Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 49 (1999), No. 3, 585–606

Persistent URL: `http://dml.cz/dmlcz/127512`

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$\mathscr{D}$-MODULES, CONTACT VALUED CALCULUS AND POINCARE-CARTAN FORM

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(Received October 30, 1996)

1. Introduction and notation

The geometric formulation of the calculus of variations has been carried out in the last decades in different fashion (see, for instance, [1], [2], [3], [4], [6], [7], [9], [11], [12], [13], [16]). These methods, used in the process of generalizing from mechanics to the higher order case and several variables, may seem somewhat artificial. For instance, in the construction of a general Poincaré-Cartan form, one utilizes a method involving determination of coefficients under certain prescribed conditions, and other adhoc techniques. It is not always clear what the real significance is of the role played by new objects (e.g. a linear connection on the base manifold) that are extrinsic to the problem.

We offer a new approach that unifies the classical methods and their generalizations to field theories of higher order and the study of the variational bicomplex in a natural and rather simple way. We make use of two basic tools; first, a decomposition of operators that is trivial in the first order case. This is where the above mentioned connection comes in. Second, we introduce a formal covariant derivation law which is the geometric analogue of the derivative of variations with respect to time in mechanics.

For a differentiable manifold M, we make the key remark that the algebra Ω^n of differential forms of maximal degree is a right module over the ring $\mathscr{D}$ of differential operators. With each linear connection on M we associate a morphism whose effect is to decompose the higher order tangent fields $\mathscr{T}^k$, $k > 1$ ($\mathscr{T}^k \subset \mathscr{D}$) as a composition of ordinary tangent fields with differential operators of order $k-1$. With this we achieve a factorization, via the exterior differential, of the action of $\mathscr{T}^k$ on the $\mathscr{D}$-module Ω^n. Mutatis mutandis, it is possible to generalize the method to the valued case $\Omega^n \otimes \mathscr{A}$, when $\mathscr{A}$ is a left $\mathscr{D}$-module through the action of the covariant derivative associated with a derivation law.

Let $\mathscr{C}$ be the contact module on the jet spaces of a fibred manifold with base manifold M. By making use of the formal or total derivative, we define a covariant derivation law on M with values in $\mathscr{C}$. This law yields a differential $\mathscr{C}$-valued calculus on M that allows us to recover in a natural way the main objects and manipulations of the classical variational calculus and then to generalize it to the higher order and several variables case; in particular, the Euler-Lagrange form appears as a result of a certain action on the Lagrangian density.

The integration by parts formula (called 'decomposition formula' in [6]) relates the differential of the Lagrangian density, the Euler-Lagrange form and an exact differential form. As shown in [6], to find a Poincaré-Cartan form is equivalent to finding an 'integral' of this exact form. We do this by means of a constructive process which is based on the factorization of the action of $\mathscr{T}^k$ on $\Omega^n \otimes \mathscr{C}$.

Obviously, the $\mathscr{C}$-valued calculus admits extension to a calculus with values in the exterior algebra of $\mathscr{C}$. This extension allows us to define the so-called variational bicomplex $\Phi^{r,s}$. We give a canonical construction (without reference to the coordinate expressions) of the Euler-Lagrange resolution. Finally, with the same technique with which we have built the Poincaré-Cartan form (relative to a specific linear connection on M), we give the decomposition of the spaces $\Phi^{r,n}$ ($\dim M = n$) explicitly finding an 'integral' in $\Phi^{r,n-1}$ of the exact term in this decomposition.

The paper is structured as follows. In §2 we see how a linear connection on M produces a notion of degree in $\mathscr{D}$. By making use of the homogeneous components of §2 we find in §3 the decomposition morphism of $\mathscr{T}^k$. In §4 we recall some properties of the $\mathscr{D}$-modules. In §5 we will see how to use the decomposition morphism (§3) to factorize the action of $\mathscr{T}^k$ on Ω^n. In §6 we make some remarks on Weil's definition of the jet spaces and the vertical lift of tangent vectors. In §7 we recall the definition of the formal or total derivatives. With the results of §7 we give in §8 the definitions of the structure form, the contact module $\mathscr{C}$ and the $\mathscr{C}$-valued differential calculus. In §9 we recover the fundamental elements of the calculus of variations, generalizing them to higher order and several variables, and we give our construction of the Poincaré-Cartan form. Finally, in §10, we apply the above techniques to the Euler-Lagrange resolution.

We now fix notation and a few conventions that we will use later. For a differentiable manifold M we will write $\mathscr{C}^\infty(M)$ for the ring of infinitely differentiable functions, TM for the tangent bundle, T^*M for the cotangent bundle and $\wedge^p M$ for the bundle of exterior forms of degree p. If $\pi\colon N \longrightarrow M$ is a projection between differentiable manifolds, we will write $T^v(N/M)$ for the bundle of vertical tangent vectors of π; if $F \longrightarrow M$ and $G \longrightarrow M$ are vector bundles, we will write F_N for the pullback π^*F and $F \otimes_M G$ for the tensor product ; when there is no risk of confusion we will understand that the tensor product is taken before a suitable pullback. Calligraphic letters will be reserved for the modules of sections of the corresponding vector bundles; for instance, $\mathscr{T}M$ will mean the $\mathscr{C}^\infty(M)$-module of sections of TM, i.e. the module of tangent vector fields. As an exception $\Omega^p(M)$ will mean the $\mathscr{C}^\infty(M)$-module of sections of $\wedge^p M$.

2. Graduation of differential operators

Let M be a differentiable manifold, $\dim M = n$ and $\mathfrak{m}_x$ the maximal ideal of functions vanishing at a point x of M.

A *differential operator* of order k on M is, by definition, an $\mathbb{R}$-linear morphism $P\colon \mathscr{C}^\infty(M) \longrightarrow \mathscr{C}^\infty(M)$ that sends $\mathfrak{m}_x^{k+1}$ to $\mathfrak{m}_x$ for each $x \in M$. We will denote the set of differential operators of order k on M by $\mathscr{D}^k$.

Let $p\colon M \times \mathbb{R} \longrightarrow M$ be the projection onto the first factor, $J^k p$ the fiber bundle of k-jets of sections of p (i.e. the functions on M) and $\mathscr{J}^k p$ the module of the corresponding sections. The map $j^k\colon \mathscr{C}^\infty(M) \longrightarrow \mathscr{J}^k p$, associating with every function its k-th Taylor expansion, represents $\mathscr{D}^k$ in the following sense: for every $P \in \mathscr{D}^k$ there is a unique morphism $\overline{P}\colon \mathscr{J}^k p \longrightarrow \mathscr{C}^\infty(M)$ such that $P = \overline{P} \circ j^k$. In other words, $\mathscr{D}^k$ is the dual $\mathscr{C}^\infty(M)$-module of $\mathscr{J}^k p$. Therefore $\mathscr{D}^k$ is locally free, with rank $\binom{n+k}{n}$, and, if $(x_1, \ldots, x_n)$ is a local chart on M, $\mathscr{D}^k$ is generated by $\partial^\alpha = (\frac{\partial}{\partial x_1})^{\alpha_1} \circ \ldots \circ (\frac{\partial}{\partial x_n})^{\alpha_n}$ as $\alpha = (\alpha_1, \ldots, \alpha_n)$ runs through the multi-indexes $|\alpha| \leqslant k$ (∂^0 means the constant function 1). If $k < r$, then there is a natural immersion $\mathscr{D}^k \subset \mathscr{D}^r$ (observe that $\mathscr{D}^0 = \mathscr{C}^\infty(M)$).

Besides the structure mentioned, every $\mathscr{D}^k$ has a second canonical $\mathscr{C}^\infty(M)$-module structure. If $g \in \mathscr{C}^\infty(M)$ and $P \in \mathscr{D}^k$, we define the operator $g * P \in \mathscr{D}^k$ according to the following rule: $(g * P)f = P(gf)$for every $f \in \mathscr{C}^\infty(M)$. This new structure for $\mathscr{D}^k$ will be denoted by $\overline{\mathscr{D}^k}$. The $\mathscr{C}^\infty(M)$-module $\overline{\mathscr{D}^k}$ is also locally free with the same local generators.

The subbundle $T^{*,k}M$ of $\mathscr{J}^k p$ comprised of those k-jets whose projections over $M \times \mathbb{R}$ are zero is, by definition, the k-th *cotangent fibre bundle* of M. The fibre of each $x \in M$ is $T_x^{*,k}M = \mathfrak{m}_x/\mathfrak{m}_x^{k+1} \subset \mathscr{C}^\infty(M)/\mathfrak{m}_x^{k+1} = J_x^k p$. The module of sections of $T^{*,k}$ will be denoted by $\mathscr{T}^{*,k}$.

The dual of $T^{*,k}$ is called the k-th *tangent fiber bundle* of M. We will denote it by $T^k M$. Equivalently, $T^k M$ is the incident of $\mathbb{R}$ in $J^k p$. The module of sections of $T^k M$ will be denoted by $\mathscr{T}^k M$. The elements of $\mathscr{T}^k M$ can be characterized by Leibniz's rule of derivations of products. On the other hand $\mathscr{T}^k M$ is, at the same time, a quotient and a submodule of $\mathscr{D}^k$; in fact we have the decomposition

$$\mathscr{D}^k \simeq \mathscr{C}^\infty(M) \oplus \mathscr{T}^k M$$

where we put $P = P(1) + (P - P(1))$ for each $P \in \mathscr{D}^k$.

For the sake of simplicity, we will write $\mathscr{T}^k$ instead of $\mathscr{T}^k M$ and $\mathscr{T}$ instead of $\mathscr{T}^1 M$. For $k \geqslant 0$, the quotient of the $\mathscr{C}^\infty(M)$-modules $\mathscr{D}^k/\mathscr{D}^{k-1}$ has the fibre $(\mathfrak{m}_x^k/\mathfrak{m}_x^{k+1})^* \simeq S^k(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ at each point $x \in M$, where $(\ )^*$ means taking the dual and S^k means the k-th tensor symmetric product.

From the above, making use of the isomorphism $(\mathfrak{m}_x/\mathfrak{m}_x^2)^* \simeq T_x$, we infer the exact sequence

$$0 \longrightarrow \mathscr{D}^{k-1} \longrightarrow \mathscr{D}^k \longrightarrow S^k\mathscr{T} \longrightarrow 0.$$

The image under the projection $\mathscr{D}^k \longrightarrow S^k\mathscr{T}$ of an element $P \in \mathscr{D}^k$ is known as the *symbol* of P.

Remark 2.1. The same exact sequence holds for the tangent bundles (taking into account the inclusions $\mathscr{T}^k \subset \mathscr{D}^k$):

$$0 \longrightarrow \mathscr{T}^{k-1} \longrightarrow \mathscr{T}^k \longrightarrow S^k\mathscr{T} \longrightarrow 0.$$

We can consider $\mathscr{C}^\infty(M)$ and $S^k\mathscr{T}$ as the homogeneous minimal and maximal degree components of $\mathscr{D}^k$, respectively. We will show how to define the remaining components for a specific linear connection.

Lemma 2.2. *A linear connection ∇ on M gives the module $\mathscr{D}^k$ a graduation via an isomorphism $\bigoplus\limits_{r=0}^{k} S^k\mathscr{T} \simeq \mathscr{D}^k$ (where $S^0\mathscr{T} = \mathscr{C}^\infty(M)$).*

Proof. It suffices to find sections s_r for the exact sequences

$$0 \longrightarrow \mathscr{D}^{r-1} \longrightarrow \mathscr{D}^r \longrightarrow S^r\mathscr{T} \longrightarrow 0 \quad r = 1, \ldots, k.$$

For $r = 1$, the sequence is

$$0 \longrightarrow \mathscr{C}^\infty(M) \longrightarrow \mathscr{D}^1 \longrightarrow \mathscr{T} \longrightarrow 0$$

and we take the section $s_1\colon \mathscr{T} \longrightarrow \mathscr{D}^1$ as the natural inclusion. For $r > 1$ we define the section $s_r\colon S^r\mathscr{T} \longrightarrow \mathscr{D}^r$ by the rule

$$\begin{aligned} s_r(D_1 \ldots D_r) = \frac{1}{r}\sum_{i=1}^{r} (D_i \circ s_{r-1}(D_1 \ldots D_{i-1}D_{i+1} \ldots D_r) \\ - s_{r-1}(D_i^\nabla(D_1 \ldots D_{i-1}D_{i+1} \ldots D_r))) \end{aligned}$$

where $D_1 \dots D_r$ is the symmetric product of r tangent vector fields D_i and D_i^∇ is the covariant derivative that ∇ induces over $S^{r-1}\mathscr{T}$.

It is easy to check that each s_r is well defined, $\mathscr{C}^\infty(M)$-linear and it is also a section of $\mathscr{D}^r \longrightarrow S^r\mathscr{T}$. $\square$

See [5] for another construction of this graduation of $\mathscr{D}^k$.

We obtain the expression of this in coordinates by letting $(x_1, \dots, x_n)$ be a local chart for M, $\partial_i = \frac{\partial}{\partial x_i}$, $\partial^{\bullet\alpha} = \partial_1^{\alpha_1} \dots \partial_n^{\alpha} \in S^r\mathscr{T}$, $\partial^\alpha = \partial_1^{\alpha_1} \circ \dots \circ \partial_n^{\alpha} \in \mathscr{D}^r$ with $\alpha = (\alpha_1, \dots, \alpha_n), |\alpha| = r$ and ∇ the local linear connection defined by $(x_1, \dots, x_n)$. Then $s_r(\partial^{\bullet\alpha}) = \partial^\alpha$.

3. Decomposition morphism of higher order tangent bundles

If P is a differential operator of order $k-1$ and D is a tangent vector field, the composition $P \circ D$ belongs to $\mathscr{T}^k$. This type of compositions defines a $\mathscr{C}^\infty(M)$-module morphism that we will explain in what follows.

Define a map

$$G\colon \overline{\mathscr{D}^{k-1}} \otimes_M \mathscr{T} \longrightarrow \mathscr{T}^k$$

where $G(P \otimes D) = P \circ D$ and extending by linearity.

The map G is well defined by the choice of the structure for $\mathscr{D}^{k-1}$. However, G is not $\mathscr{C}^\infty(M)$-linear. Now let $\mathscr{N}_k$ be the $\mathscr{C}^\infty(M)$-module obtained from $\overline{\mathscr{D}^{k-1}}_M \otimes \mathscr{T}$ by defining the product by functions as follows: for each $f \in \mathscr{C}^\infty(M)$, we put $f \bullet (P \otimes D) = (fP) \otimes D$; i.e., only multiplying f by the first factor and throughout the first structure for $\mathscr{D}^{k-1}$. Over $\mathscr{N}_k$, the map G is $\mathscr{C}^\infty(M)$-linear.

Definition 3.1. Henceforth $G\colon \mathscr{N}_k \longrightarrow \mathscr{T}^k$ shall be known as the composition morphism.

We are interested in determining a section of the morphism G for subsequent use. The following is the precise statement of this.

Theorem 3.2. *Each linear connection ∇ on M produces a section $H_\nabla\colon \mathscr{T}^k \longrightarrow \mathscr{N}_k$ of G, i.e. $G \circ H_\nabla = Id$. We will call H_∇ the decomposition morphism.*

P r o o f. It suffices to define the images by H_∇ for each of the homogeneous components of the graduation produced by ∇ (Lemma 2.2). Let $D_1, ..., D_r$ be vector fields on M.

For $r = 1$, we define $H_\nabla(s_1(D_1)) = D_1$.

For $r > 1$, we put

$$H_\nabla(s_r(D_1 \dots D_r)) = \frac{1}{r}\sum_i (D_i \circ H_\nabla(s_{r-1}(D_1 \dots D_{i-1}D_{i+1} \dots D_r) \\ - H_\nabla(s_{r-1}(D_i^\nabla(D_1 \dots D_{i-1}D_{i+1} \dots D_r)))$$

where the symbol '$\circ$' means composing D_i with the first factor of the tensor product.

As with s_r, it is easy to see that H_∇ is well defined and $\mathscr{C}^\infty(M)$-linear. We will verify that H_∇ is a section of G:

$$\begin{aligned} G \circ H_\nabla(s_r(D_1 \dots D_r)) &= \frac{1}{r}\sum_i (G(D_i \circ H_\nabla s_{r-1}(D_1 \dots D_{i-1}D_{i+1} \dots D_r)) \\ &\quad - GH_\nabla s_{r-1}(D_i^\nabla(D_1 \dots D_{i-1}D_{i+1} \dots D_r))) \\ &= \frac{1}{r}\sum_i (D_i \circ GH_\nabla s_{r-1}(D_1 \dots D_{i-1}D_{i+1} \dots D_r) \\ &\quad - GH_\nabla s_{r-1}(D_i^\nabla(D_1 \dots D_{i-1}D_{i+1} \dots D_r))). \end{aligned}$$

By the induction hypothesis, $GH_\nabla s_{r-1} = s_{r-1}$, hence

$$\begin{aligned} G \circ H_\nabla(s_r(D_1 \dots D_r)) &= \frac{1}{r}\sum_i (D_i \circ s_{r-1}(D_1 \dots D_{i-1}D_{i+1} \dots D_r) \\ &\quad - s_{r-1}(D_i^\nabla(D_1 \dots D_{i-1}D_{i+1} \dots D_r))) \\ &= s_r(D_1 \dots D_r). \end{aligned}$$

□

Remark 3.3. For orders $k = 1$ and $k = 2$ the morphism H_∇ is independent of the connection ∇. Indeed, in order 1 it is clearly independent, and in order 2 we have

$$H(D_1 \circ D_2) = \frac{1}{2}(D_1 \otimes D_2 + D_2 \otimes D_1 + 1 \otimes [D_1, D_2])$$

where $[D_1, D_2]$ is the Lie bracket of D_1 and D_2. In general it can be deduced from the construction of H_∇ that H_∇ depends only on the $(k-2)$-jet of the symmetric connection associated with ∇.

In local coordinates, if ∇ is the connection that the local chart $(x_1, \dots, x_n)$ induces on M, we have that $H_\nabla(\partial^\alpha) = \sum\limits_i \frac{\alpha_i}{|\alpha|}\partial^{\alpha-\varepsilon_i} \otimes \partial_i$ where ε_i is the multi-index $(0, \dots, 0, \overset{(i)}{1}, 0, \dots, 0)$.

We have taken certain definitions and properties of $\mathscr{D}$-modules from Schneiders [18]. These properties will be used in the subsequent sections.

Definition 4.1. The injective limit of the system $(\mathscr{D}^k)_{k\in\mathbb{N}}$, where the inclusions $\mathscr{D}^k \subseteq \mathscr{D}^h$ for $k \leqslant h$ are considered, will be called the ring of the differential operators (of the manifold M) and denoted by $\mathscr{D}$.

The ring $\mathscr{D}$ inherits the $\mathscr{C}^\infty(M)$-module structure of $\mathscr{D}^k$. On the other hand, $\mathscr{D}$ is a non-commutative ring under composition of operators. Therefore, we need to distinguish between left and right $\mathscr{D}$-modules. The proof of the next results can be found in [18].

Proposition 4.2. *Let $\mathscr{A}$ be a $\mathscr{C}^\infty(M)$-module and let $\alpha\colon \mathscr{C}^\infty(M) \longrightarrow \mathrm{End}_{\mathbb{R}}\,\mathscr{A}$ be the associated multiplication morphism. Let us assume that there exists a map $\chi\colon \mathscr{T} \longrightarrow \mathrm{End}_{\mathbb{R}}\,\mathscr{A}$ defining an $\mathbb{R}$-linear action of $\mathscr{T}$ on $\mathscr{A}$ such that:*

1) $[\chi(D), \alpha(f)] = \alpha(Df)$ *(resp.* $-\alpha(Df)$*),*
2) $[\chi(D_1), \chi(D_2)] = \chi([D_1, D_2])$ *(resp.* $-\chi([D_2, D_1])$*),*
3) $\alpha(f) \circ \chi(D) = \chi(fD)$ *(resp.* $\chi(D) \circ \alpha(f) = \chi(fD)$*)*

for every $D_1, D_2, D_3 \in \mathscr{T}$ and every $f \in \mathscr{C}^\infty(M)$.

Then there exists a unique structure of the left (resp. right) $\mathscr{D}$-module enlarging the actions α and χ.

Corollary 4.3. *The exterior algebra $\Omega^n(M)$ comprised of the forms of maximal degree possesses a unique structure of the right $\mathscr{D}$-module that extends its structure, α, of the $\mathscr{C}^\infty(M)$-module, and also extends the action of $\mathscr{T}$ given by $\omega \cdot D = -L_D\omega$ for any $\omega \in \Omega^n(M), D \in \mathscr{T}$ (L_D being the Lie derivative).*

Corollary 4.4. *Let $\mathscr{A}_1, \mathscr{A}_2$ be two left $\mathscr{D}$-modules and $\mathscr{B}$ a right $\mathscr{D}$-module. Then:*

1) *The action of $\mathscr{T}$ on the $\mathscr{C}^\infty(M)$-module $\mathscr{A}_1 \otimes_M \mathscr{A}_2$ given by $D(a_1 \otimes a_2) = Da_1 \otimes a_2 + a_1 \otimes Da_2$ for any $D \in \mathscr{T}$, $a_1 \in \mathscr{A}_1$, $a_2 \in \mathscr{A}_2$, extends to a unique structure of the left $\mathscr{D}$-module on $\mathscr{A}_1 \otimes_M \mathscr{A}_2$.*
2) *The action of $\mathscr{T}$ on the $\mathscr{C}^\infty(M)$-module $\mathscr{B} \otimes_M \mathscr{A}_2$ given by $(b \otimes a) \cdot D = b \cdot D \otimes a - b \otimes Da$ for any $D \in \mathscr{T}$, $b \in \mathscr{B}$, $a \in \mathscr{A}_2$, extends to a unique structure of the right $\mathscr{D}$-module on $\mathscr{B} \otimes_M \mathscr{A}_2$.*

Let D be a (tangent) vector field on M and ω a form of maximal degree. Then for the action defined in 4.3 we have $\omega\cdot D=-L_D\omega=d(-i_D\omega)$ ($i_D\omega$ being the inner contraction of ω with D), in other words, we can build an explicit integral for the exact form $\omega\cdot D$. We want now to extend this result for the action of higher order vector fields $P\in\mathscr{T}^k$, $k>1$. The key to this resides in the decomposition morphism (Theorem 3.2).

Theorem 5.1. *Given a linear connection ∇ on M, for each $P\in\mathscr{T}^k$, there is an $\mathbb{R}$-linear map $\Phi_P\colon\ \Omega^n(M)\longrightarrow\Omega^{n-1}(M)$ that makes the next diagram commutative:*

$$\begin{tikzcd}
\Omega^n(M) \arrow[r, "\cdot P"] \arrow[d, "\Phi_P"'] & \Omega^n(M) \\
\Omega^{n-1}(M) \arrow[r, equal] & \Omega^{n-1}(M) \arrow[u, "d"']
\end{tikzcd}$$

where $\cdot P$ is the action of P and d is the exterior differential. In other words, for any $\omega\in\Omega^n(M)$, $\Phi_P\omega$ is an integral of $\omega\cdot P$.

Proof. If $H_\nabla(P)=\sum Q\otimes D\in\mathscr{N}_k$ then we can define $\Phi_P\omega=-\sum i_D(\omega\cdot Q)$ which gives

$$d\Phi_P\omega=d\sum -i_D(\omega\cdot Q)=\omega\cdot\Big(\sum Q\circ D\Big)=\omega\cdot(GH_\nabla P)=\omega\cdot P,$$

where we have made use of the fact that $L_D=di_D$ on $\Omega^n(M)$. $\square$

Remark 5.2. When P is of order 1 or 2, the decomposition morphism is independent of the linear connection, as a result of which Φ_P is also independent.

On the other hand, provided that $\dim M=1$, $\overline{\nabla}$ is another connection and $\overline{\Phi}_P$ is the associated operator, then $\overline{\Phi}_P$ can only differ from Φ_P in a constant term. Both Φ_P and $\overline{\Phi}_P$ depend linearly on P, thus we deduce that this constant term is zero. Therefore, if $\dim M=1$ for any order k, Φ_P is also independent of the connection.

Remark 5.3. The result obtained in the theorem is automatically generalizable to the following ('valued') case: let $\mathscr{A}$ be a $\mathscr{C}^\infty(M)$-module with a covariant derivation law such that the Lie covariant derivative converts $\mathscr{A}$ into a left $\mathscr{D}$-module (Lemma 4.2). The $\mathscr{C}^\infty(M)$-module $\Omega^n(M)\otimes_M\mathscr{A}$ is now a right $\mathscr{D}$-module (Corollaries 4.3 and 4.4). Explicitly: if $D\in\mathscr{T}$, $\omega\in\Omega^n(M)$ and $a\in\mathscr{A}$ we put

$$(\omega\otimes a)\cdot D=\omega\cdot D\otimes a-\omega\otimes Da=-L_D\omega\otimes a-\omega\otimes L_Da=-L_D(\omega\otimes a),$$

where L_D means the covariant derivative of a by D and $L_D(\omega\otimes a)$ is the Lie covariant derivative. Hence Theorem 5.1 holds, with the same proof, when one replaces $\Omega^n(M)$ by $\Omega^n(M)\otimes\mathscr{A}$, and d by the correspondent exterior covariant differential.

Corollary 5.4. *Let $L\colon \mathscr{T}^k\otimes_M\Omega^n(M)\longrightarrow\Omega^n(M)$ be the morphism associated with the action of $\mathscr{T}^k\subset\mathscr{D}$ on $\Omega^n(M)$. Given a fixed linear connection there exists a map Φ that makes the next diagram commutative:*

$$\begin{tikzcd}
\mathscr{T}^k\otimes_M\Omega^n(M) \arrow[r, "L"] \arrow[d, "\Phi"'] & \Omega^n(M) \\
\Omega^{n-1}(M) \arrow[r, equal] & \Omega^{n-1}(M) \arrow[u, "d"']
\end{tikzcd}$$

Proof. Define $\Phi(P\otimes\omega)=\Phi_P\omega$ with $P\in\mathscr{T}^k$, $\omega\in\Omega^n(M)$ and Φ_P the operator defined in 5.1. Extending Φ by linearity the proof is complete. □

Remark 5.5. In the same way as 5.1, Corollary 5.4 is valid in the 'valued' case.

6. Jet spaces and vertical lift

Let $\pi\colon E\longrightarrow M$ be a fibred manifold and $J^k=J^k(E/M)$ the space of k-jets of sections of π. It is known that there is an isomorphism

$$T^*M\otimes_{J^1}T^v(E/M)\simeq T^v(J^1/E),$$

i.e., if p^1 is a 1-jet of J^1 that projects over $p\in E$ and $x\in M$, with each pair (ω_x,D_p), comprised by $\omega_x\in T^*_x(M)$ and $D_p\in T^v_p(E/M)$, we can associate canonically a tangent vector in $T^v_{p^1}(J^1/E)$.

It will be shown in this section how to generalize this operation in higher orders. The result obtained is equivalent to [7] ((14) of §1). Nevertheless, our calculations make use of an alternative construction of the jet spaces [14], following Weil's suggestions (see [20] and [8]).

In agreement with [14], a k-jet $p^k\in J^k$ over a point $x\in M$ is a morphism of algebras $\mathscr{C}^\infty(E)\longrightarrow\mathscr{C}^\infty(M)/\mathfrak{m}_x^{k+1}$ such that, restricted to $C^\infty(M)$, this is the quotient by $\mathfrak{m}_x^{k+1}$.

In categorical terminology we would say that p^k is 'a point of E with values in $\mathscr{C}^\infty(M)/\mathfrak{m}_x^{k+1}$'. The principal advantages of this construction stem from always working over the same ring $\mathscr{C}^\infty(E)$.

In accordance with this, the tangent space to J^k at p^k vertical to M, $T^v_{p^k}(J^k/M)$, is identified canonically with the derivations over $\mathscr{C}^\infty(M)$ of the ring $\mathscr{C}^\infty(E)$ having values in $\mathscr{C}^\infty(M)/\mathfrak{m}_x^{k+1}$, where the latter is a $\mathscr{C}^\infty(E)$-module throughout p^k (see [14] for details).

Now let $D_{p^k} \in T^v_{p^k}(J^k/M)$ and $\omega_x \in T_x^{*,k+1}M = \mathfrak{m}_x/\mathfrak{m}_x^{k+2}$. Let us denote the multiplication by ω_x on $\mathscr{C}^\infty(M)/\mathfrak{m}_x^{k+1}$ by $\cdot\omega_x$. We have a composition

$$\mathscr{C}^\infty(E) \xrightarrow{D_{p^k}} \mathscr{C}^\infty(M)/\mathfrak{m}_x^{k+1} \xrightarrow{\cdot\omega_x} \mathfrak{m}_x/\mathfrak{m}_x^{k+2} \subset \mathscr{C}^\infty(M)/\mathfrak{m}_x^{k+2}.$$

Further, if p^{k+1} is a $(k+1)$-jet over p^k, $(\cdot\omega_x)\circ D_{p^k}$ can be identified as above with a tangent vector at p^{k+1} which is vertical to $J^{k+1} \longrightarrow E$ (and not only to $J^{k+1} \longrightarrow M$).

Thereby this defines a morphism:

$$(*) \qquad \mathscr{T}^{*,k+1} \otimes_{J^{k+1}} \mathscr{T}^v(J^k/M) \longrightarrow \mathscr{T}^v(J^{k+1}/E)$$

(recall that the calligraphic letters mean modules of sections).

Definition 6.1. Henceforth the morphism

$$\delta\colon\ \mathscr{T}^v(J^k/M)_{J^{k+1}} \longrightarrow \mathscr{T}^{k+1} \otimes_{J^{k+1}} \mathscr{T}^v(J^{k+1}/E)$$

obtained from $(*)$ by transposing $\mathscr{T}^{*,k+1}$ shall be called the vertical lift.

To express this in coordinates, let $(x_1,\dots,x_n)$ be a local chart on M and let $(y_1,\dots,y_m)$ be local coordinates on the fibres of $\pi\colon\ E \longrightarrow M$. These charts produce the usual fibred coordinates $(x_i, y_{j\alpha})$, $i=1,\dots,n$, $j=1,\dots m$, $|\alpha| \leqslant k$ on J^k. Then

$$\delta(\frac{\partial}{\partial y_{j\beta}}) = \sum_{1\leqslant|\alpha|\leqslant k, |\alpha+\beta|\leqslant k+1} \binom{\alpha+\beta}{\beta} \partial^\alpha \otimes \frac{\partial}{\partial y_{j\alpha+\beta}},$$

where $\partial^\alpha = (\frac{\partial}{\partial x_1})^{\alpha_1}\circ\dots\circ(\frac{\partial}{\partial x_n})^{\alpha_n}$, $\alpha = (\alpha_1,\dots,\alpha_n)$.

7. The formal derivative

In this section the concept of 'formal' (also known as 'total') derivative will be defined (refer to [10] or [15]).

By means of the immersion $J^{k+1} \subset J^1J^k$, we associate with each $p^{k+1} \in J^{k+1}$ an element of J^1J^k that will be denoted by $p^{k,1}$. Following [14], $p^{k,1}$ is a morphism $\mathscr{C}^\infty(J^k) \longrightarrow \mathscr{C}^\infty(M)/\mathfrak{m}_x^2$. Thus for each function $f \in \mathscr{C}^\infty(J^k)$ we have $p^{k,1}f \in \mathscr{C}^\infty(M)/\mathfrak{m}_x^2$.

Remark 7.1. With the usual construction of jet spaces, p^{k+1} is the $(k+1)$-jet of some local section s in x of $\pi\colon E \longrightarrow M$, i.e. $p^{k+1} = j^{k+1}_x s$. Let $\bar{s} = j^k s$ be the local k-jet prolongation of s. Then $p^{k,1} = j^1_x \bar{s}$ and $p^{k,1} f$ is the class of the function $f \circ \bar{s}$ modulo $\mathfrak{m}_x^2$.

Definition 7.2. Each tangent vector field $D \in \mathscr{T}$ produces a derivative

$$\widehat{D}\colon \mathscr{C}^\infty(J^k) \longrightarrow \mathscr{C}^\infty(J^{k+1}) \qquad k \geqslant 0$$

defined by the rule $(\widehat{D}f)p^{k+1} = D_x(p^{k,1}f) \in \mathscr{C}^\infty(M)/\mathfrak{m}_x \simeq \mathbb{R}$. We will call $\widehat{D}$ the holonomic lift of D to the jets. Similarly, we can consider $\widehat{D}$ as a section of $(TJ^k)_{J^{k+1}}$ or as a map $J^{k+1} \longrightarrow TJ^k$ over J^k. Later on we will use the notation $\widehat{D} = b_{k+1}(D)$.

Remark 7.3. The holonomic lifts to different orders are compatible with each other. This fact allows us to use the notation $\widehat{D}$ without reference to k.

Let $(x_i, y_{j\alpha})$ be fibred coordinates as above. Then $\widehat{\frac{\partial}{\partial x_i}} = \frac{\partial}{\partial x_i} + \sum\limits_{|\alpha| \leqslant k} y_{j\alpha+\varepsilon_i} \frac{\partial}{\partial y_{j\alpha}}$.

Definition 7.4. Let D be a vector field on M. We will call the map defined below the formal inner contraction with respect to D:

$$i_{\widehat{D}}\colon \Omega^p(J^k) \longrightarrow \Omega^{p-1}(J^{k+1})$$

defined by the rule $(i_{\widehat{D}}\sigma)_{p^{k+1}} = i_{\widehat{D}_{p^{k+1}}} \sigma_{p^k}$, for any $\sigma \in \Omega^p(J^k)$ and $p^{k+1} \in J^{k+1}$ over p^k; the second member of the equality being an ordinary contraction (observe that $\widehat{D}_{p^{k+1}}$ is in $T_{p^k}J^k$).

Definition 7.5. We will denote the formal Lie derivative with respect to the vector field D on M as the derivative

$$L_{\widehat{D}}\colon \Omega^p(J^k) \longrightarrow \Omega^p(J^{k+1})$$

defined by Cartan's formula $L_{\widehat{D}} = i_{\widehat{D}} d + d i_{\widehat{D}}$.

The formal Lie derivative verifies the usual properties of the ordinary Lie derivative (see [10], [15]).

In this section we will recall the definition of the contact module and then see how the restriction of the formal Lie derivative (Definition 7.5) produces a covariant derivation law.

The holonomic lift, $b_k\colon T(M)_{J^k} \longrightarrow T(J^{k-1})_{J^k}$, is a canonical splitting of the exact sequence

$$0 \longrightarrow T^v(J^{k-1}/M)_{J^k} \longrightarrow T(J^{k-1})_{J^k} \longrightarrow T(M)_{J^k} \longrightarrow 0.$$

Definition 8.1. The retract θ^k associated with b_k is known as the structure form of J^k. Through pull-back to J^k, θ^k defines a section of the bundle $T^*J^k \otimes_{J^k} T^v(J^{k-1}/M)$.

With the above notation, in a fibred local chart $(x_i, y_{j\alpha})$ we have

$$\theta^k = \sum_{j,0\leqslant|\alpha|\leqslant k-1} \theta_{j\alpha} \otimes \frac{\partial}{\partial y_{j\alpha}} \qquad \text{where } \theta_{j\alpha} = dy_{j\alpha} - y_{j\alpha+\varepsilon_i} dx_i.$$

As is known, a section $\bar{s}$ of the projection $J^k \longrightarrow M$ is holonomic (i.e., is the k-jet prolongation of some section of $\pi\colon E \longrightarrow M$) if and only if θ^k vanishes on the image of $\bar{s}$.

Definition 8.2. We will call the $\mathscr{C}^\infty(J^k)$-module comprised by those 1-forms of J^k that become zero over the holonomic sections as the contact module of J^k, denoted by $\mathscr{C}_k$.

The module $\mathscr{C}_k$ is generated by the components of θ^k (the above '$\theta_{j\alpha}$-s'). The fibre bundle associated with $\mathscr{C}_k$ will be denoted by C_k.

It is useful to observe that the dual map of θ^k produces an immersion

$$(\theta^k)^*\colon T^v(J^{k-1}/M)^*_{J^k} \longrightarrow (T^*J^{k-1})_{J^k},$$

which identifies the dual of $T^v(J^{k-1}/M)_{J^k}$ with C_k. In particular, the holonomic lifts of tangent vectors on M are incident with C_k.

Denote the natural projections $J^k \longrightarrow J^r$, $0 \leqslant r \leqslant k$, by π^k_r. Each $\mathscr{C}_r$ with $r \leqslant k$ induces a submodule of $\mathscr{C}_k$ via the pull back by π^k_r.

Definition 8.3. We will call the injective limit of the system $(\mathscr{C}_k, \pi^k_r)$ the contact module $\mathscr{C}$:

$$\mathscr{C} = \varinjlim \mathscr{C}_k.$$

The restriction of the formal Lie derivative in Definition 7.5 to $\mathscr{C}_k \subset \Omega^1(J^k)$ takes values in $\mathscr{C}_{k+1}$. Indeed, if s is a local section of $\pi\colon E \longrightarrow M$, $j^{k+1}s$ is the $(k+1)$-prolongation and $\sigma \in \mathscr{C}_k$, then because $i_{\widehat{D}}\sigma = 0$ we have $(j^{k+1}s)^* L_{\widehat{D}}\sigma = (j^{k+1}s)^* i_{\widehat{D}} d\sigma = i_D d(j^k s)^*\sigma = i_D d0 = 0$, hence $L_{\widehat{D}}\sigma \in \mathscr{C}_{k+1}$.

In a local chart $(x_i, y_{j\alpha})$ as above, we have $L_{\widehat{\frac{\partial}{\partial x_i}}}\theta_{j\alpha} = \theta_{j\alpha+\varepsilon_i}$.

Because of the compatibility of the holonomic lifts with the projections π^r_k it is possible to define, for any tangent vector field D on M, the *formal Lie derivative*

$$L_{\widehat{D}}\colon \mathscr{C} \longrightarrow \mathscr{C}.$$

Recall that $\mathscr{C}$ is, in particular, a $\mathscr{C}^\infty(M)$-module.

Proposition 8.4. *The assignment $D \longrightarrow L_{\widehat{D}}$ produces a covariant derivation law on the $\mathscr{C}^\infty(M)$-module $\mathscr{C}$.*

Proof. If $f \in \mathscr{C}^\infty(M)$, $D \in \mathscr{T}$ and $\sigma \in \mathscr{C}$ we have

$$L_{\widehat{D}}(f\sigma) = Df \cdot \sigma + fL_{\widehat{D}}\sigma \text{ and } L_{\widehat{fD}}\sigma = L_{f\widehat{D}}\sigma = i_{f\widehat{D}}d\sigma = f(i_{\widehat{D}}d\sigma) = fL_{\widehat{D}}\sigma.$$

$\square$

This law allows us to define a $\mathscr{C}$-valued differential calculus on M. In particular, we will have a *formal covariant exterior differential*

$$\widehat{d}\colon \Omega^p(M) \otimes_M \mathscr{C} \longrightarrow \Omega^{p+1}(M) \otimes_M \mathscr{C}$$

and a *formal covariant Lie derivative*

$$\widehat{L}_D\colon \Omega^p(M) \otimes_M \mathscr{C} \longrightarrow \Omega^p(M) \otimes_M \mathscr{C}$$

(by putting $\widehat{L}_D = i_D\widehat{d} + \widehat{d}i_D$).

Remark 8.5. $\widehat{d}$ and $\widehat{L}_D$ satisfy the relations

$$\widehat{d}(\Omega^p(M) \otimes_M \mathscr{C}_k) \subset \Omega^{p+1}(M) \otimes_M \mathscr{C}_{k+1}$$

and

$$\widehat{L}_D(\Omega^p(M) \otimes_M \mathscr{C}_k) \subset \Omega^p(M) \otimes_M \mathscr{C}_{k+1}.$$

Moreover, in local coordinates we have $\widehat{d}\theta_{j\alpha} = \sum_i dx_i \otimes \theta_{j\alpha+\varepsilon_i}$.

Proposition 8.6. *The module $\mathscr{C}$ has the structure of a left $\mathscr{D}$-module, precisely that defined by the $\widehat{L}_D$ action.*

P r o o f. Let $\alpha\colon \mathscr{C}^\infty(M) \longrightarrow \operatorname{End}_{\mathbb{R}} \mathscr{C}$ be the multiplication corresponding to the structure of the $\mathscr{C}^\infty(M)$-module of $\mathscr{C}$ and let us define $\chi\colon \mathscr{T} \longrightarrow \operatorname{End}_{\mathbb{R}} \mathscr{C}$ by $\chi(D) = \widehat{L}_D$ for any $D \in \mathscr{T}$.

We must check properties 1), 2) and 3) of Proposition 4.2 as follows:

For $D, D_1, D_2 \in \mathscr{T}$, $f \in \mathscr{C}^\infty(M)$ and $\sigma \in \mathscr{C}$,

1) $[\chi(D), \alpha(f)]\sigma = \widehat{L}_D(f\sigma) - f\widehat{L}_D\sigma = Df \cdot \sigma + f\widehat{L}_D\sigma - f\widehat{L}_D\sigma = \alpha(Df)\sigma$.

2) $[\chi(D_1), \chi(D_2)]\sigma = \widehat{L}_{D_1}\widehat{L}_{D_2}\sigma - \widehat{L}_{D_2}\widehat{L}_{D_1}\sigma = \widehat{L}_{[D_1,D_2]}\sigma = \chi([D_1, D_2])\sigma$.

3) $\alpha(f) \circ \chi(D)\sigma = f\widehat{L}_D\sigma = fi_D\widehat{d}\sigma = i_{fD}\widehat{d}\sigma = \widehat{L}_{fD}\sigma = \chi(fD)\sigma$. □

Corollary 8.7. *$\Omega^n(M) \otimes_M \mathscr{C}$ has a right $\mathscr{D}$-module structure ($n = \dim M$).*

P r o o f. According to Corollaries 4.3 and 4.4, the former structure is obtained by extending the following action of $\mathscr{T}$: for any $\omega \in \Omega^n(M)$, $\sigma \in \mathscr{C}$ and $D \in \mathscr{T}$ we put $(\omega \otimes \sigma) \cdot D = \omega \cdot D \otimes \sigma - \omega \otimes D \cdot \sigma = -L_D\omega \otimes \sigma - \omega \otimes \widehat{L}_D\sigma = -\widehat{L}_D(\omega \otimes \sigma)$. □

In a fibred local chart $(x_i, y_{j\alpha})$, let $D = \frac{\partial}{\partial x_i} = \partial_i \in \mathscr{T}$, $P = \partial^\beta \in \mathscr{D}$ and $\eta = \sum\limits_{j\alpha} \eta_{j\alpha}\theta_{j\alpha} \otimes dx_1 \wedge \ldots \wedge dx_n \in \Omega^n(M) \otimes_M \mathscr{C}$. Then

$$\begin{aligned}
\eta \cdot D &= -\sum_{j\alpha} \widehat{L}_{\partial_i}(\eta_{j\alpha}\theta_{j\alpha}) \otimes dx_1 \wedge \ldots \wedge dx_n \\
&= -\sum_{j\alpha} ((\widehat{\partial}_i\eta_{j\alpha})\theta_{j\alpha} + \eta_{j\alpha}\theta_{j\alpha+\varepsilon_i}) \otimes dx_1 \wedge \ldots \wedge dx_n
\end{aligned}$$

and analogously

$$\begin{aligned}
\eta \cdot P &= \sum_{j\alpha} (-\widehat{L}_\partial)^\beta(\eta_{j\alpha}\theta_{j\alpha}) \otimes dx_1 \wedge \ldots \wedge dx_n = \\
&= \sum_{j\alpha} (-1)^{|\beta|} \sum_{\gamma+\nu=\beta} \binom{\beta}{\gamma} (\widehat{\partial}^\gamma \eta_{j\alpha})\theta_{j\alpha+\nu} \otimes dx_1 \wedge \ldots \wedge dx_n
\end{aligned}$$

where $(-\widehat{L}_\partial)^\beta = (-\widehat{L}_{\partial_1})^{\beta_1} \circ \ldots \circ (-\widehat{L}_{\partial_n})^{\beta_n}$.

Remark 8.8. The $\mathscr{C}$-valued differential calculus is extensible in an obvious way to a calculus with values in $\wedge^k\mathscr{C}$, $S^k\mathscr{C}$, etc.

Remark 8.9. If $\eta \in \Omega^n(M) \otimes_M \mathscr{C}_k$ and $P \in \mathscr{D}^h$ or $\in \mathscr{T}^h$ then $\eta \cdot P \in \Omega^n(M) \otimes_M \mathscr{C}_{k+h}$.

Remark 8.10. Corollary 8.7 and Remark 5.3 say that the factorization Theorem 5.1 is available for the $\Omega^n(M) \otimes_M \mathscr{C}$ case.

The passage of the $\mathscr{C}$-valued calculus to the ordinary one is carried out via the morphism

$$\eta\colon\ \Omega^p(M)\otimes_M \mathscr{C}_k \longrightarrow \Omega^{p+1}(J^k).$$

This process consists of a combination of introducing $\Omega^p(M)$ and $\mathscr{C}_k$ into $\Omega^{\cdot}(J^k)$ and then alternating. This is equivalent to taking the exterior product, multiplying by the structure form θ^k relative to the bilinear product that the duality between $\mathscr{C}_k$ and $\mathscr{T}^v(J^{k-1}/M)_{J^k}$ produces (see Definition 8.2).

Definition 8.11. We call η the antisymmetry operator.

The morphism η relates $\widehat{d}$ to a certain differential on J^k.

Definition 8.12. The unique antiderivation of degree 1,

$$d_H\colon\ \Omega^p(J^k) \longrightarrow \Omega^{p+1}(J^{k+1}),$$

such that

1) for $p=0$, $d_H = H\circ d$, where $H\colon\ \Omega^1(J^k) \longrightarrow \Omega^1(M)_{J^{k+1}} \subset \Omega^1(J^{k+1})$ is the transposed operator of the holonomic lift b_k (Definition 7.2);
2) $d_h\circ d = -d\circ d_H$

will be called the horizontal differential.

In local coordinates $(x_i, y_{j\alpha})$ we have

$$d_H x_i = dx_i, \qquad d_H dx_i = 0, \qquad d_H y_{j\alpha} = y_{j\alpha+\varepsilon_i}dx_i,$$
$$d_H dy_{j\alpha} = \sum_i dx_i\wedge dy_{j\alpha+\varepsilon_i}, \qquad d_H\theta_{j\alpha} = \sum_i dx_i\wedge\theta_{j\alpha+\varepsilon_i}.$$

It is a simple process to check the following:

Proposition 8.13. *On $\Omega^p(M)\otimes_M\mathscr{C}_k$, d_H and $\widehat{d}$ satisfy $\eta\circ\widehat{d} = d_H\circ\eta$.*

9. Construction of the Poincaré-Cartan form

In the classical variational calculus the equations for the critical sections are found by using integration by parts. Briefly, if the problem is defined by a lagrangian function $\mathscr{L} = \mathscr{L}(t, q_i, \dot q_i)$, where t, q_i and $\dot q_i$ are the time, position and velocity coordinates of a mechanical system, respectively, the variation of $\mathscr{L}\,\mathrm{d}t$ is written as

$$(*) \qquad \delta(\mathscr{L}\,\mathrm{d}t) = \left(\frac{\partial\mathscr{L}}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathscr{L}}{\partial\dot q_i}\right)\delta q_i\,\mathrm{d}t + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{L}}{\partial\dot q_i}\delta q_i\,\mathrm{d}t\right)$$

by making use of $\frac{\mathrm{d}}{\mathrm{d}t}\delta q_i = \delta\dot{q}_i$ (this 'δ' does not relate to the vertical lift).

The construction that we give below is based on an adequate interpretation of the elements in the above formula. The variation is viewed as a type of vertical differential in such a way that with the obvious notation, $\delta q_i = dq_i - \dot{q}_i\,\mathrm{d}t$, $\delta\dot{q}_i = d\dot{q}_i - \ddot{q}_i\,\mathrm{d}t$, $\delta\mathscr{L} = \frac{\partial\mathscr{L}}{\partial q_i}\delta q_i + \frac{\partial\mathscr{L}}{\partial\dot{q}_i}\delta\dot{q}_i$ are contact forms; the second term of the second member of (*) is the formal derivative with $\frac{\mathrm{d}}{\mathrm{d}t}$ of $\frac{\partial}{\partial\dot{q}_i}\delta q_i\,\mathrm{d}t$, and the last is simply the result of applying the vertical lift to the structure form $(dq_i - \dot{q}_i\,\mathrm{d}t)\otimes\frac{\partial}{\partial q_i}$ and then to acting on $\mathscr{L}\,\mathrm{d}t$.

From the above we deduce that the $\mathscr{C}$-valued differential calculus of §8 appears as the best framework for the generalization of the calculus of variations to several independent variables and the higher order case.

In general, let λ be an element of $\Omega^n(M)_{J^k}$, i.e. a *Lagrangian density* of order k over $\pi\colon E \longrightarrow M$, and let $\theta^k \in \mathscr{C}_k \otimes_{J^k} \mathscr{T}^v(J^{k-1}/M)$ be the structure form on J^k.

Definition 9.1. We will call the 1-form $\bar{\theta}^k = \delta\circ\theta^k \in \mathscr{C}_k \otimes_{J^k} \mathscr{T}^k \otimes_{J^k} \mathscr{T}^v(J^k/E)$ the vertical lift of the structure form (δ is the vertical lift morphism defined in 6.1).

We will consider also $\bar{\theta}^k(\lambda) \in \mathscr{T}^k \otimes_M \mathscr{C}_k \otimes_M \Omega^n(M)$, the result of contracting $d\lambda$ with the index of $\bar{\theta}^k$ in $\mathscr{T}^v(J^k/E)$ (i.e., to 'derive λ by $\bar{\theta}^k$').

Now let us denote by L the morphism given by the action of $\mathscr{T}^k \subset \mathscr{D}$ on $\mathscr{C}_k \otimes_M \Omega^n(M)$ (Corollary 8.7 and Remark 8.9):

$$L\colon\ \mathscr{T}^k \otimes_M \mathscr{C}_k \otimes_M \Omega^n(M) \longrightarrow \mathscr{C}_{2k} \otimes_M \Omega^n(M).$$

Definition 9.2. Let $A_\lambda = L(\bar{\theta}^k(\lambda)) \in \mathscr{C}_{2k} \otimes_M \Omega^n(M)$ and let us define the vertical differential of λ as $d^v\lambda = \theta^{k+1}(\lambda) \in \mathscr{C}_{k+1} \otimes_M \Omega^n(M)$ (in the same sense as the definition of $\bar{\theta}^k(\lambda)$). Then we will call the $\mathscr{C}$-valued form $\mathscr{E} = A_\lambda + d^v\lambda$ the Euler-Lagrange form associated with the Lagrangian density λ.

Remark 9.3. Definition 9.2, written in the form $d^v\lambda = \mathscr{E} - A_\lambda$, is the identity corresponding to integration by parts (*) in the classical variational calculus.

Let $(x_i, y_{j\alpha})$ with $0 \leqslant |\alpha| \leqslant k-1$ be fibred local coordinates in J^{k-1} and analogously, in J^k with $0 \leqslant |\alpha| \leqslant k$, and let $\lambda = \mathscr{L}\omega$, where $\mathscr{L} = \mathscr{L}(x_i, y_{j\alpha}) \in \mathscr{C}^\infty(J^k)$

and $\omega = dx_1 \wedge \ldots \wedge dx_n$. Then

$$\theta^k = \sum_{0 \leqslant |\alpha| \leqslant k-1; j} \theta_{j\alpha} \otimes \frac{\partial}{\partial y_{j\alpha}},$$

$$\bar{\theta}^k = \sum_{1 \leqslant |\beta|; |\alpha+\beta| \leqslant k-1; j} \binom{\alpha+\beta}{\beta} \theta_{j\alpha} \otimes \partial^\beta \otimes \frac{\partial}{\partial y_{j\alpha+\beta}},$$

$$\theta^{k+1}(\lambda) = \sum_{0 \leqslant |\alpha| \leqslant k; j} \frac{\partial \mathscr{L}}{\partial y_{j\alpha}} \theta_{j\alpha} \otimes \omega,$$

$$\bar{\theta}^k(\lambda) = \sum_{1 \leqslant |\beta|; |\alpha+\beta| \leqslant k-1; j} \binom{\alpha+\beta}{\beta} \partial^\beta \otimes \frac{\partial \mathscr{L}}{\partial y_{j\alpha+\beta}} \theta_{j\alpha} \otimes \omega,$$

$$\begin{aligned} A_\lambda &= \sum_{1 \leqslant |\beta|; |\alpha+\beta| \leqslant k-1; j} \binom{\alpha+\beta}{\beta} \left(\frac{\partial \mathscr{L}}{\partial y_{j\alpha+\beta}} \theta_{j\alpha} \otimes \omega \right) \cdot \partial^\beta \\ &= \sum_{1 \leqslant |\beta|; |\alpha+\beta| \leqslant k-1; j} (-1)^{|\beta|} \binom{\alpha+\beta}{\beta} \sum_{\gamma+\tau=\beta} \binom{\beta}{\gamma} (-1)^{|\tau|} \widehat{\partial}^\tau \left(\frac{\partial \mathscr{L}}{\partial y_{j\alpha+\beta}} \right) \theta_{j\alpha+\gamma} \otimes \omega \end{aligned}$$

and

$$\mathscr{E} = \sum_{0 \leqslant |\alpha|; j} (-1)^{|\alpha|} \widehat{\partial}^\alpha \left(\frac{\partial \mathscr{L}}{\partial y_{j\alpha}} \right) \theta_j \otimes \omega \qquad \text{where } \theta_j = \theta_{j0}.$$

Taking into account the coordinate expression for $\mathscr{E}$ we obtain

Proposition 9.4. *The Euler-Lagrange form $\mathscr{E}$ belongs to $(\mathscr{C}_1)_{J^{2k}} \otimes_M \Omega^n(M)$.*

The utility of the integration by parts (Remark 9.3) depends critically on the fact that A_λ is an exact differential.

Theorem 9.5. *By fixing a linear connection ∇ on M, there is a $\mathscr{C}$-valued $(n-1)$-form*

$$\Theta_1 \in \mathscr{C}_{2k-1} \otimes_M \Omega^{n-1}(M) \qquad \textit{such that } \ A_\lambda = \widehat{d}\Theta_1$$

from which it can be deduced that $d^v\lambda = \mathscr{E} - \widehat{d}\Theta_1$.

Proof. The operator (depending on ∇)

$$\Phi\colon \mathscr{T}^k \otimes_M \mathscr{C}_k \otimes_M \Omega^n(M) \longrightarrow \mathscr{C}_{2k-1} \otimes_M \Omega^{n-1}(M)$$

is constructed as in Corollary 5.4 in such a way that $L = \widehat{d} \circ \Phi$. Hence $A_\lambda = L\bar{\theta}^k(\lambda) = \widehat{d}(\Phi\bar{\theta}^k(\lambda))$. By putting $\Theta_1 = \Phi\bar{\theta}^k(\lambda)$ the proof is complete. $\square$

Via the antisymmetry operator η (Definition 8.11), the objects $\mathscr{E}$, Θ_1 and $d^v\lambda$ can be understood as forms in $\Omega^p(J^r)$ (for suitable values of p and r).

We will use the following notation:

$$\overline{\mathscr{E}} = \eta \circ \mathscr{E}, \qquad \overline{\Theta}_1 = \eta \circ \Theta_1 \quad \text{and} \quad \overline{d^v\lambda} = \eta \circ d^v\lambda = d\lambda.$$

Definition 9.6. We will call the n-form $\Theta \in \Omega^n(J^{2k-2})_{J^{2k-1}}$ defined by $\Theta = \overline{\Theta}_1 + \lambda$ the Poincaré-Cartan form associated with the Lagrangian density λ (and with the linear connection ∇).

Remark 9.7. In fact $\Theta \in \Omega^n(J^{k-1})_{J^{2k-1}}$ because $\mathscr{L}$ depends only on the $y_{j\alpha}$ with $|\alpha| \leqslant k$.

In a fibred local chart $(x_i, y_{j\alpha})$, if ∇ is the local connection produced by (x_i) on M and if H_∇ is the associated decomposition morphism (3.2), we have

$$H_\nabla \bar{\theta}^k(\lambda) = \sum_{\substack{1\leqslant|\beta| \\ |\alpha+\beta|\leqslant k \\ j;i}} \binom{\alpha+\beta}{\beta} \frac{\beta_i}{|\beta|} \partial_i \otimes \partial^{\beta-\varepsilon_i} \otimes \frac{\partial \mathscr{L}}{\partial y_{j\alpha+\beta}} \theta_{j\alpha} \otimes \omega,$$

$$\begin{aligned}
\Theta_1 &= \sum_{\substack{1\leqslant|\beta| \\ |\alpha+\beta|\leqslant k \\ j;i}} \binom{\alpha+\beta}{\beta} \frac{\beta_i}{|\beta|} (-1)^{|\beta|-1} (\widehat{L}_\partial)^{\beta-\varepsilon_i} \left(\frac{\partial \mathscr{L}}{\partial y_{j\alpha+\beta}} \theta_{j\alpha} \right) \otimes \omega_i \\
&= \sum_{\substack{0\leqslant|\alpha|,|\gamma|,|\tau|\leqslant k-1 \\ j;i}} \frac{(\alpha+\gamma+\tau+\varepsilon_i)!}{\alpha!\gamma!\tau!} \frac{(-1)^{|\gamma|+|\tau|}}{|\gamma|+|\tau|+1} \widehat{\partial}^\tau \left(\frac{\partial \mathscr{L}}{\partial y_{j\alpha+\gamma+\tau+\varepsilon_i}} \right) \theta_{j\alpha+\gamma} \otimes \omega_i \\
&= \sum_{\substack{0\leqslant|\sigma|\leqslant k-1 \\ 0\leqslant|\tau|\leqslant k-1 \\ j;i}} \frac{(\sigma+\tau+\varepsilon_i)!}{\sigma!\tau!} (-1)^{|\tau|} \left(\sum_{\alpha+\gamma=\sigma} \binom{\sigma}{\gamma} \frac{(-1)^{|\gamma|}}{|\gamma|+|\tau|+1} \right) \cdot \\
&\qquad\qquad \cdot \widehat{\partial}^\tau \left(\frac{\partial \mathscr{L}}{\partial y_{j\sigma+\tau+\varepsilon_i}} \right) \theta_{j\sigma} \otimes \omega_i
\end{aligned}$$

where $\omega_i = i_{\partial_i}\omega$. Finally, we make use of the combinatorial identity

$$\sum_{\alpha+\gamma=\sigma} \binom{\sigma}{\gamma} \frac{(-1)^{|\gamma|}}{|\gamma|+|\tau|+1} = \frac{|\sigma|!|\tau|!}{(|\sigma|+|\tau|+1)!}$$

to obtain

$$\begin{aligned}
\Theta &= \Theta_1 + \lambda \\
&= \sum_{\substack{0\leqslant|\sigma|,|\tau|\leqslant k-1 \\ j;i}} \frac{(\sigma+\tau+\varepsilon_i)!|\sigma|!|\tau|!}{\sigma!\tau!(|\sigma|+|\tau|+1)!} (-1)^{|\tau|} \widehat{\partial}^\tau \left(\frac{\partial \mathscr{L}}{\partial y_{j\sigma+\tau+\varepsilon_i}} \right) \theta_{j\sigma} \wedge \omega_i + \mathscr{L}\omega.
\end{aligned}$$

The above expression coincides with that obtained in [12] and [16].

Theorem 9.8. $d\lambda = \overline{\mathscr{E}} - d_H\Theta$.

P r o o f. This is a direct consequence of Theorem 9.5 and the relations

$$(\eta \circ \widehat{d})\Theta_1 = (d_H \circ \eta)\Theta_1 = d_H\overline{\Theta}_1, \qquad d_H\lambda = 0. \qquad \square$$

Remark 9.9. The property enunciated in the last theorem or, equivalently, the correspondent one in Theorem 9.5 is known as 'decomposition formula' in [7]. This formula makes it possible to find the equations for the critical sections in a variational problem defined for a given Lagrangian density (see [7]).

As demonstrated in [6], any other form verifying 'decomposition formula' as $\overline{\Theta}_1$ is of the form $\overline{\Theta}_1 + G$, where $G \in (\mathscr{C}_k)_{J^{2k-2}} \otimes_M \Omega^{n-2}(M)$.

Remark 9.10. There is no total agreement on this topic. One can consult [4] and [13] for a discussion concerning the construction of the Poincaré-Cartan form. See [1], [2], [3], [9], [11], [12] and [16] for other references of the geometric formulation of the calculus of variations (naturally, the above list is not exhaustive).

10. Application to the Euler-Lagrange resolution

The $\mathscr{C}$-valued differential calculus of §8 allows us to define intrinsically the elements of the Euler-Lagrange resolution. We will follow Tulczyjew's paper [19] (it can also be consulted in [17]).

We fix the projection $\pi\colon E \longrightarrow M$ and work on the infinity jet space $J^\infty = \operatorname{proj\,lim} J^k$.

Let us consider the tensor products $\Phi^{r,s} = \wedge^r\mathscr{C} \otimes_M \Omega^s(M)$. Via the antisymmetry operator, the $\Phi^{r,s}$ spaces are identified with the $(r+s)$-forms on J^∞ that are r times vertical and s times horizontal.

The formal Lie derivative (Proposition 8.4) over $\mathscr{C}$ extends naturally to the whole exterior algebra $\overset{\bullet}{\bigwedge}\mathscr{C}$. Analogously, we have a valued formal differential

$$\widehat{d}\colon \Phi^{r,s} \longrightarrow \Phi^{r,s+1}.$$

In the same way as in Proposition 8.6, one checks that $\overset{\bullet}{\bigwedge}\mathscr{C} \otimes_M \Omega^n(M)$ ($\dim M = n$) is a right $\mathscr{D}$-module. Let us denote by L the action of $\mathscr{D}$ over $\overset{\bullet}{\bigwedge}\mathscr{C} \otimes_M \Omega^n(M)$, i.e.

$$\begin{aligned} L\colon \mathscr{D} \otimes_M \overset{\bullet}{\bigwedge}\mathscr{C} \otimes_M \Omega^n(M) &\longrightarrow \overset{\bullet}{\bigwedge}\mathscr{C} \otimes_M \Omega^n(M) \\ P \otimes \sigma \otimes \omega &\longmapsto (\sigma \otimes \omega) \bullet P \end{aligned}$$

where $\bullet$ means 'action'.

Remark 10.1. $\widehat{d}$ corresponds, via the antisymmetry operator, to the horizontal differential d_H of [19]. Moreover, this notation is coherent with Definition 8.12.

On the other hand, the vertical lift morphism (Definition 6.1) can be dualized giving us a morphism $\delta_k^*\colon \mathscr{C}_{k+1} \longrightarrow \mathscr{C}_k \otimes_{J^{k+1}} \mathscr{T}^k$. With notation as above, in local coordinates we have

$$\delta_k^*\theta_{j\alpha} = \sum_{\substack{\beta+\gamma=\alpha\\ 0\leqslant|\beta|\leqslant k-1\\ 1\leqslant|\gamma|\leqslant k}} \binom{\alpha}{\gamma}\partial^\gamma \otimes \theta_{h\beta}.$$

The morphisms δ_k^* are compatible with the inclusions $\mathscr{C}_k \subset \mathscr{C}_r$, $\mathscr{T}^k \subset \mathscr{T}^r$, $k < r$. In this way it is possible to define δ_k^* on the injective limit, thus obtaining a map

$$\delta^*\colon \mathscr{C} \longrightarrow \mathscr{C} \otimes_M \mathscr{T}^\infty$$

where $\mathscr{T}^\infty = \operatorname{proj\,lim} \mathscr{T}^k = \{P \in \mathscr{D}/P(1) = 0\}$. The local expression of δ^* is the same as that of δ_k^*. Now we extend δ^* to a derivation over the exterior algebra of $\mathscr{C}$. We get the map (keeping the notation)

$$\delta^*\colon \overset{\bullet}{\bigwedge}\mathscr{C} \longrightarrow \overset{\bullet}{\bigwedge}\mathscr{C} \otimes_M \mathscr{T}^\infty.$$

The composition $L \circ \delta^*$ is denoted by $\overline{\tau}$:

$$\overline{\tau}\colon \overset{\bullet}{\bigwedge}\mathscr{C} \otimes_M \Omega^n(M) \longrightarrow \overset{\bullet}{\bigwedge}\mathscr{C} \otimes_M \Omega^n(M).$$

We will make use of the notation

$$\begin{aligned}
J &= (j_1, \dots, j_r),\ j_i \in \mathbb{N},\ A = (\alpha_1, \dots, \alpha_r) \text{ with } \alpha_i \text{ a multi-index},\\
\theta_{JA} &= \theta_{j_1\alpha_1} \wedge \dots \wedge \theta_{j_r\alpha_r},\\
\theta_{JA,i} &= \theta_{j_1\alpha_1} \wedge \dots \wedge \theta_{j_{i-1}\alpha_{i-1}} \wedge \theta_{j_{i+1}\alpha_{i+1}} \wedge \dots \wedge \theta_{j_r\alpha_r} \quad \text{and}\\
\omega &= dx_1 \wedge \dots \wedge dx_n.
\end{aligned}$$

In this way the elements of $\wedge^r\mathscr{C} \otimes_M \Omega^n(M)$ have the form

$$\mu = \sum_{J,A} \mu_{JA}\theta_{JA} \otimes \omega \qquad \text{with } \mu_{JA} \in \mathscr{C}^\infty(J^\infty).$$

Therefore

$$\begin{aligned}
\delta^*\mu &= \sum_{i=1}^r \sum_{J,A} (-1)^{r-i} \sum_{\substack{\gamma_i+\beta_i=\alpha_i\\ |\gamma_i|\geqslant 1}} \partial^{\gamma_i} \otimes \left(\mu_{JA}\theta_{JA,i} \wedge \binom{\alpha_i+\beta_i}{\gamma_i}\theta_{j_i\beta_i}\right) \otimes \omega\\
&= \sum_{i=1}^r \sum_{J,A} (-1)^{r-i} \sum_{\substack{\gamma_i+\beta_i=\alpha_i\\ |\gamma_i|\geqslant 0}} \partial^{\gamma_i} \otimes \left(\mu_{JA}\theta_{JA,i} \wedge \binom{\alpha_i+\beta_i}{\gamma_i}\theta_{j_i\beta_i}\right) \otimes \omega - r\mu
\end{aligned}$$

and

$$\overline{\tau}\mu = L\delta^*\mu = \sum_{i,J,A}(-1)^{r-i}(-1)^{|\alpha_i|}(\widehat{L}_\partial)^{\alpha_i}(\mu_{JA}\theta_{JA,i})\wedge\theta_{j_i0}\otimes\omega - r\mu.$$

Since for any $\mu\in\Phi^{r,s}$, $\overline{\tau}$ is obtained by applying L, we deduce (Corollary 5.4, Remark 5.5) that for each linear connection ∇ on M, there exists an element $F_\mu\in\Phi^{r,n-1}$ such that $\overline{\tau}\mu=\widehat{d}F_\mu$. Now let us modify the definition of $\overline{\tau}$ over each $\Phi^{r,n}$ by putting $\overline{\tau}_r=\overline{\tau}+r\cdot Id$, where Id is the identity map. In local coordinates, if μ is as above, we have

$$\overline{\tau}_r\mu = \sum_{i,J,A}(-1)^{r-i}(-1)^{|\alpha_i|}(\widehat{L}_\partial)^{\alpha_i}(\mu_{JA}\wedge\theta_{JA,i})\wedge\theta_{j_i0}\otimes\omega.$$

Moreover, $\overline{\tau}_r\mu=\widehat{d}F_\mu+r\cdot\mu$. Finally, if we put $\tau_r=\frac{1}{r}\overline{\tau}_r$ and $F'_\mu=\frac{1}{r}F_\mu$, then

$$\tau_r\mu=\widehat{d}F'_\mu+\mu.$$

One can deduce as in [19] that $\tau_r\circ\widehat{d}=0$. By applying this equality we deduce that $\tau_r\circ\tau_r\mu=\tau_r\circ\widehat{d}F'_\mu+\tau_r\mu=\tau_r\mu$. In other words, τ_r is a projector in $\Phi^{r,n}$.

Theorem 10.2. *The subspace $\Lambda^r=\tau_r\Phi^{r,n}$ is a complement of $\widehat{d}\Phi^{r,n-1}$ in $\Phi^{r,n}$:*

$$\Phi^{r,n}\simeq\Lambda^r\oplus\widehat{d}\Phi^{r,n-1}.$$

Proof. With each $\mu\in\Phi^{r,n}$ we assign the decomposition $\mu=\tau_r\mu-\widehat{d}F'_\mu$. The rest follows since τ_r is a projector and $\tau_r\circ\widehat{d}=0$. □

Remark 10.3. Even though F'_μ depends on the choice of the connection, the element $\widehat{d}F'_\mu$ is canonically defined.

Remark 10.4. The above theorem can be viewed as a generalization of the procedure in Theorem 9.5 (by taking there $\mu=d^v\mathscr{L}\omega$). As in Theorem 9.5, it is easy to find the explicit expression of F'_μ.

Spreading the vertical differential d^v to every $\Phi^{r,n}$ as in Definition 9.2 and following [19] we can define the variational operators $\nu^r=\tau_{r+1}\circ d^v$. Ultimately, the Euler-Lagrange resolution is the sequence (see [19])

$$0\longrightarrow\Phi^{0,0}\xrightarrow{\widehat{d}}\Phi^{0,1}\xrightarrow{\widehat{d}}\dots\xrightarrow{\widehat{d}}\Phi^{0,n}\xrightarrow{\nu^0}\wedge^1\xrightarrow{\nu^1}\dots\xrightarrow{\nu^{r-1}}\wedge^r\xrightarrow{\nu^r}\wedge^{r+1}\dots$$

The operator ν^0 is known as the Euler-Lagrange operator and ν^1 is known as the Helmholtz-Sonin operator (inverse problem of the calculus of variations).

Acknowledgements. The author wishes to thank Prof. Muñoz Díaz and Prof. Rodríguez Lombardero for stimulating discussions and continuous encouragement during all stages of this work.

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