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ON SOME GENERALIZED SISTER CELINE'S POLYNOMIALS

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Abstract. Certain generalizations of Sister Celine's polynomials are given which include most of the known polynomials as their special cases. Besides, generating functions and integral representations of these generalized polynomials are derived and a relation between generalized Laguerre polynomials and generalized Bateman's polynomials is established.

Keywords: Ultraspherical type generalization of Bateman's polynomials, ultraspherical type generalization of Pasternak's polynomials, Jacobi type generalization of Bateman's polynomials, Jacobi type generalization of Pasternak's polynomials. Sister Celine's polynomial, Hahn polynomial, Generalized Hermite polynomial, Krawtchouk's polynomial, Meixner's polynomial, Charlier polynomial, Sylvester's polynomial, Gottlieb's polynomial, Konhauser's polynomial, generating functions, integral relations

1. Introduction

In 1947, Sister Celine (Fasenmyer [2]) concentrated on polynomials generated by

$$(1.1)\qquad (1-t)^{-1}{}_pF_q\left[\begin{matrix} a_1,\dots,a_p; & \\ & \frac{-4xt}{(1-t)^2} \\ b_1,\dots,b_q; & \end{matrix}\right]=\sum_{n=0}^{\infty} f_n\left[\begin{matrix} a_1,\dots,a_p; & \\ & x \\ b_1,\dots,b_q; & \end{matrix}\right]t^n,$$

which yields

$$(1.2)\qquad f_n\left[\begin{matrix} a_1,\dots,a_p; & \\ & x \\ b_1,\dots,b_q; & \end{matrix}\right]={}_{p+2}F_{q+2}\left[\begin{matrix} -n,n+1,a_1,\dots,a_p; & \\ & x \\ 1,\ \ \frac{1}{2},\ \ \ b_1,\dots,b_q; & \end{matrix}\right]$$

Her polynomials include as special cases Legendre's polynomials $P_n(1-2x)$, some special Jacobi polynomials, Rice's $H_n(\xi,p,v)$, Bateman's $Z_n(x)$ and $F_n(z)$ and Pasternak's $F_n^m(z)$ which is a generalization of Bateman's $F_n(z)$. The simple Bessel's polynomial is also included.

In this paper a generalization of (1.2) is given which includes Legendre's polynomials $P_n(x)$, Gegenbauer's polynomials $C_n^{\nu}(x)$, ultraspherical polynomials, Rice's polynomials $H_n(\xi, p, v)$, Bateman's polynomials $Z_n(x)$ and $F_n(z)$, Pasternak's polynomials $F_n^m(z)$, simple Bessel's polynomials $y_n(x)$, and provides ultraspherical type generalizations of Bateman's polynomial $Z_n(x)$ and $F_n(z)$ and Pasternak's polynomial $F_n^m(z)$.

Of these generalizations, the one for $Z_n(x)$ was given by Bateman himself but those of $F_n(z)$, $F_n^m(z)$ are believed to be new in literature.

A generating function for this generalized polynomial and some other interesting results have been obtained.

A further generalization has also been established which in addition to all polynomials included in the above generalization, also includes Jacobi's polynomials $P_n^{(\alpha,\beta)}(x)$, generalized Rice's polynomials $H_n^{(\alpha,\beta)}(\xi, p, v)$ due to Khandekar [3], generalized Bessel's polynomials $y_n(a, b, x)$ and Hahn's polynomials $Q_n(x; \alpha, \beta, N)$ and provides Jacobi type generalizations of Bateman's polynomials $Z_n(x)$ and $F_n(z)$ and Pasternak's polynomials $F_n^m(z)$ which are believed to be new in literature.

Still another generalization include also Konhauser's polynomials $Z_n^{\alpha}(x; k)$ which are not included in the above mentioned two generalizations.

A special case of this generalization has also been studied which includes Laguerre polynomials $L_n^{(\alpha)}(x)$, Hermite polynomials $H_n(x)$, generalized Hermite polynomials $g_n^m(x, h)$ due to Gould and Hopper, Krawtchouk's polynomial $K_n(x; p, N)$, Meixner's polynomials $M_n(x; \beta, c)$, Charlier polynomials $C_n^{(a)}$, Sylvester's polynomials $\varphi_n(x)$ and Gottlieb's polynomials $\ell_n(x; \lambda)$.

2. Generalized polynomials

Consider the polynomials defined by

(2.1)
$$f_n(k, \lambda; a_1, \ldots, a_p; b_1, \ldots, b_q; x) = {}_{p+k+1}F_{q+k+1}\left[\begin{matrix}\Delta(k, -n), n+\lambda, a_1, \ldots, a_p; \\ \\ \Delta(k+1, \lambda), b_1, \ldots, b_q;\end{matrix} \quad x\right].$$

For particular values of the parameters and special arguments, the polynomial (2.1) reduces to the following known as well as new polynomials:

(2.2) (i) $f_n(1, 1; a_1, \ldots, a_p; b_1, \ldots, b_q; x)$
$$= {}_{p+2}F_{q+2}\left[\begin{matrix}-n, n+1, a_1, \ldots, a_p; \\ \\ \frac{1}{2}, 1, b_1, \ldots, b_q;\end{matrix} \quad x\right]$$

which is Sister Celine's polynomial (1.2),

$$(2.3)\qquad \text{(ii)}\quad f_n(1,1;\tfrac{1}{2};-;\tfrac{1-x}{2}) = {}_2F_1\left[\begin{matrix}-n, n+1; & \\ & \frac{1-x}{2}; \\ 1; & \end{matrix}\right]$$

which is Legendre's polynomials $P_n(x)$,

$$(2.4)\qquad \text{(iii)}\quad f_n(1,2\nu;\nu;-;\tfrac{1-x}{2}) = {}_2F_1\left[\begin{matrix}-n, 2\nu+n; & \\ & \frac{1-x}{2} \\ \nu+\frac{1}{2}; & \end{matrix}\right] = \frac{n!}{(2\nu)_n}C_n^{\nu}(x)$$

where $C_n^{\nu}(x)$ is Gegenbauer's polynomial,

$$(2.5)\qquad \text{(iv)}\quad f_n(1,1+2\alpha;\alpha+\tfrac{1}{2};-;\tfrac{1-x}{2}) = {}_2F_1\left[\begin{matrix}-n, n+2\alpha+1; & \\ & \frac{1-x}{2} \\ \alpha+1; & \end{matrix}\right]$$
$$= \frac{n!}{(1+\alpha)_n}P_n^{(\alpha,\alpha)}(x)$$

where $P_n^{(\alpha,\alpha)}(x)$ is ultraspherical polynomial,

$$(2.6)\qquad \text{(v)}\quad f_n(1,1;\xi,\tfrac{1}{2};p;v) = {}_3F_2\left[\begin{matrix}-n, n+1, \xi; & v \\ 1, p; & \end{matrix}\right]$$

which is Rice's polynomial $H_n(\xi, p, v)$,

$$(2.7)\qquad \text{(vi)}\quad f_n(1,1;\tfrac{1}{2};1;x) = {}_2F_2\left[\begin{matrix}-n, n+1; & \\ & x \\ 1,\quad 1; & \end{matrix}\right]$$

which is Bateman's polynomial $Z_n(x)$,

$$(2.8)\qquad \text{(vii)}\quad f_n(1,1;\tfrac{1}{2},\tfrac{1+x}{2};1;1) = {}_3F_2\left[\begin{matrix}-n, n+1, \frac{1}{2}(1+z); & \\ & 1 \\ 1, 1; & \end{matrix}\right]$$

which is another Bateman's polynomial $F_n(z)$,

$$(2.9)\qquad \text{(viii)}\quad f_n(1,2\nu;\nu;1+b;t) = {}_2F_2\left[\begin{matrix}-n, n+2\nu; & \\ & t \\ \nu+\frac{1}{2}, 1+b; & \end{matrix}\right]$$

which is a generalization of Bateman's polynomial $Z_n(x)$ and was given by Bateman himself. We shall adopt for it the symbol $Z_n^{(\nu)}(b,t)$. For $\nu=\frac{1}{2}$, $b=0$ it reduces to $Z_n(t)$;

$$(2.10)\qquad \text{(ix)}\quad f_n(1,1+2\alpha;\alpha+\tfrac{1}{2},\tfrac{1+z}{2};p;1) = {}_3F_2\left[\begin{matrix}-n,n+2\alpha+1,\frac{1}{2}(1+z); \\ 1+\alpha,p;\end{matrix}\; 1\right]$$

which is an ultraspherical type generalization of Bateman's polynomial $F_n(z)$ and is believed to be new. We adopt for it the symbol $F_n^{(\alpha,\alpha)}(p,z)$. For $\alpha=0$, $p=1$ it reduces to Bateman's polynomial $F_n(z)$;

$$(2.11)\qquad \text{(x)}\quad f_n(1,1;\tfrac{1}{2},\tfrac{1+z+m}{2};m+1;1) = {}_3F_2\left[\begin{matrix}-n,n+1,\frac{1}{2}(1+z+m); \\ 1,\quad m+1;\end{matrix}\; 1\right]$$

which is Pasternak's polynomial $F_n^m(z)$,

$$(2.12)\qquad \text{(xi)}\quad f_n(1,1+2\alpha;\alpha+\tfrac{1}{2},\tfrac{1+z+m}{2};m+1;1) = {}_3F_2\left[\begin{matrix}-n,n+2\alpha+1,\frac{1}{2}(1+z+m); \\ 1+\alpha,\quad m+1;\end{matrix}\; 1\right]$$

which is an ultraspherical type generalization of Pasternak's polynomial and is believed to be new. We adopt for it the symbol $F_{n,m}^{(\alpha,\alpha)}(z)$. For $\alpha=0$ it reduces to Pasternak's polynomial $F_n^m(z)$;

$$(2.13)\qquad \text{(xii)}\quad f_n(1,1;\tfrac{1}{2},1;-;-\tfrac{x}{2}) = {}_2F_0\left[\begin{matrix}-n,n+1; \\ -\quad ;\end{matrix}\; -x\right]$$

which is simple Bessel's polynomial $y_n(x)$.

Next, consider the following generalization of (2.1):

$$(2.14)\qquad f_n(k,\lambda,\mu;a_1,\dots,a_p;b_1,\dots,b_q;x) = {}_{p+k+1}F_{q+k+1}\left[\begin{matrix}\Delta(k,-n),n+\lambda,a_1,\dots,a_p; \\ \Delta(k+1,\mu),b_1,\dots,b_q;\end{matrix}\; x\right]$$

For $\mu=\lambda$, (2.14) becomes (2.1) and hence (2.14) includes as special cases all polynomials which are included in (2.1).

Besides, (2.14) includes some more known as well as new polynomials as special cases which are not included in (2.1). These are as given below:

$$(2.15)\quad \text{(xiii)}\quad f_n(1, 1+\alpha+\beta, 1+2\alpha; \alpha+\tfrac{1}{2}; -; \tfrac{1-x}{2}) = {}_2F_1\left[\begin{matrix} -n, n+\alpha+\beta+1; \\ 1+\alpha; \end{matrix} \ \tfrac{1-x}{2}\right] = \frac{n!}{(1+\alpha)_n} P_n^{(\alpha,\beta)}(x)$$

where $P_n^{(\alpha,\beta)}$ is Jacobi's polynomial,

$$(2.16)\quad \text{(xiv)}\quad f_n(1, 1+\alpha+\beta, 1+2\alpha; \xi, \alpha+\tfrac{1}{2}; p; v) = {}_3F_2\left[\begin{matrix} -n, n+\alpha+\beta+1, \xi; \\ 1+\alpha, p; \end{matrix} \ v\right] = \frac{n!}{(1+\alpha)_n} H_n^{(\alpha,\beta)}(\xi, p, v)$$

where $H_n^{(\alpha,\beta)}(\xi, p, v)$ is the generalized Rice's polynomial due to Khandekar [1],

$$(2.17)\quad \text{(xv)}\quad f_n(1, 1+\alpha+\beta, 1+2\alpha; \alpha+\tfrac{1}{2}; 1+b; x) = {}_2F_2\left[\begin{matrix} -n, n+\alpha+\beta+1; \\ \alpha+1, b+1; \end{matrix} \ x\right]$$

which is a Jacobi type generalization of Bateman's polynomial $Z_n(x)$ and is believed to be new. For $\alpha = \beta = \nu - \frac{1}{2}$ it reduces to (2.9). Also, for $\alpha = \beta = b = 0$ it becomes Bateman's polynomial $Z_n(x)$. We adopt the symbol $Z_n^{(\alpha,\beta)}(b, x)$ to denote the polynomial (2.17).

$$(2.18)\quad \text{(xvi)}\quad f_n(1, 1+\alpha+\beta, 1+2\alpha; \alpha+\tfrac{1}{2}, \tfrac{1+z}{2}; p; 1) = {}_3F_2\left[\begin{matrix} -n, n+\alpha+\beta+1, \tfrac{1}{2}(1+z); \\ 1+\alpha, p; \end{matrix} \ 1\right]$$

which is a Jacobi type generalization of Bateman's polynomial $F_n(z)$ and is believed to be new. For $\beta = \alpha$ it reduces to (2.10). Also, for $\alpha = \beta = 0$, $p = 1$ it becomes $F_n(z)$. We adopt the symbol $F_n^{(\alpha,\beta)}(p, z)$ to denote the polynomial (2.18).

$$(2.19)\quad \text{(xvii)}\quad f_n(1, 1+\alpha+\beta, 1+2\alpha; \alpha+\tfrac{1}{2}, \tfrac{1+z+m}{2}; m+1; 1) = {}_3F_2\left[\begin{matrix} -n, n+\alpha+\beta+1, \tfrac{1}{2}(1+z+m); \\ 1+\alpha, \quad m+1; \end{matrix} \ 1\right]$$

which is a Jacobi type generalization of Pasternak's polynomial $F_n^m(z)$ and is believed to be new. For $\alpha = \beta$ it reduces to (2.12) and for $\alpha = \beta = 0$ it becomes Pasternak's polynomial $F_n^m(z)$. We adopt the symbol $F_{n,m}^{(\alpha,\beta)}(z)$ to denote the polynomial (2.19).

$$(2.20) \qquad \text{(xviii)}\ f_n(1, a-1, 1; \tfrac{1}{2}, 1; -; -\tfrac{x}{b}) = {}_2F_0 \left[\begin{matrix} -n, a-1+n; \\ \\ -; \end{matrix} \quad -\tfrac{x}{b} \right]$$

which is a generalized Bessel's polynomial $y_n(a, b, x)$;

$$(2.21) \qquad \text{(xix)} \quad f_n(1, 1+\alpha+\beta, 1+2\alpha; \alpha+\tfrac{1}{2}, -x; -N; 1) = {}_3F_2 \left[\begin{matrix} -n, n+\alpha+\beta+1, -x; \\ \\ 1+\alpha, -N; \end{matrix} \quad 1 \right]$$

which, for $\alpha, \beta > -1$, $n, x = 0, 1, 2, \ldots, N$, is Hahn's polynomial $Q_n(x; \alpha, \beta, N)$.

Finally, consider the following generalization of (2.14):

$$(2.22) \qquad f_n(r, s; \lambda, \mu; a_1, \ldots, a_p; b_1, \ldots, b_q; x) = {}_{p+r+1}F_{q+s+1} \left[\begin{matrix} \Delta(r, -n), n+\lambda, a_1, \ldots, a_p; \\ \\ \Delta(s+1, \mu), \qquad b_1, \ldots, b_q; \end{matrix} \quad x \right].$$

For $r = s = k$, (2.22) becomes (2.14) and hence all polynomials which are included in (2.14) are also included in (2.22) as special cases. Besides, a special case of (2.22) includes Konhauser's polynomial $Z_n^\alpha(x; k)$ which is not included in the similar special case of (2.14).

We now consider special cases of (2.14) and (2.22). To this end we first replace x by $\frac{x}{\lambda}$ and let $|\lambda| \to \infty$ in (2.14), obtaining

$$(2.23) \qquad f_n(k, \mu; a_1, \ldots, a_p; b_1, \ldots, b_q; x) = {}_{p+k}F_{q+k+1} \left[\begin{matrix} \Delta(k, -n), a_1, \ldots, a_p; \\ \\ \Delta(k+1, \mu), b_1, \ldots, b_q; \end{matrix} \quad x \right].$$

The polynomial (2.23) includes as special cases the following polynomials:

$$(2.24) \qquad \text{(xx)} \quad f_n(1; -; -; 1+\alpha; x) = {}_1F_1 \left[\begin{matrix} -n; \\ \\ 1+\alpha; \end{matrix} \quad x \right] = \frac{n!}{(1+\alpha)_n} L_n^{(\alpha)}(x)$$

where $L_n^{(\alpha)}(x)$ is Laguerre polynomial;

$$(2.25)\qquad \text{(xxi)}\quad f_n(2;-;-;-;-\tfrac{1}{x^2}) = {}_2F_0\left[\begin{matrix}-\frac{n}{2}, \frac{-n+1}{2}; & -\frac{1}{x^2}\\ -; & \end{matrix}\right] = \frac{1}{(2x)^n}H_n(x)$$

where $H_n(x)$ is Hermite's polynomial;

$$(2.26)\qquad \text{(xxii)}\quad f_n(m;-;-;-;h(-\tfrac{m}{x})^m)$$
$$= {}_mF_0\left[\begin{matrix}\Delta(m,-n); & h(-\frac{m}{x})^m\\ -; & \end{matrix}\right] = \frac{1}{x^n}g_n^m(x,h)$$

where $g_n^m(x,h)$ is Gould and Hopper's generalization of Hermite's polynomial;

$$(2.27)\qquad \text{(xxiii)}\quad f_n(1;-;-x;-N;p^{-1}) = {}_2F_1\left[\begin{matrix}-n,-x; & \\ & p^{-1}\\ -N; & \end{matrix}\right]$$

which, for $0 < p < 1$, $x = 0,1,\ldots,N$, is Krawtchouk's polynomial $K_n(x;p,N)$;

$$(2.28)\qquad \text{(xxiv)}\quad f_n(1;-;-x;\beta;1-c^{-1}) = {}_2F_1\left[\begin{matrix}-n,-x; & \\ & 1-c^{-1}\\ \beta; & \end{matrix}\right]$$

which, for $\beta > 0$, $0 < c < 1$, $x = 0,1,2,\ldots$, is Meixner's polynomial $M_n(x;\beta,c)$;

$$(2.29)\qquad \text{(xxv)}\quad f_n(1;-;-x;-;-\tfrac{1}{a}) = {}_2F_0\left[\begin{matrix}-n,-x; & \\ & -\frac{1}{a}\\ -; & \end{matrix}\right] = (-\tfrac{1}{a})^n C_n^{(a)}(x)$$

where $C_n^{(a)}(x)$ is Charlier's polynomial;

$$(2.30)\qquad \text{(xxvi)}\quad f_n(1;-;x;-;-x^{-1}) = {}_2F_0\left[\begin{matrix}-n,x; & \\ & ;-x^{-1}\\ -; & \end{matrix}\right] = \frac{n!}{x^n}\varphi_n(x)$$

where $\varphi_n(x)$ is Sylvester's polynomial;

$$(2.31)\qquad \text{(xxvii)}\quad f_n(1;-;-x;1;1-\mathrm{e}^{\lambda}) = {}_2F_1\left[\begin{matrix}-n,-x; & \\ & 1-\mathrm{e}^{\lambda}\\ 1; & \end{matrix}\right]$$
$$= \mathrm{e}^{n\lambda}\ell_n(x;\lambda)$$

where $\ell_n(x;\lambda)$ is Gottlieb's polynomial.

We now replace x by $\frac{x}{\lambda}$ and let $|\lambda| \to \infty$ in (2.22), obtaining

$$f_n(r, s; \mu; a_1, \dots, a_p; b_1, \dots, b_q; x) = {}_{p+r}F_{q+s+1}\left[\begin{matrix} \Delta(r, -n), a_1, \dots, a_p; \\ \\ \Delta(s+1, \mu), b_1, \dots, b_q; \end{matrix} \; x\right]. \tag{2.32}$$

For $r = s = k$, (2.32) becomes (2.23) and hence all polynomials included in (2.23) are also included in (2.32) as special cases.

Further, (2.32) includes as special case Kounhauser's polynomial $Z_n^\alpha(x; k)$ which is not included in (2.23):

$$\text{(xxviii)} \quad f_n(1, k-1; \alpha+1; -; -; (\tfrac{x}{k})^k) = {}_1F_1\left[\begin{matrix} -n; \\ \\ \Delta(k, \alpha+1); \end{matrix} \; (\tfrac{x}{k})^k\right] = \frac{n!}{(\alpha+1)_{kn}} Z_n^\alpha(x; k) \tag{2.33}$$

where $Z_n^\alpha(x; k)$ is Konhauser's polynomial.

3. Generating functions

Let $\Psi(u)$ have a formal power-series expansion

$$\Psi(u) = \sum_{n=0}^{\infty} \gamma_n u^n, \ \ \gamma_0 \neq 0. \tag{3.1}$$

Define polynomials $f_n(k; x)$ by

$$(1-t)^{-\lambda} \Psi\Big(\frac{(-1)^k (k+1)^{k+1} x t^k}{k^k (1-t)^{k+1}}\Big) = \sum_{n=0}^{\infty} f_n(k; x) t^n. \tag{3.2}$$

Theorem 1. *The polynomials $f_n(k;x)$ defined by (3.1) and (3.2) have the following properties:*

$$(3.3)\qquad f_n(k;x)=\frac{(\lambda)_n}{n!}\sum_{r=0}^{[\frac{n}{k}]}\frac{(-\frac{n}{k})_r(\frac{-n+1}{k})_r\dots(\frac{-n+k-1}{k})_r\gamma_r x^r}{(\frac{\lambda}{k+1})_r(\frac{\lambda+1}{k+1})_r\dots(\frac{\lambda+k}{k+1})_r},$$

$$(3.4)\qquad kxf_n'(k;x)-n\,f_n(k;x)$$
$$=-(\lambda+n-1)f_{n-1}(k;x)-xf_{n-1}'(k;x),\quad n\geqslant 1,$$

$$(3.5)\qquad xf_n'(k;x)-n\,f_n(k;x)$$
$$=-\lambda\sum_{r=0}^{n-1}f_r(k;x)-2x\sum_{r=0}^{n-1}f_r'(k;x)-(k-1)x\sum_{r=0}^{n}f_r'(k;x),\quad n\geqslant 1,$$

$$(3.6)\qquad xf_n'(k;x)-n\,f_n(k;x)$$
$$=\sum_{r=0}^{n-1}(-1)^{n-r}(\lambda+2r)f_r(k;x)-(k-1)x\sum_{r=0}^{n}(-1)^{n-r}f_r'(k;x),\quad n\geqslant 1.$$

P r o o f. To obtain (3.3), consider

$$\sum_{n=0}^{\infty}f_n(k;x)t^n=\sum_{r=0}^{\infty}\Big[\frac{(-1)^k(k+1)^{k+1}xt^k}{k^k}\Big]^r\frac{\gamma_r}{(1-t)^{\lambda+(k+1)r}}$$
$$=\sum_{r=0}^{\infty}\Big[\frac{(-1)^k(k+1)^{k+1}xt^k}{k^k}\Big]^r\gamma_r\sum_{n=0}^{\infty}\frac{(\lambda+(k+1)r)_n t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{(\lambda)_{n+(k+1)r}\gamma_r}{(\lambda)_{(k+1)r}n!}\Big[\frac{(-1)^k(k+1)^{k+1}x}{k^k}\Big]^r t^{n+kr}$$
$$=\sum_{n=0}^{\infty}\sum_{r=0}^{[\frac{n}{k}]}\frac{(\lambda)_{n+r}\gamma_r(-1)^{kr}x^r t^n}{(\frac{\lambda}{k+1})_r(\frac{\lambda+1}{k+1})_r\dots(\frac{\lambda+k}{k+1})_r(n-kr)!k^{kr}}$$
$$=\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\Big[\sum_{r=0}^{[\frac{n}{k}]}\frac{(-\frac{n}{k})_r\dots(\frac{-n+k-1}{k})_r(\lambda+n)_r\gamma_r x^r}{(\frac{\lambda}{k+1})_r(\frac{\lambda+1}{k+1})_r\dots(\frac{\lambda+k}{k+1})_r}\Big]t^n$$

from which (3.3) follows by equating coefficients at t^n.

In order to derive (3.4), (3.5), and (3.6), put

$$(3.7)\qquad F=(1-t)^{-\lambda}\Psi\Big(\frac{(-1)^k(k+1)^{k+1}xt^k}{k^k(1-t)^{k+1}}\Big).$$

Then

$$(3.8)\qquad \frac{\partial F}{\partial x}=(1-t)^{-\lambda}\frac{(-1)^k(k+1)^{k+1}t^k}{k^k(1-t)^{k+1}}\Psi',$$

$$(3.9)\qquad \frac{\partial F}{\partial t}=\lambda(1-t)^{-\lambda-1}\Psi+\frac{(-1)^k(k+1)^{k+1}x(k+t)t^{k-1}}{k^k(1-t)^{\lambda+k+2}}\Psi'.$$

From (3.7), (3.8) and (3.9) we obtain that F satisfies the partial differential equation

$$x(k+t)\frac{\partial F}{\partial x} - t(1-t)\frac{\partial F}{\partial t} = -\lambda t F. \tag{3.10}$$

Equation (3.10) can be put in the forms

$$kx\frac{\partial F}{\partial x} - t\frac{\partial F}{\partial t} = -\lambda t F - t^2\frac{\partial F}{\partial t} - xt\frac{\partial F}{\partial x}, \tag{3.11}$$

$$x\frac{\partial F}{\partial x} - t\frac{\partial F}{\partial t} = -\frac{\lambda t}{1-t}F - \frac{2xt}{1-t}\frac{\partial F}{\partial x} - \frac{(k-1)x}{1-t}\frac{\partial F}{\partial x}, \tag{3.12}$$

$$x\frac{\partial F}{\partial x} - t\frac{\partial F}{\partial t} = -\frac{\lambda t}{1+t}F - \frac{2t^2}{1+t}\frac{\partial F}{\partial t} - \frac{(k-1)x}{1+t}\frac{\partial F}{\partial x}. \tag{3.13}$$

Since $F = \sum_{n=0}^{\infty} f_n(k;x)t^n$, equation (3.11) yields

$$\begin{aligned}
&\sum_{n=0}^{\infty}[kxf_n'(k;x) - nf_n(k;x)]t^n \\
&\quad = -\lambda\sum_{n=0}^{\infty} f_n(k;x)t^{n+1} - \sum_{n=0}^{\infty} nf_n(k;x)t^{n+1} - \sum_{n=0}^{\infty} xf_n'(k;x)t^{n+1} \\
&\quad = -\sum_{n=1}^{\infty}(\lambda+n-1)f_{n-1}(k;x)t^n - \sum_{n=1}^{\infty} xf_{n-1}'(k;x)t^n,
\end{aligned}$$

which leads to (3.4).

Equation (3.12) yields

$$\begin{aligned}
&\sum_{n=0}^{\infty}[xf_n'(k;x) - n\ f_n(k;x)]t^n \\
&\quad = -\lambda\Big(\sum_{n=0}^{\infty} t^{n+1}\Big)\Big(\sum_{r=0}^{\infty} f_r(k;x)t^r\Big) - 2x\Big(\sum_{n=0}^{\infty} t^{n+1}\Big)\Big(\sum_{r=0}^{\infty} f_r'(k;x)t^r\Big) \\
&\qquad - (k-1)x\Big(\sum_{n=0}^{\infty} t^n\Big)\Big(\sum_{r=0}^{\infty} f_r'(k;x)t^r\Big)
\end{aligned}$$

$$
\begin{aligned}
&= -\lambda \sum_{n=0}^{\infty}\sum_{r=0}^{n} f_r(k;x)t^{n+1} - 2x\sum_{n=0}^{\infty}\sum_{r=0}^{n} f'_r(k;x)t^{n+1} \\
&\quad - (k-1)x\sum_{n=0}^{\infty}\sum_{r=0}^{n} f'_r(k;x)t^{n} \\
&= -\lambda \sum_{n=1}^{\infty}\sum_{r=0}^{n-1} f_r(k;x)t^{n} - 2x\sum_{n=1}^{\infty}\sum_{r=0}^{n-1} f'_r(k;x)t^{n} \\
&\quad - (k-1)x\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} f'_r(k;x)t^{n},
\end{aligned}
$$

which leads to (3.5).

From (3.13) we obtain

$$
\begin{aligned}
&\sum_{n=0}^{\infty} [xf'_n(k;x) - n\ f_n(k;x)]t^n \\
&= -\lambda\Big(\sum_{n=0}^{\infty}(-1)^n t^{n+1}\Big)\Big(\sum_{r=0}^{\infty} f_r(k;x)t^r\Big) - 2\Big(\sum_{n=0}^{\infty}(-1)^n t^{n+1}\Big)\Big(\sum_{r=0}^{\infty} r\ f_r(k;x)t^r\Big) \\
&\quad - (k-1)\Big(\sum_{n=0}^{\infty}(-1)^n t^{n}\Big)\Big(\sum_{r=0}^{\infty} f'_r(k;x)t^r\Big) \\
&= -\lambda\sum_{n=0}^{\infty}\sum_{r=0}^{n}(-1)^{n-r} f_r(k;x)t^{n+1} - 2\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}(-1)^{n-r} r\ f_r(k;x)t^{n+1} \\
&\quad - (k-1)x\sum_{n=0}^{\infty}\sum_{r=0}^{n}(-1)^{n-r} f'_r(k;x)t^{n} \\
&= \sum_{n=1}^{\infty}\sum_{r=0}^{n}(-1)^{n-r}(\lambda+2r) f_r(k;x)t^{n} - (k-1)x\sum_{n=0}^{\infty}\sum_{r=0}^{n}(-1)^{n-r} f'_r(k;x)t^{n},
\end{aligned}
$$

which gives (3.6).

Thus if $\Psi(u)$ is a generalized hypergeometric function

$$
\Psi(u) = {}_pF_q\left[\begin{matrix} a_1,\ldots,a_p; \\ \\ b_1,\ldots,b_q; \end{matrix}\quad u\right]
$$

then the functions $f_n(k;x)$ defined by (3.2) but without the factor $\frac{(\lambda)_n}{n!}$ are precisely the polynomials $f_n(k,\lambda;a_1,\ldots,a_p;b_1,\ldots,b_q;x)$ given by (2.1). It is necessary to note that in using the generating function we implicitly demand that the parameters a_i and b_j be independent of n.

Thus for (2.1) we have the generating relation

$$(3.14)\qquad (1-t)^{-\lambda}{}_pF_q\left[\begin{matrix}a_1,\dots,a_p;\\ \\ b_1,\dots,b_q;\end{matrix}\ \frac{(-1)^k(k+1)^{k+1}xt^k}{k^k(1-t)^{k+1}}\right] = \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}f_n(k,\lambda;a_1,\dots,a_p;b_1,\dots,b_q;x)t^n.$$

The generating relation for (2.14) is

$$(3.15)\qquad (1-t)^{-\lambda}{}_{p+k+1}F_{q+k+1}\left[\begin{matrix}\Delta(k+1,\lambda),a_1,\dots,a_p;\\ \\ \Delta(k+1,\mu),b_1,\dots,b_q;\end{matrix}\ \frac{(-1)^k(k+1)^{k+1}xt^k}{k^k(1-t)^{k+1}}\right] = \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}f_n(k,\lambda,\mu;a_1,\dots,a_p;b_1,\dots,b_q;x)t^n.$$

The generating relation for (2.22) is

$$(3.16)\qquad (1-t)^{-\lambda}{}_{p+r+1}F_{q+s+1}\left[\begin{matrix}\Delta(r+1,\lambda),a_1,\dots,a_p;\\ \\ \Delta(s+1,\mu),b_1,\dots,b_q;\end{matrix}\ \frac{(-1)^r(r+1)^{r+1}xt^r}{r^r(1-t)^{r+1}}\right] = \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}f_n(r,s;\lambda,\mu;a_1,\dots,a_p;b_1,\dots,b_q;x)t^n.$$

Since the proofs of (3.15) and (3.16) proceed along similar lines we prove (3.16) only briefly. □

P r o o f of (3.16). The right hand side of (3.16) equals

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{r}]}\frac{(-\frac{n}{r})_k(\frac{-n+1}{r})_k\dots(\frac{-n+r-1}{r})_k(\lambda)_{n+k}(a_1)_k\dots(a_p)_kx^kt^n}{n!k!(\frac{\mu}{s+1})_k(\frac{\mu+1}{s+1})_k\dots(\frac{\mu+s}{s+1})_k(b_1)_k\dots(b_q)_k}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{r}]}\frac{(-1)^{kr}(\lambda)_{n+k}(a_1)_k\dots(a_p)_kx^kt^n}{(n-rk)!r^{rk}k!(\frac{\mu}{s+1})_k\dots(\frac{\mu+s}{s+1})_k(b_1)_k\dots(b_q)_k}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{kr}(\lambda)_{n+(r+1)k}(a_1)_k\dots(a_p)_kx^kt^{n+rk}}{n!r^{rk}k!(\frac{\mu}{s+1})_k\dots(\frac{\mu+s}{s+1})_k(b_1)_k\dots(b_q)_k}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^{kr}(\lambda)_{(r+1)k}(a_1)_k\dots(a_p)_kx^kt^{rk}(1-t)^{-\lambda-(r+1)k}}{k!r^{rk}(\frac{\mu}{s+1})_k(\frac{\mu+1}{s+1})_k\dots(\frac{\mu+s}{s+1})_k(b_1)_k\dots(b_q)_k}$$

$$= (1-t)^{-\lambda}{}_{p+r+1}F_{q+s+1}\left[\begin{matrix}\Delta(r+1,\lambda),a_1,\dots,a_p;\\ \\ \Delta(s+1,\mu),b_1,\dots,b_q;\end{matrix}\ \frac{(-1)^r(r+1)^{r+1}xt^r}{r^r(1-t)^{r+1}}\right],$$

which completes the proof of (3.16).

A generating function for (2.23) is

$$\mathrm{e}^t {}_pF_{q+k+1}\left[\begin{matrix} a_1, a_2, \dots, a_p; \\ \Delta(k+1,\mu), b_1, \dots, b_q; \end{matrix} \; \frac{(-1)^k x t^k}{k^k}\right] = \sum_{n=0}^{\infty} f_n(k;\mu;a_1,\dots,a_p;b_1,\dots,b_q;x)\frac{t^n}{n!}. \tag{3.17}$$

Similarly, a generating function for (2.32) is

$$\mathrm{e}^t {}_pF_{q+s+1}\left[\begin{matrix} a_1, a_2, \dots, a_p; \\ \Delta(s+1,\mu), b_1, \dots, b_q; \end{matrix} \; \frac{(-1)^r x t^r}{r^r}\right] = \sum_{n=0}^{\infty} f_n(r,s;\mu;a_1,\dots,a_p;b_1,\dots,b_q;x)\frac{t^n}{n!}. \tag{3.18}$$

Proof of (3.17) and (3.18) are easy and hence are omitted. □

4. Integral relations

It is easy to obtain the following integral relations for (2.22):

$$\frac{\mathrm{i}}{2\sqrt{\pi}}\int_{\infty}^{(0+)} (-t)^{-\frac12}\mathrm{e}^{-t} f_n(r,s;\lambda,\mu;\tfrac12,a_1,\dots,a_p;b_1,\dots,b_q;-\tfrac{x}{t})\,\mathrm{d}t = f_n(r,s;\lambda,\mu;a_1,\dots,a_p;b_1,\dots,b_q;x), \tag{4.1}$$

$$\frac{1}{\sqrt{\pi}}\int_0^{\infty} t^{-\frac12}\mathrm{e}^{-t} f_n(r,s;\lambda,\mu;a_1,\dots,a_p;\tfrac12,b_1,\dots,b_q;xt)\,\mathrm{d}t = f_n(r,s;\lambda,\mu;a_1,\dots,a_p;b_1,\dots,b_q;x), \tag{4.2}$$

$$\frac{\Gamma(b_1)}{\Gamma(a_1)\Gamma(b_1-a_1)}\int_0^1 t^{a_1-1}(1-t)^{b_1-a_1-1} f_n(r,s;\lambda,\mu;a_2,\dots,a_p;b_2,\dots,b_q;xt)\,\mathrm{d}t = f_n(r,s;\lambda,\mu;a_1,\dots,a_p;b_1,\dots,b_q;x). \tag{4.3}$$

As an immediate generalization of (4.1) and (4.2) we obtain

$$\frac{\mathrm{i}}{2\sin\nu\pi\Gamma(1-\nu)}\int_{\infty}^{(0+)} (-t)^{-\nu}\mathrm{e}^{-t} f_n(r,s;\lambda,\mu;\nu,a_1,\dots,a_p;b_1,\dots,b_q;-\tfrac{x}{t})\,\mathrm{d}t = f_n(r,s;\lambda,\mu;a_1,\dots,a_p;b_1,\dots,b_q;x), \tag{4.4}$$

$$\frac{1}{\Gamma(\nu)}\int_0^{\infty} t^{\nu-1}\mathrm{e}^{-t} f_n(r,s;\lambda,\mu;a_1,\dots,a_p;\nu,b_1,\dots,b_q;xt)\,\mathrm{d}t = f_n(r,s;\lambda,\mu;a_1,\dots,a_p;b_1,\dots,b_q;x). \tag{4.5}$$

Putting $r = s = k$ in (4.1), (4.2), (4.3), (4.4) and (4.5), we get the corresponding integral relations for (2.14), while by putting $r = s = k$ and $\lambda = \mu$ in (4.1), (4.2), (4.3), (4.4) and (4.5) the corresponding integral relations for (2.1) are obtained.

Some special cases of interest are:

$$\frac{1}{\Gamma(\nu)}\int_0^\infty t^{\nu-1}\mathrm{e}^{-t} f_n(1,1;a_1,\dots,a_p;\nu,b_1,\dots,b_q;xt)\,\mathrm{d}t = {}_{p+2}F_{q+2}\left[\begin{matrix} -n, n+1, a_1,\dots,a_p; \\ 1, \frac{1}{2}, \quad b_1,\dots,b_q; \end{matrix}\; x\right] \tag{4.6}$$

which is Sister Celine's polynomial,

$$\frac{\mathrm{i}}{2\sin\nu\pi\Gamma(1-\nu)}\int_\infty^{(0+)} (-t)^{-\nu}\mathrm{e}^{-t}\,{}_{p+2}F_{q+2}\left[\begin{matrix} -n, n+1, a_1,\dots,a_p; \\ 1, \frac{1}{2}, \; b_1,\dots,b_q; \end{matrix}\; -\frac{x}{t}\right]\mathrm{d}t = f_n(1,1;a_1,\dots,a_p;\nu,b_1,\dots,b_q;x), \tag{4.7}$$

$$\frac{1}{\sqrt{\pi}}\int_0^\infty t^{-\frac{1}{2}}\mathrm{e}^{-t} f_n(1,1;-;-;xt)\,\mathrm{d}t = P_n(1-2x), \tag{4.8}$$

$$\frac{\mathrm{i}}{2\sqrt{\pi}}\int_\infty^{(0+)} (-t)^{-\frac{1}{2}}\mathrm{e}^{-t} P_n\Big(\frac{2x+t}{t}\Big)\,\mathrm{d}t = f_n(1,1;-;-;x), \tag{4.9}$$

$$\frac{1}{\Gamma(\nu)}\int_0^\infty t^{\nu-1}\mathrm{e}^{-t} f_n(1,2\nu;-;-;xt)\,\mathrm{d}t = \frac{n!}{(2\nu)_n} C_n^\nu(1-2x), \tag{4.10}$$

$$\frac{\mathrm{i}}{2\sin\nu\pi\Gamma(1-\nu)}\int_\infty^{(0+)} (-t)^{-\nu}\mathrm{e}^{-t} C_n^\nu\Big(\frac{2x+t}{t}\Big)\,\mathrm{d}t = \frac{(2\nu)_n}{n!} f_n(1,2\nu;-;-;x), \tag{4.11}$$

$$\frac{1}{\Gamma(\alpha+\frac{1}{2})}\int_0^\infty t^{\alpha-\frac{1}{2}}\mathrm{e}^{-t} f_n(1,2\alpha+1;-;-;xt)\,\mathrm{d}t = \frac{n!}{(1+\alpha)_n} P_n^{(\alpha,\alpha)}(1-2x), \tag{4.12}$$

$$\frac{\mathrm{i}}{2\sin(\alpha+\frac{1}{2})\pi\Gamma(\frac{1}{2}-\alpha)}\int_\infty^{(0+)} (-t)^{-\alpha-\frac{1}{2}}\mathrm{e}^{-t} P_N^{(\alpha,\alpha)}\Big(\frac{(2x+t)}{t}\Big)\,\mathrm{d}t = \frac{(1+\alpha)_n}{n!} f_n(1,2\alpha+1;-;-;x), \tag{4.13}$$

$$\frac{1}{\sqrt{\pi}}\int_0^\infty t^{-\frac{1}{2}}\mathrm{e}^{-t} f_n(1,1;\xi;p;vt)\,\mathrm{d}t = H_n(\xi,p,v), \tag{4.14}$$

(4.15) $$\frac{\mathrm{i}}{2\sqrt{\pi}}\int_{\infty}^{(0+)}(-t)^{-\frac{1}{2}}\mathrm{e}^{-t}H_n(\xi,p,-\tfrac{v}{t})\,\mathrm{d}t$$
$$= f_n(1,1;\xi;p;v),$$

(4.16) $$\frac{1}{\sqrt{\pi}}\int_0^{\infty}t^{-\frac{1}{2}}\mathrm{e}^{-t}f_n(1,1;-;1;xt)\,\mathrm{d}t = Z_n(x),$$

(4.17) $$\frac{i}{2\sqrt{\pi}}\int_{\infty}^{(0+)}(-t)^{-\frac{1}{2}}\mathrm{e}^{-t}Z_n(-\tfrac{x}{t})\,\mathrm{d}t = f_n(1,1;-;1;x),$$

(4.17a) $$\frac{1}{\sqrt{\pi}}\int_0^{\infty}t^{-\frac{1}{2}}\mathrm{e}^{-t}f_n(1,1;\tfrac{1}{2}+\tfrac{1}{2}z;1;t)\,\mathrm{d}t = F_n(z),$$

(4.18) $$\frac{1}{\Gamma(\nu)}\int_0^{\infty}t^{\nu-1}\mathrm{e}^{-t}f_n(1,2\nu;-;1+b;xt)\,\mathrm{d}t$$
$$= Z_n^{(\nu)}(b,x),$$

(4.19) $$\frac{\mathrm{i}}{2\sin\nu\pi\Gamma(1-\nu)}\int_{\infty}^{(0+)}(-t)^{-\nu}\mathrm{e}^{-t}Z_n^{(\nu)}(b,-\tfrac{x}{t})\,\mathrm{d}t$$
$$= f_n(1,2\nu;-;1+b;x),$$

(4.20) $$\frac{1}{\Gamma(\alpha+\frac{1}{2})}\int_0^{\infty}t^{\alpha-\frac{1}{2}}\mathrm{e}^{-t}f_n(1,2\alpha+1;\tfrac{1}{2}+\tfrac{1}{2}z;p;t)\,\mathrm{d}t$$
$$= F_n^{(\alpha,\alpha)}(p,z),$$

(4.21) $$\frac{1}{\sqrt{\pi}}\int_0^{\infty}t^{-\frac{1}{2}}\mathrm{e}^{-t}f_n(1,1;\tfrac{1}{2}+\tfrac{1}{2}z+\tfrac{1}{2}m;m+1;t)\,\mathrm{d}t$$
$$= F_n^{m}(z),$$

(4.22) $$\frac{1}{\Gamma(\alpha+\frac{1}{2})}\int_0^{\infty}t^{\alpha-\frac{1}{2}}\mathrm{e}^{-t}f_n(1,2\alpha+1;\tfrac{1}{2}+\tfrac{1}{2}z+\tfrac{1}{2}m;m+1;t)\,\mathrm{d}t$$
$$= F_{n,m}^{(\alpha,\alpha)}(z),$$

(4.23) $$\frac{1}{\sqrt{\pi}}\int_0^{\infty}t^{-\frac{1}{2}}\mathrm{e}^{-t}f_n(1,1;1;-;-\tfrac{1}{2}xt)\,\mathrm{d}t = y_n(x),$$

(4.24) $$\frac{\mathrm{i}}{2\sqrt{\pi}}\int_{\infty}^{(0+)}(-t)^{-\frac{1}{2}}\mathrm{e}^{-t}y_n(\tfrac{2x}{t})\,\mathrm{d}t = f_n(1,1;1;-;x),$$

(4.25) $$\frac{1}{\Gamma(\nu)}\int_0^{\infty}t^{\nu-1}\mathrm{e}^{-t}f_n(1,a-1,2\nu;\nu+\tfrac{1}{2};-;-\tfrac{1}{b}xt)\,\mathrm{d}t$$
$$= y_n(a,b,x)$$

(4.26) $$\frac{\mathrm{i}}{2\sin\nu\pi\Gamma(1-\nu)}\int_{\infty}^{(0+)}(-t)^{-\nu}\mathrm{e}^{-t}y_n(a,b,\tfrac{bx}{t})\,\mathrm{d}t$$
$$= f_n(1,a-1,2\nu;\nu+\tfrac{1}{2};-;x),$$

(4.27) $$\frac{1}{\Gamma(\alpha+\frac{1}{2})}\int_0^{\infty}t^{\alpha-\frac{1}{2}}\mathrm{e}^{-t}f_n(1,\alpha+\beta+1,2\alpha+1;-;b+1;xt)\,\mathrm{d}t$$
$$= Z_n^{(\alpha,\beta)}(b,x),$$

$$(4.28)\quad \frac{\mathrm{i}}{2\sin(\alpha+\frac12)\pi\Gamma(\frac12-\alpha)}\int_{\infty}^{(0+)}(-t)^{-\alpha-\frac12}\mathrm{e}^{-t}Z_n^{(\alpha,\beta)}(b,-\tfrac{x}{t})\,\mathrm{d}t$$
$$= f_n(1,1+\alpha+\beta,\ 2\alpha+1;-;b+1;x),$$

$$(4.29)\quad \frac{1}{\Gamma(\alpha+\frac12)}\int_0^{\infty}t^{\alpha-\frac12}\mathrm{e}^{-t}f_n(1,\alpha+\beta+1,2\alpha+1;\tfrac12+\tfrac12 z;p;t)\,\mathrm{d}t$$
$$= F_n^{(\alpha,\beta)}(p,z),$$

$$(4.30)\quad \frac{1}{\Gamma(\alpha+\frac12)}\int_0^{\infty}t^{\alpha-\frac12}\mathrm{e}^{-t}f_n(1,\alpha+\beta+1,2\alpha+1;\tfrac12+\tfrac12 z+\tfrac12 m;m+1;t)\,\mathrm{d}t$$
$$= F_{n,m}^{(\alpha,\beta)}(z),$$

$$(4.31)\quad \frac{1}{\Gamma(\alpha+\frac12)}\int_0^{\infty}t^{\alpha-\frac12}\mathrm{e}^{-t}f_n(1,\alpha+\beta+1,2\alpha+1;-x;-N;t)\,\mathrm{d}t$$
$$= Q_n(x;\alpha,\beta,N),$$

$$(4.32)\quad \frac{1}{\Gamma(\alpha+\frac12)}\int_0^{\infty}t^{\alpha-\frac12}\mathrm{e}^{-t}f_n(1;2\alpha+1;-;-;xt)\,\mathrm{d}t$$
$$= \frac{n!}{(1+\alpha)_n}L_n^{(\alpha)}(x),$$

$$(4.33)\quad \frac{\mathrm{i}}{2\sin(\alpha+\frac12)\pi\Gamma(\frac12-\alpha)}\int_{\infty}^{(0+)}(-t)^{-\alpha-\frac12}\mathrm{e}^{-t}L_n^{(\alpha)}(-\tfrac{x}{t})\,\mathrm{d}t$$
$$= \frac{(\alpha+1)_n}{n!}f_n(1;2\alpha+1;-;-;x),$$

$$(4.34)\quad \frac{1}{\Gamma(\nu)}\int_0^{\infty}t^{\nu-1}\mathrm{e}^{-t}f_n(2;3\nu;\nu+\tfrac13,\nu+\tfrac23;-\tfrac{t}{x^2})\,\mathrm{d}t$$
$$= \frac{1}{(2x)^n}H_n(x),$$

$$(4.35)\quad \frac{\mathrm{i}}{2\sin\nu\pi\Gamma(1-\nu)}\int_{\infty}^{(0+)}(-1)^{-\nu}t^{-\nu-\frac{n}{2}}H_n(\sqrt{\tfrac{t}{x}})\,\mathrm{d}t$$
$$= (\tfrac{2}{\sqrt{x}})^n f_n(2;3\nu;\nu+\tfrac13,\nu+\tfrac23;x),$$

$$(4.36)\quad \frac{1}{\Gamma(\nu)}\int_0^{\infty}t^{\nu-1}\mathrm{e}^{-t}f_n(m;(m+1)\nu;\nu+\tfrac{1}{m+1},\ldots,\nu+\tfrac{m}{m+1};h(-\tfrac{m}{x})^m t)\,\mathrm{d}t$$
$$= \frac{1}{x^n}g_n^m(x,h),$$

$$(4.36)\quad \frac{\mathrm{i}}{2\sin\nu\pi\Gamma(1-\nu)}\int_{\infty}^{(0+)}(-1)^{-\nu}t^{-\nu-\frac{n}{m}}g_n^m(-m-(\tfrac{ht}{x})^{\frac1m},h)\,\mathrm{d}t$$
$$= \{-m(-\tfrac{h}{x})^{\frac1m}\}^n f_n(m;(m+1)\nu;\nu+\tfrac{1}{m+1},\ldots,\nu+\tfrac{m}{m+1};x),$$

$$(4.37)\quad \frac{1}{\Gamma(\nu)}\int_0^{\infty}t^{\nu-1}\mathrm{e}^{-t}f_n(1;2\nu;\nu+\tfrac12,-x;-N;p^{-1}t)\,\mathrm{d}t$$
$$= K_n(x;p,N),$$

$$(4.38)\qquad \frac{\mathrm{i}}{2\sin\nu\pi\Gamma(1-\nu)}\int_{\infty}^{(0+)}(-t)^{-\nu}\mathrm{e}^{-t}K_n(x;-p^{-1}t,N)\,\mathrm{d}t$$
$$= f_n(1;2\nu;\nu+\tfrac{1}{2},-x;-N;p),$$

$$(4.39)\qquad \frac{1}{\Gamma(\nu)}\int_0^{\infty} t^{\nu-1}\mathrm{e}^{-t}f_n(1;2\nu;\nu+\tfrac{1}{2},-x;\beta;(1-c^{-1})t)\,\mathrm{d}t$$
$$= M_n(x;\beta,c),$$

$$(4.40)\qquad \frac{\mathrm{i}}{2\sin\nu\pi\Gamma(1-\nu)}\int_{0}^{(0+)}(-t)^{-\nu}\mathrm{e}^{-t}M_n(x;\beta,\tfrac{t}{c+t})\,\mathrm{d}t$$
$$= f_n(1;2\nu;\nu+\tfrac{1}{2},-x;\beta;c),$$

$$(4.41)\qquad \frac{1}{\Gamma(\nu)}\int_0^{\infty} t^{\nu-1}\mathrm{e}^{-t}f_n(1;2\nu;\nu+\tfrac{1}{2},x;-;-x^{-1}t)\,\mathrm{d}t$$
$$= \frac{n!}{x^n}\varphi_n(x),$$

$$(4.42)\qquad \frac{1}{\Gamma(\nu)}\int_0^{\infty} t^{\nu-1}\mathrm{e}^{-t}f_n(1;2\nu;\nu+\tfrac{1}{2},-x;-;-\tfrac{1}{a}t)\,\mathrm{d}t$$
$$= (-\tfrac{1}{a})^n C_n^{(a)}(x),$$

$$(4.43)\qquad \frac{\mathrm{i}}{2\sin\nu\pi\Gamma(1-\nu)}\int_{\infty}^{(0+)}(-1)^{-\nu}t^{-\nu-n}C_n^{(t/a)}(x)\,\mathrm{d}t$$
$$= (-\tfrac{1}{a})^n f_n(1;2\nu;\nu+\tfrac{1}{2},-x;-;a),$$

$$(4.44)\qquad \frac{1}{\sqrt{\pi}}\int_0^{\infty} t^{-\frac{1}{2}}\mathrm{e}^{-t}f_n(1;1;-x;-;(1-\mathrm{e}^{-\lambda})t)\,\mathrm{d}t$$
$$= \mathrm{e}^{n\lambda}\ell_n(x;\lambda),$$

$$(4.45)\qquad \frac{\mathrm{i}}{2\sqrt{\pi}}\int_{\infty}^{(0+)}(-t)^{-\frac{1}{2}}(1+\tfrac{}{t})^{-n}\mathrm{e}^{-t}{}_n(x;-\log(1+\tfrac{}{t}))\,\mathrm{d}t$$
$$= f_n(1;1;-x;-;\,),$$

$$(4.46)\qquad \frac{1}{\Gamma(\nu)}\int_0^{\infty} t^{\nu-1}\mathrm{e}^{-t}f_n(1,k-1;\alpha+1;-;\nu;(\tfrac{x}{k})^k t)\,\mathrm{d}t$$
$$= \frac{n!}{(1+\alpha)_{kn}}Z_n^{\alpha}(x;k),$$

$$(4.47)\qquad \frac{\mathrm{i}}{2\sin\nu\pi\Gamma(1-\nu)}\int_{\infty}^{(0+)}(-t)^{-\nu}\mathrm{e}^{-t}Z_n^{\alpha}(k(-\tfrac{x}{t})^{\frac{1}{k}};k)\,\mathrm{d}t$$
$$= \frac{(1+\alpha)_{kn}}{n!}f_n(1,k-1;\alpha+1;-;\nu;x).$$

5. Laguerre's $L_n^{(\alpha)}(x)$ and Bateman's polynomial $Z_n^{(\nu,b)}(x)$-

Using (4.5) we obtain at once that

$$\begin{aligned}\frac{1}{\Gamma(\nu)}\int_0^\infty & t^{\nu-1}\mathrm{e}^{-t} f_n(1, 1+\alpha; \tfrac{1+\alpha}{2}; \nu, 1+\alpha; x^2 t)\,\mathrm{d}t \\ &= f_n(1, 1+\alpha; \tfrac{1+\alpha}{2}; 1+\alpha; x^2).\end{aligned} \tag{5.1}$$

Ramanujan's formula which gives the product of two ${}_1F_1$'s as an ${}_2F_3$ and which was proved by Preece [6] reads as follows:

$$ {}_1F_1\left[\begin{matrix}\alpha; & \\ & x \\ \beta; & \end{matrix}\right] {}_1F_1\left[\begin{matrix}\alpha; & \\ & -x \\ \beta; & \end{matrix}\right] = {}_2F_3\left[\begin{matrix}\alpha, \beta-\alpha; & \\ & \frac{x^2}{4} \\ \beta, \frac{1}{2}\beta, \frac{1}{2}\beta+\frac{1}{2}; & \end{matrix}\right]. \tag{5.2}$$

It yields at once

$$L_n^{(\alpha)}(x) L_n^{(\alpha)}(-x) = \left\{\frac{(1+\alpha)_n}{n!}\right\}^2 f_n(1, 1+\alpha; -; 1+\alpha; \tfrac{x^2}{4}). \tag{5.3}$$

However, Bateman's $Z_n^{(\nu)}(b,t)$ which is a generalization of his polynomial $Z_n(t)$ is given by

$$Z_n^{(\nu)}(b,t) = f_n(1, 2\nu; \nu; 1+b; t). \tag{5.4}$$

With these results it is easy to write (5.1) in an interesting form

$$\begin{aligned} & Z_n^{(\frac{1}{2}+\frac{1}{2}\alpha)}(\alpha, x^2) \\ & \quad = \frac{(n!)^2}{2^\alpha \Gamma(\frac{1+\alpha}{2})\{(1+\alpha)_n\}^2}\int_0^\infty t^\alpha \mathrm{e}^{-t^2/4} L_n^{(\alpha)}(xt) L_n^{(\alpha)}(-xt)\,\mathrm{d}t.\end{aligned} \tag{5.5}$$

For $\alpha = 0$, (5.5) reduces to

$$Z_n(x^2) = \frac{1}{\sqrt{\pi}}\int_0^\infty \mathrm{e}^{-t^2/4} L_n(xt) L_n(-xt)\,\mathrm{d}t \tag{5.6}$$

where $L_n(x)$ is the simple Laguerre polynomial. Formula (5.6) appears in Sister Celine's work, Fasenmyer [2].

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