqslant |A||B|K_u(\{k_n\}_n) \leqslant \|f\| + \varepsilon$ for a suitable factorization $f = AB$ of f. On the other hand, by definition we have $K_u(\{f_n\}) \geqslant \|f\|$ for each w'-$\sum f_n = f$.

$\mathscr{S}(X,Y)$ has the property $(\mathscr{U})$ with respect to the factorization norm $\|\cdot\|$ on $\mathscr{L}(X,Y)$ and has all the properties we have demanded for the class $\mathscr{S}(X,Y)$. We shall only show that $\mathscr{S}(X,Y)$ is closed in the sense described in the definition. Thus let $\{f_{np}\}_n \in \mathscr{S}(X,Y)$ be such that $\sum_p \|f_{np}\| = \sum_p |\!|\!|f_{np}|\!|\!| \leqslant \sum K_u(\{f_{np}\}_n) \leqslant C$, w'-$\sum_n f_{np} = f_p$ and let $f_{np} = b_p k_{np} a_p$ where $f_p = a_p b_p$ are the factorizations through Banach spaces Z_p, Z_p having a countable finite-dimensional unconditional decomposition given by the projections $\{k_{np}\}$. Having in mind the definition of the norm $\|\cdot\|$ we assume that $|a_p||b_p|K_u(\{k_{np}\}_n) < \|f_p\| + \frac{\varepsilon}{2^p}$, where $K_u(\{k_{np}\}_n) = 1$. According to [Pie, Lemma 8.6.4.] we may further assume that $1 = |b_1| \geqslant |b_2| \geqslant \ldots \geqslant 0$, $\lim_p |b_p| = 0$ and that $\sum |a_p| \leqslant \sum \|f_p\| + \varepsilon$. Let $Z = \left(\bigoplus_{n=1}^{\infty} Z_p\right)_{c_0}$ be the c_0-sum of Z_p's with the sup norm. Let q_p be the projections of Z onto Z_p and let i_p be the imbeddings of Z_p into Z. Let us write $\mathscr{K}_{np} = i_p k_{np} q_p \in \mathscr{K}(Z)$, $A_p = a_p q_p \in \mathscr{L}(Z,Y)$ and $B_p = i_p b_p \in \mathscr{L}(X,Z)$. Then Z also has a 1-unconditional finite-dimensional decomposition $Z = \bigoplus_{n,p=1}^{\infty} K_{np} = \bigoplus_{i=1}^{\infty} K_{n_i p_i}$ where $\{K_{n_i p_i}\}$ is any reordering of $\{K_{np}\}$ into a sequence. Let $B\colon X \to Z$ be defined by $Bx = \sum B_p \in Z$ and let $A\colon Z \to Y$, $Az = \sum_p A_p z$. Then evidently $|B| \leqslant 1$ and $|A| \leqslant \sum |A_p| \leqslant \sum |a_p| \leqslant \sum_p \|f_p\| + \varepsilon$. We easily see that $f_{np} = AK_{np}B$ are factorizations through Z. Now

$$w'\text{-}\sum_i f_{n_i p_i} = A \circ \left(w'\text{-}\sum_i K_{n_i p_i}\right) \circ B = AB$$

and thus $\{f_{n_i p_i}\}_i = \{AK_{n_i p_i}B\}_i \in \mathscr{S}(X,Y)$.

Similarly we observe that $\mathscr{S}(X,Y)$ is closed under addition. Indeed,

$$f_{n1} + f_{n2} = A(K_{n1} + K_{n2})B$$

and thus w'-$\sum_n (f_{n1} + f_{n2}) = f_1 + f_2$ and by definition $\{f_{n1} + f_{n2}\}_n \in \mathscr{S}(X,Y)$ and

$$\|f_1 + f_2\| \leqslant |A||B|K_u(\{K_{n1} + K_{n2}\}_n).$$

Having in mind that $|A| \leqslant \sum_{p=1}^{2} |f_p| + \varepsilon$ and that

$$K_u(\{K_{n1}+K_{n2}\}_n) \leqslant \max_p K_u(\{K_{np}\}_n) \leqslant 1$$

we see that $\|\cdot\|$ is a norm.

Finally, we observe that the class $\mathscr{S}$ has the property $\mathscr{U}$ with respect to the norm $\|\cdot\|$. Indeed,

$$\sum_{i=1}^{n} \varepsilon_i f_i = A \circ \left(\sum_{i=1}^{n} \varepsilon_i k_i\right) \circ B \text{ and } f - 2\sum_{i=1}^{n} \eta_i f_i = A \circ \left(\mathrm{Id}_Z - 2\sum_{i=1}^{n} \eta_i k_i\right) \circ B$$

and thus

$$\max\left\{\left\|\sum_{i=1}^{n} \varepsilon_i f_i\right\|, \left\|f - 2\sum_{i=1}^{n} \eta_i f_i\right\|\right\} \leqslant |A| \cdot |B| \cdot K_u(\{k_n\}_n)$$

for all $n > 1$, all $|\varepsilon_i| \leqslant 1$ and all $0 \leqslant \eta_i \leqslant 1$. □

Remark 1. We could alternatively have used instead of $K_u(\{f_n\})$ the number

$$\tilde{K}_u(\{f_n\}) = \limsup_n \left\{\max\left(\left\|\sum_{i=1}^{n} \varepsilon_i f_i\right\|, \left\|f - 2\sum_{i=1}^{n} \eta_i f_i\right\|\right);\ |\varepsilon_i| \leqslant 1, 0 \leqslant \eta_i \leqslant 1\right\}.$$

Also we could have defined an equivalent norm $|\!|\!|\cdot|\!|\!|_1 = \inf K_u(\{f_n\})$ where the $K_u(\{f_n\})$ is built from the sup norm $|\cdot|$. Corollary 1 holds also for this norm, i.e. $\mathscr{K}(X,Y)$ is a u-ideal in $(\mathscr{L}(X,Y), |\!|\!|\cdot|\!|\!|)$. $|\!|\!|\cdot|\!|\!|$ is equivalent to the norm $\|\cdot\|$ and thus to the sup norm $|\cdot|$. The same holds for the norms built from $\tilde{K}_u(\{f_n\})$.

Remark 2. The corollary applies in particular when every $f \in \mathscr{L}(X,Y)$ is factorable through a Hilbert space. This is for example the case of $\mathscr{L}(P,P^*)$, where P is any Pisier space [Pi], [J1]. Note that the canonical basis in the Hilbert space is 1-unconditional so that we can easily see that the norm $|\!|\!|\cdot|\!|\!|$ is equal to the usual γ_2 norm, i.e. the factorization norm through a Hilbert space. Thus

$\mathscr{K}(P,P^*)$ is a u-ideal in $\mathscr{L}(P,P^*)$, where the latter space is equipped with the γ_2 norm.

This strengthens a result in [J1, J2].

Remark 3. In Proposition 1 we have supposed that each $f \in \mathscr{L}(X,Y)$ is suitably factorable. Nevertheless Proposition 1 remains also valid for smaller classes of operators:

Proposition 1a. *Let X, Y be Banach spaces, let $\mathscr{Z}(X,Y) \subset \mathscr{L}(X,Y)$ be a vector subspace (not necessarily closed in the sup norm $|\cdot|$) and let $\mathscr{S}(X,Y)$ be an ($\mathscr{S}$)-class of sequences $\{f_n\} \subset \mathscr{K}(X,Y)$ which has the property ($\mathscr{U}$) with respect to the norm $\|\cdot\|$. Suppose that for every $f \in \mathscr{Z}(X,Y)$ there is a sequence $\{f_n\} \subset \mathscr{Z}(X,Y) \cap \mathscr{K}(X,Y)$, $\{f_n\} \in \mathscr{S}(X,Y)$ with $w' - \sum f_n = f$ and let $\|\cdot\| \geqslant |\cdot|$ be a complete operator ideal norm on $\mathscr{Z}(X,Y)$. Then the norm $|\!|\!|\cdot|\!|\!|$,*

$$|\!|\!|f|\!|\!| = \inf\{K_u(\{f_n\});\ w'\text{-}\sum f_n = f, \{f_n\} \in \mathscr{S}(X,Y), \{f_n\} \subset \mathscr{K}(X,Y) \cap \mathscr{Z}(X,Y)\}$$

for $f \in \mathscr{Z}(X,Y)$ is an equivalent norm on $(\mathscr{Z}(X,Y), \|\cdot\|)$ and $\mathscr{Z}(X,Y) \cap \mathscr{K}(X,Y)$ is a u-ideal in $(\mathscr{Z}(X,Y), |\!|\!|\cdot|\!|\!|)$.

Proof. For the proof that $\mathscr{Z}(X,Y) \cap \mathscr{K}(X,Y)$ is a u-ideal in $(\mathscr{Z}(X,Y), |\!|\!|\cdot|\!|\!|)$ we only have to substitute $\mathscr{L}$ by $\mathscr{Z}$ and $\mathscr{K}$ by $\mathscr{K} \cap \mathscr{Z}$ in the proof of Proposition 1. So it remains to observe that $|\!|\!|\cdot|\!|\!|$ is a norm equivalent to the norm $\|\cdot\|$. Again we show that $(\mathscr{Z}, |\!|\!|\cdot|\!|\!|)$ is complete. We follow the proof of Proposition 1 having in mind that $|\cdot| \leqslant \|\cdot\| \leqslant |\!|\!|\cdot|\!|\!|$. Thus $\sum f_p$ converges in the Banach space $(\mathscr{Z}, \|\cdot\|)$ and similarly $\sum\limits_p f_{np} \in \mathscr{K} \cap \mathscr{Z}$ exists because $\mathscr{K}$ is $|\cdot|$ complete and $\mathscr{L}$ is $\|\cdot\|$ complete. This completes the sketch of the proof. □

Variants of Proposition 1a where the norm $\|\cdot\|$ is a factorization norm on $\mathscr{Z}(X,Y)$ are also possible. Again we denote by $|\cdot|$ the sup norm.

Proposition 2. *Let $(\mathscr{A}, |\cdot|_{\mathscr{A}})$ be a normed operator ideal and let $\mathscr{Z}(X,Y)$ be a class of all operators $f\colon X \to Y$ which are factorable through a Banach space $Z = Z_f$, $f = AB$ where $B \in \mathscr{L}(X,Z)$, $A \in \mathscr{A}(Z,Y)$. Suppose further that there is a sequence $\{k_n\} \subset \mathscr{K}(Z_f)$ such that $K_u(\{k_n\}_n)$ is finite and such that $w'\text{-}\sum k_n = \mathrm{Id}_{Z_f}$. Let the ($\mathscr{S}$)-class $\mathscr{S}(X,Y)$ consist of all sequences $\{f_n\}$ of the form $f_n = Ak_nB$, where $f \in \mathscr{Z}(X,Y)$ and $f = AB$ is the above described factorization of f through Z_f with $A \in \mathscr{A}(Z,Y)$. (Note that Z_f may vary with f.) Let the norm $\|\cdot\|$ on $\mathscr{Z}(X,Y)$ be defined by*

$$\|f\| = \inf |A|_{\mathscr{A}} \cdot |B| \cdot K_u(\{k_n\}_n) \quad \text{for } f \in \mathscr{Z}(X,Y).$$

The infimum is taken over all above described factorizations $f = AB$, $w'\text{-}\sum k_n = \mathrm{Id}_{Z_f}$, $K_u(\{k_n\}_n) < \infty$ and $A \in \mathscr{A}$.

Then $\mathscr{S}(X,Y)$ is an ($\mathscr{S}$)-class which has the property ($\mathscr{U}$) with respect to the operator norm $\|\cdot\|$ on $\mathscr{Z}(X,Y)$.

Thus Proposition 1a *yields that $\mathscr{K}(X,Y) \cap \mathscr{Z}(X,Y)$ is a u-ideal in $(\mathscr{Z}(X,Y), |\!|\!|\cdot|\!|\!|) = (\mathscr{Z}(X,Y), \|\cdot\|)$.*

Proof. First we observe that if $f \in \mathscr{Z}(X,Y)$ and $f_n = Ak_nB$ are as in the proposition then $f_n \in \mathscr{K}(X,Y) \cap \mathscr{Z}(X,Y)$ and $w'\text{-}\sum f_n = f$. Next we notice that $\|\cdot\|$ is a norm and that $\mathscr{S}(X,Y)$ is an $(\mathscr{S})$-class with the property $(\mathscr{U})$. The argument is the same as in the proof of Corollary 1. We just write $|A|_{\mathscr{A}}$ instead of $|\mathscr{A}|$ for $A \in \mathscr{A}(Z_f, Y)$. □

Remark 4. Proposition 2 remains valid if we suppose the factorizations $f = AB$ in the form $A \in \mathscr{L}(Z,Y)$ and $B \in \mathscr{B}(X,Z)$ where $(\mathscr{B}, |\cdot|_{\mathscr{B}})$ is a normed operator ideal. Of course the norm $\|\cdot\|$ on $\mathscr{Z}(X,Y)$ is then defined by

$$\|f\| = \inf |A| \cdot |B|_{\mathscr{B}} \cdot K_u(\{k_n\}_n).$$

Proposition 2 has for example the following two special cases:

A) *Let X, Y be Banach spaces and let $\Gamma_2(X,Y) \subset \mathscr{L}(X,Y)$ be the set of all operators from $\mathscr{L}(X,Y)$ which are factorable through a Hilbert space. Then $\mathscr{K}(X,Y) \cap \Gamma_2(X,Y)$ is a u-ideal in* $(\Gamma_2(X,Y), \gamma_2)$.

Notice that as in Remark 2 we have $\|\!|\cdot|\!\| = \gamma_2$.

It is well known that every 2-summing operator $f\colon X \to Y$ may be factored through a Hilbert space H as $f = AB$ where $A\colon X \to H$ is 2-summing. Let $(\mathscr{P}_2, P_2)$ denote the operator ideal of 2-summing operators [Pie]. Then Proposition 2 with Remark 3 give

B) *$\mathscr{K}(X,Y) \cap \mathscr{P}_2(X,Y)$ is a u-ideal in $(\mathscr{P}_2, P_2)(X,Y)$ for any Banach spaces X, Y.*

Acknowledgement. The author is indebted to Dirk Werner for valuable discussions on subjects closely connected with the paper.

References

[E,J] *G. Emmanuele, K. John*: Some remarks on the position of the space $\mathscr{K}(X,Y)$ inside the space $\mathscr{W}(X,Y)$. New Zealand J. Math. *26(2)* (1997), 183–189.

[F] *J. H. Fourie*: Projections on operator duals. Quaestiones Math. *21* (1998), 11–18.

[G,K,S] *G. Godefroy, N. J. Kalton, P. D. Saphar*: Unconditional ideals in Banach spaces. Studia Math. *104 (1)* (1993), 13–59.

[H,W,W] *P. Harmand, D. Werner, W. Werner*: M-ideals in Banach Spaces and Banach Algebras. Lecture Notes in Math., 1547, Springer, Berlin, 1993.

[J1] *K. John*: On the space $\mathscr{K}(P,P^*)$ of compact operators on Pisier space P. Note di Matematica *12* (1992), 69–75.

[J2] *K. John*: On a result of J. Johnson. Czechoslovak Math. Journal *45* (1995), 235–240.

[Jo] *J. Johnson*: Remarks on Banach spaces of compact operators. J. Funct. Analysis *32* (1979), 304–311.

[Ka] *N. J. Kalton*: Spaces of compact operators. Math. Annalen *208* (1974), 267–278.

[Li] *Å. Lima*: Property (wM^*) and the unconditional metric approximation property. Studia Math. *113 (3)* (1995), 249–263.

[LT,I] *J. Lindenstrauss, L. Tzafriri*: Classical Banach Spaces, Sequence Spaces. EMG 92 Springer Verlag (1977).

[LT,II] *J. Lindenstrauss, L. Tzafriri*: Classical Banach Spaces, Function Spaces. EMG 97 Springer Verlag (1979).

[Pie] *A. Pietsch*: Operator ideals. Berlin, Deutscher Verlag der Wissenschaften, 1978.

[Pi] *J. Pisier*: Counterexamples to a conjecture of Grothendieck. Acta Mat. *151* (1983), 180–208.

[Ru] *W. Ruess*: Duality and Geometry of spaces of compact operators. Functional Analysis: Surveys and Recent Results III. Math. Studies 90, North Holland, 1984, pp. 59–78.

Author's address: Mathematical Institute of Czech Academy of Sciences, Žitná 25, 115 67 Praha 1, Czech Republic, e-mail: Kjohn@math.cas.cz.