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CANTOR-BERNSTEIN THEOREM FOR MV-ALGEBRAS

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Theorems of Cantor-Bernstein type have been proved by Sikorski [9] and Tarski [11] for Boolean σ-algebras, and by the author [3], [5] for some classes of lattice ordered groups.

In the present paper we prove a result of Cantor-Bernstein type for a class of complete MV-algebras. This class is defined by means of properties of singular elements.

1. Preliminaries; main result

For MV-algebras we apply the terminology and notation from [2] and [4]. Thus an MV-algebra is a system $\mathscr{A} = (A, \oplus, *, \neg, 0, 1)$, where A is a nonempty set, $\oplus, *$ are binary operations, $\neg$ is a unary operation and $0, 1$ are nulary operations on A such that the identities (m_1)-(m_8) from [2] are satisfied.

If we put

$$x \vee y = (x * \neg y) \oplus y, \quad x \wedge y = \neg(\neg x \vee \neg y)$$

for each $x, y \in A$, then $\mathscr{L}(\mathscr{A}) = (A; \vee, \wedge)$ turns out to be a distributive lattice with the least element 0 and the greatest element 1.

Let G be an abelian lattice ordered group with a strong unit u. Let A be the interval $[0, u]$ of G. For each $a, b \in A$ we put

$$a \oplus b = (a + b) \wedge u, \quad \neg a = u - a, \quad 1 = u, \quad a * b = \neg(\neg a \oplus \neg b).$$

Then $\mathscr{A} = (A; \oplus, *, \neg, 0, 1)$ is an MV-algebra. We denote it by $\mathscr{A}_0(G, u)$.

For each MV-algebra $\mathscr{A}$ there exists an abelian lattice ordered group G with a strong unit u such that $\mathscr{A} = \mathscr{A}_0(G, u)$.

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(For the above results cf. [8] and [2].)

Recall that if G is a lattice ordered group and u is a positive element of G such that for each $g \in G$ there is a positive integer n with $g \leqslant nu$, then u is called a strong unit of G.

Given an MV-algebra $\mathscr{A}$ we always consider the partial order $\leqslant$ on A which is inherited from the lattice $\mathscr{L}(\mathscr{A})$. Also, without loss of generality we can suppose that an abelian lattice ordered group G with a strong unit u such that $\mathscr{A} = \mathscr{A}_0(G,u)$ is given. In such a case, the partial order $\leqslant$ on A inherited from $\mathscr{L}(\mathscr{A})$ is the same as that inherited from G.

The MV-algebra $\mathscr{A}$ is said to be complete if the lattice $\mathscr{L}(\mathscr{A})$ is complete.

Let φ be an isomorphism of a lattice L_1 into a lattice L_2. If $\varphi(L_1)$ is a convex sublattice of L_2, then φ is called a convex isomorphism.

An element $s \in A$ will be said to be singular in $\mathscr{A}$ if, whenever $x \in A$ and $x \leqslant s$, then x has a complement in the interval $[0,s]$ of $\mathscr{L}(\mathscr{A})$. This is equivalent with the condition that the interval $[0,s]$ of $\mathscr{L}(\mathscr{A})$ is a Boolean algebra.

Consider the following condition for an MV-algebra $\mathscr{A}$:

($*$) Each singular element of $\mathscr{A}$ has a complement in $\mathscr{L}(\mathscr{A})$.

In the present paper the following result will be proved:

(A) *Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be complete MV-algebras satisfying the condition* ($*$). *Assume that*

(i) *there exists a convex isomorphism φ_1 of $\mathscr{L}(\mathscr{A}_1)$ into $\mathscr{L}(\mathscr{A}_2)$;*

(ii) *there exists a convex isomorphism φ_2 of $\mathscr{L}(\mathscr{A}_2)$ into $\mathscr{L}(\mathscr{A}_1)$.*

Then the MV-algebras $\mathscr{A}_1$ and $\mathscr{A}_2$ are isomorphic.

Next, a result of Cantor-Bernstein type from [5] concerning complete lattice ordered groups will be generalized in the present paper.

2. Auxiliary results

We assume that $\mathscr{A}, G$ and u are as in the previous section, i.e., $\mathscr{A} = \mathscr{A}_0(G,u)$.

2.1. Lemma. (Cf. [6].) *The MV-algebra $\mathscr{A}$ is complete if and only if G is complete.*

An element g of G with $0 \leqslant g$ is said to be singular in G if, whenever $x \in G$ such that $0 \leqslant x \leqslant g$, then $x \wedge (g - x) = 0$. (Cf. [1].) Equivalently, an element $g \in G^+$ is singular in G if and only if the interval $[0,g]$ of $\mathscr{L}(G)$ is a Boolean algebra. (Cf. [3], 2.2.)

2.2. Lemma. *Let s be a singular element of G. If $b_1, b_2, \ldots, b_n \in G^+$, $n \geqslant 2$ and $s = b_1 + b_2 + \ldots + b_n$, then $b_i \wedge b_j = 0$ whenever i and j are distinct elements of $\{1, 2, \ldots, n\}$.*

Proof. We proceed by induction on n. For $n = 2$ the assertion follows from the definition of singular elements of G. Suppose that $n > 2$ and that the assertion holds for $n - 1$.

Since $0 \leqslant b_1 + b_2 + \ldots + b_{n-1} \leqslant s$, the element $b_1 + b_2 + \ldots + b_{n-1}$ is singular and hence $b_i \wedge b_j = 0$ whenever i and j are distinct elements of the set $\{1, 2, \ldots, n-1\}$. Next, $s = (b_1 + \ldots + b_{n-1}) + b_n$, whence

$$(b_1 + \ldots + b_{n-1}) \wedge b_n = 0$$

and this yields that $b_i \wedge b_n = 0$ for $i = 1, 2, \ldots, n-1$. □

2.3. Lemma. *Let $g \in G$. Then the following conditions are equivalent:*

(i) *g is a singular element of G.*
(ii) *g belongs to A and it is a singular element of $\mathscr{A}$.*

Proof. It is obvious that (ii)⇒(i). Conversely, let (i) be valid. There exists a positive integer n such that $g \leqslant nu$. Hence there are elements $a_1, a_2, \ldots, a_n$ in A with $g = a_1 + a_2 + \ldots + a_n$. Thus according to 2.2 the relation

$$g = a_1 \vee a_2 \vee \ldots \vee a_n \tag{1}$$

is valid in $\mathscr{L}(G)$. Since $\mathscr{L}(\mathscr{A})$ is a sublattice of $\mathscr{L}(G)$ we infer from (1) that $g \in A$. Then it is clear that g is a singular element of $\mathscr{A}$. □

An MV-algebra $\mathscr{A}$ is called singular if each strictly positive element of A exceeds a strictly positive singular element of $\mathscr{A}$. The notion of the singular lattice ordered group is defined analogously.

The following lemma is an immediate consequence of 2.3.

2.4. Lemma. *An MV-algebra $\mathscr{A}$ is singular if and only if the lattice ordered group G is singular.*

For $X \subseteq G$ we put

$$X^\delta = \{g \in G \colon |g| \wedge |x| = 0 \quad \text{for each } x \in X\}.$$

2.5. Lemma. *Let X be a set of singular elements of G. Then the lattice ordered group $X^{\delta\delta}$ is singular.*

The proof is simple, it will be omitted.

For direct product decompositions of MV-algebras we apply the notation as in [6].

2.6. Lemma. *Assume that $\mathscr{A}$ is a complete MV-algebra. Then there exists a direct product decomposition $\mathscr{A} = \mathscr{A}_1 \times \mathscr{A}_2$ such that $\mathscr{A}_1$ is singular and $\mathscr{A}_2$ has no singular element distinct from* 0.

Proof. Let S be the set of all singular elements of G. Put $G_1 = S^{\delta\delta}$ and $G_2 = S^{\delta}$. In view of 2.1, G is complete. Then according to the well-known Riesz Theorem we have

$$G = G_1 \times G_2. \tag{2}$$

In view of 2.5, G_1 is singular. It is clear that G_2 has no strictly positive singular element. The relation (2) and [4], 3.2 yield that there exists a direct product decomposition

$$\mathscr{A} = \mathscr{A}_1 \times \mathscr{A}_2, \tag{3}$$

where $A_1 = G_1 \cap A$ and $A_2 = G_2 \cap A$. If u_i $(i = \{1, 2\})$ is a component of u in G_i in the direct product decomposition (2), then

$$\mathscr{A}_i = \mathscr{A}_0(G_i, u_i).$$

Hence in view of 2.4, $\mathscr{A}_1$ is singular. Next, according to 2.3, $\mathscr{A}_2$ has no strictly positive singular element. □

2.7. Lemma. *Let $\mathscr{A}$, $\mathscr{A}_1$ and $\mathscr{A}_2$ be as in* 2.6. *Then $\mathscr{A}$ satisfies the condition* $(*)$ *if and only if $\mathscr{A}_1$ satisfies this condition.*

Proof. Assume that $\mathscr{A}$ satisfies the condition $(*)$. Let u_1 and u_2 be as in the proof of 2.6. Then u_1 is a strong unit of $\mathscr{A}_1$. Let s be a singular element of $\mathscr{A}_1$. Hence s is a singular element of $\mathscr{A}$ and thus $s \leqslant u$; since $s \in A_1$ we obtain that $s \leqslant u_1$. Next, there exists a relative complement s_1 of s in the interval $[0, u]$. We denote by s_{11} the component of s_1 in $\mathscr{A}_1$. Hence s_{11} is a complement of s in the interval $[0, u_1]$. Therefore $(*)$ is valid for $\mathscr{A}_1$.

Conversely, assume that $\mathscr{A}_1$ satisfies the condition $(*)$. Let s be a singular element of $\mathscr{A}$. According to 2.3, s belongs to $\mathscr{A}_1$. Hence $s \leqslant u_1 \leqslant u$. Next, there exists a complement x of s in the interval $[0, u_1]$. Put

$$y = x \vee u_2.$$

Then y is a complement of s in the interval $[0, u]$. Thus $\mathscr{A}$ satisfies the condition $(*)$. □

A subset X of G^+ will be called orthogonal if $x_1 \wedge x_2 = 0$ whenever x_1 and x_2 are distinct elements of X.

As above, let S be the set of all singular elements of G. By applying Axiom of Choice we conclude that there exists a maximal orthogonal subset $\{s_i\}_{i\in I}$ of S.

Since G is complete, in view of 2.3 there exists $t \in G$ such that $t = \bigvee_{i\in I} s_i$. Clearly $t \in A$.

2.8. Lemma. *The element t is singular in G.*

Proof. Let $0 \leqslant x \leqslant t$. Hence

$$x = x \wedge t = x \wedge \left(\bigvee_{i\in I} s_i\right) = \bigvee_{i\in I}(x \wedge s_i).$$

For each $i \in I$ there exists $y_i \in [0, s_i]$ such that y_i is a complement of $x \wedge s_i$ in $[0, s_i]$. We have $\{y_i\}_{i\in I} \subseteq A$, hence there exists $y \in A$ such that $y = \bigvee_{i\in I} y_i$. Then

$$x \vee y = \bigvee_{i\in I}((x \wedge s_i) \vee y_i) = t,$$

$$x \wedge y = \left(\bigvee_{i\in I}(x \wedge s_i)\right) \wedge \left(\bigvee_{j\in I} y_j\right) = \bigvee_{i\in I}\bigvee_{j\in J}((x \wedge s_i) \wedge y_j).$$

For each $i, j \in I$ we have $(x \wedge s_i) \wedge y_j = 0$. Hence $x \wedge y = 0$. Thus y is a complement of x in $[0, t]$. Therefore t is a singular element of G. $\square$

2.9. Lemma. $t = \sup S$.

Proof. By way of contradiction, suppose that $t \neq \sup S$. Hence there exists $s \in S$ such that $s \nleqq t$. Thus $s > 0$. If $t \wedge s = 0$, then $\{s_i\}_{i\in I}$ fails to be a maximal orthogonal subset of S, which is a contradiction. Hence $0 < t \wedge s < s$. Put $t \wedge s = x$. There exists a complement y of x in the interval $[0, s]$. Clearly $0 < y < s$. Then $y \in S$ and

$$x = t \wedge s = t \wedge (x \vee y) = (t \wedge x) \vee (t \wedge y) = x \vee (t \wedge y).$$

We have $x \wedge (t \wedge y) = 0$, thus $x \vee (t \wedge y) = x + (t \wedge y)$. If $t \wedge y > 0$, then

$$x \vee (t \wedge y) > x,$$

which is a contradiction. Therefore $t \wedge y = 0$ and this is impossible since $\{s_i\}_{i\in I}$ is a maximal orthogonal subset of S. $\square$

2.10. Lemma. *Let $g \in G_1$, $t < g$. Then t has no complement in the interval $[0, g]$.*

Proof. If $t = 0$, then according to the definition of G_1 we would have $G_1 = \{0\}$ and hence $g = 0$, which is a contradiction. Thus $t > 0$. Suppose that z is a complement of t in the interval $[0, g]$. Hence $0 < z < g$ and $z \wedge t = 0$. Since G_1 is singular, there exists $s \in S$ such that $0 < s \leqslant z$. Then $z \wedge t \geqslant s$, which is a contradiction. □

2.11. Corollary. *Let $(*)$ be valid. Then $t = u_1$.*

We conclude this section by recalling some notions and results on polars of lattices and of lattice ordered groups.

Let L be a lattice with the least element 0. For $X \subseteq L$ we put

$$X^{\perp} = \{y \in L \colon\ y \wedge x = 0 \quad \text{for each } x \in X\}.$$

The set $X^{\perp}$ is said to be a polar of L. The system of all polars of L will be denoted by $\mathscr{P}_1(L)$; this system is partially ordered by the set-theoretical inclusion. Then $\mathscr{P}_1(L)$ turns out to be a Boolean algebra. If L and L_1 are isomorphic lattices, then clearly $\mathscr{P}_1(L)$ and $\mathscr{P}_1(L_1)$ are isomorphic.

2.12. Lemma. *Let L be a lattice with the least element 0 and let $x \in L$. Then $\mathscr{P}_1([0, x])$ is isomorphic to the interval $[\{0\}, \{x\}^{\perp\perp}]$ of $\mathscr{P}_1(L)$.*

The proof will be omitted (it requires similar steps as in the proof of [5], 1.2).

For a subset X of a lattice ordered group G let X^{δ} be as above. Put $\mathscr{P}(G) = \{X^{\delta} \colon\ X \subseteq G\}$. The system $\mathscr{P}(G)$ partially ordered by the set-theoretical inclusion is a Boolean algebra.

3. Proof of (A)

Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be MV-algebras such that the assumptions of (A) are satisfied. Next, let G_1 and G_2 be the corresponding lattice ordered groups with strong units u_1 and u_2, respectively.

Put $\varphi_1'(x) = \varphi_1(x) - \varphi_1(0)$. Hence φ_1' is a convex isomorphism of $\mathscr{L}(\mathscr{A}_1)$ into $\mathscr{L}(\mathscr{A}_2)$ such that $\varphi_1'(0) = 0$. Thus without loss of generality we can suppose that $\varphi_1(0) = 0$. Similarly, we can suppose that $\varphi_2(0) = 0$.

Analogously to the relation (1) and (2) of Section 2 we can write

$$\begin{aligned} &G_1 = G_{11} \times G_{12}, \quad G_2 = G_{21} \times G_{22}, \\ &\mathscr{A}_1 = \mathscr{A}_{11} \times \mathscr{A}_{12}, \quad \mathscr{A}_2 = \mathscr{A}_{21} \times \mathscr{A}_{22}, \end{aligned} \tag{1}$$

where

(i) $G_{11}, G_{21}, \mathscr{A}_{11}$ and $\mathscr{A}_{21}$ are singular,

(ii) $G_{12}, G_{22}, \mathscr{A}_{12}$ and $\mathscr{A}_{22}$ have no strictly positive singular element.

Next, we have

$$\mathscr{A}_{ij} = \mathscr{A}_0(G_{ij}, u_i(G_{ij})) \quad \text{for } i, j \in \{1, 2\},$$

where the meaning of the notation $u_i(G_{ij})$ is analogous to that applied in 2.11.

We denote by S_1 and S_2 the set of all singular elements of $\mathscr{A}_1$ or $\mathscr{A}_2$, respectively. Let t_i be the greatest element of S_i $(i = 1, 2)$; such an element does exist in view of 2.9.

3.1. Lemma. *Let $i \in \{1, 2\}$. Then S_i is a complete Boolean algebra and $S_i \subseteq A_{i1}$. If $0 < x \in A_{i2}$, then the interval $[0, x]$ fails to be a Boolean algebra.*

P r o o f. The first assertion is a consequence of 2.8 and 2.9, the second follows from (ii) above. □

3.2. Lemma. *$\varphi_1(S_1)$ is a convex subset of S_2, and $\varphi_2(S_2)$ is a convex subset of S_1.*

P r o o f. This follows from 3.1 and from the fact that S_i is a convex subset of $\mathscr{A}_{i1}$ $(i = 1, 2)$. □

We shall apply the following result (cf. [10], p. 193):

(S). *Let A and B be Boolean algebras, $a \in A$, $b \in B$. If B is isomorphic to the interval $[0, a]$ of A and A is isomorphic to the interval $[0, b]$ of B, then A and B are isomorphic.*

Now, 3.2 and (S) yield

3.3. Lemma. *There exists an isomorphism φ_0 of the Boolean algebra S_1 onto the Boolean algebra S_2.*

3.4. Lemma. *Let $0 < g_1 \in G_{11}$. There exists a positive integer n and uniquely determined elements $s_0, s_1, s_2, \ldots, s_n \in S_1$ such that*

$$g_1 = \sum i s_i \quad (i = 1, 2, \ldots, n),$$
$$t_1 = \bigvee s_i \quad (i = 0, 1, 2, \ldots, n)$$

and the set $\{s_0, s_1, s_2, \ldots, s_n\}$ is orthogonal.

P r o o f. In view of the results of Section 2, t_1 is a singular element in G_{11} and, at the same time, it is a strong unit in G_{11}. The assertion now follows from [3], Theorem 3.2. □

An analogous result is valid for G_{21}. Since a lattice ordered group is determined up to isomorphism by its positive cone, from 3.3 and 3.4 we infer:

3.5. Lemma. *There exists an isomorphism φ_{01} of G_{11} onto G_{21} such that $\varphi_{01}(t_1) = t_2$.*

From the assumptions of (A) and from the conditions (i), (ii) above we obtain

3.6. Lemma. *$\varphi_1(A_{12})$ is a convex sublattice of $\mathscr{L}(\mathscr{A}_{22})$, and $\varphi_2(A_{22})$ is a convex sublattice of $\mathscr{L}(\mathscr{A}_{12})$.*

3.6.1. Lemma. (i) *There exists a convex isomorphism φ_{10} of $\mathscr{L}(G_{12})$ into $\mathscr{L}(G_{22})$ such that $\varphi_{10}(x) = \varphi_1(x)$ for each $x \in A_{12}$.*

(ii) *There exists a convex isomorphism φ_{20} of $\mathscr{L}(G_{22})$ into $\mathscr{L}(G_{12})$ such that $\varphi_{21}(y) = \varphi_2(y)$ for each $y \in A_{22}$.*

P r o o f. This is a consequence of 3.5, 3.6 and [5]. □

3.7. Lemma. *The Boolean algebras $\mathscr{P}(G_{12})$ and $\mathscr{P}(G_{22})$ are isomorphic.*

P r o o f. In view of 3.6 and 2.12, the system $\mathscr{P}_1(\mathscr{L}(\mathscr{A}_{12}))$ of polars of $\mathscr{L}(\mathscr{A}_{12})$ is isomorphic to an interval of $\mathscr{P}_1(\mathscr{L}(\mathscr{A}_{22}))$ containing the least element $\{0\}$ of $\mathscr{P}_1(\mathscr{L}(\mathscr{A}_{22}))$. Similarly, the system $\mathscr{P}_1(\mathscr{L}(\mathscr{A}_{22}))$ is isomorphic to an interval of $\mathscr{P}_1(\mathscr{L}(\mathscr{A}_{12}))$ containing the least element of $\mathscr{P}_1(\mathscr{L}(\mathscr{A}_{12}))$. Thus in view of Theorem (S) above we conclude that $\mathscr{P}_1(\mathscr{L}(\mathscr{A}_{12}))$ and $\mathscr{P}_1(\mathscr{L}(\mathscr{A}_{22}))$ are isomorphic. From this and from [5], 1.2 and 1.3 we infer that $\mathscr{P}(G_{12})$ and $\mathscr{P}(G_{22})$ are isomorphic. □

3.8. Lemma. *Both G_{12} and G_{22} are divisible.*

P r o o f. This assertion follows from the condition (ii) above and from [1], Theorem 4.9, Corollary 2. □

3.9. Lemma. *There exists an isomorphism φ_{02} of G_{12} onto G_{22} such that $\varphi_{02}(u_1(G_{12})) = u_2(G_{22})$.*

P r o o f. Both G_{12} and G_{22} are complete and have strong units $u_1(G_{12})$ and $u_2(G_{22})$, respectively. In view of 3.7 and 3.8 the assertion follows from [7], Chap. XIII, Section 3.2. □

3.10. Lemma. *There exists an isomorphism φ_{03} of G_1 onto G_2 such that $\varphi_{03}(u_1) = u_2$.*

P r o o f. This follows from (1), 3.5, 3.8 and 2.11 (according to 2.11 we have $t_i = u_i(G_{i1})$ for $i = 1, 2$). □

P r o o f o f (A). Put $\varphi = \varphi_{03}|A_1$. Then 3.10 yields that φ is an isomorphism of $\mathscr{A}_1$ onto $\mathscr{A}_2$ such that $\varphi(u_1) = u_2$. □

4. The condition $(*_1)$

The aim of the present section is to show that by a slight modification of the method of the previous section we can generalize a result from [5] concerning complete lattice ordered groups.

Let G be a lattice ordered group with a strong unit u. Consider the following condition:

$(*_1)$ If s is a singular element of G and $s \leqslant u$, then s has a relative complement in the interval $[0, s]$ of $\mathscr{L}(G)$.

In fact, if the MV-algebra $\mathscr{A} = \mathscr{A}_0(G, u)$ is taken into account, then in view of the results of Section 2 the condition $(*_1)$ is equivalent to $(*)$ (recall that $s \leqslant u$ is valid for each singular element of G).

Let S be the set of all singular elements of G. Then the relation (2) from Section 2 is valid. For $g \in G$ and $i \in \{1, 2\}$ we denote by $g(G_i)$ the component of g in G_i.

4.1. Lemma. *G satisfies the condition $(*_1)$ if and only if G_1 satisfies this condition.*

P r o o f. It suffices to apply analogous steps as in the proof of 2.4. □

In what follows we assume that G is complete. Let $\{s_i\}_{i \in I}$ be as in Section 2.

4.2. Lemma. *There exists $t \in G$ such that $t = \bigvee_{i \in I} s_i$.*

P r o o f. We have already remarked above that $s \leqslant u$ for each $s \in S$. Since G is complete, there exists $t \in [0, u]$ such that $t = \bigvee_{i \in I} s_i$. □

4.3. Lemma. *The element t is singular in G and $t = \sup S$. Moreover, $t = u(G_1)$.*

P r o o f. Cf. 2.8–2.11 (with the application of 4.1 and 4.2). □

4.4. Theorem. *Let G_1 and G_2 be complete lattice ordered groups with strong units u_1 and u_2, respectively. Assume that both G_1 and G_2 satisfy the condition $(*_1)$. Suppose that*

(i) *there exists a convex isomorphism φ_1 of $\mathscr{L}(G_1)$ into $\mathscr{L}(G_2)$;*
(ii) *there exists a convex isomorphism φ_2 of $\mathscr{L}(G_2)$ into $\mathscr{L}(G_1)$.*

Then there exists an isomorphism φ of G_1 onto G_2 such that $\varphi(u_1) = u_2$.

Proof. We proceed analogously as when proving (A) with the distinction that instead of applying the results from Section 2 we apply 4.2 and 4.3.

Let S_1 and S_2 be the set of all singular elements of G_1 or G_2, respectively. According to 4.3, S_i has a largest element; it will be denoted by t_i ($i = 1, 2$). Then 3.1–3.5 are valid.

Similarly as in 3.6.1 we have:

(i) There exists a convex isomorphism φ_{10} of $\mathscr{L}(G_{12})$ into $\mathscr{L}(G_{22})$ such that $\varphi_{10}(u_1(G_{12})) = u_2(G_{22})$.

(ii) There exists a convex isomorphism φ_{20} of $\mathscr{L}(G_{22})$ into $\mathscr{L}(G_{12})$ such that $\varphi_{20}(u_2(G_{22}) = u_1(G_{12})$.

Thus 3.8–3.10 are valid, completing the proof. □

If both G_1 and G_2 are divisible, then $S_1 = \{0\}$ and $S_2 = \{0\}$, thus they satisfy the condition $(*_1)$. Hence we have

4.5. Corollary. (Cf. [5].) *Let G_1 and G_2 be complete divisible lattice ordered groups with strong units u_1 and u_2, respectively. Suppose that the conditions* (i) *and* (ii) *from* 4.4 *are satisfied. Then there exists an isomorphism φ of G_1 onto G_2 such that $\varphi(u_1) = u_2$.*

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