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LOCALLY SYMMETRIC IMMERSIONS

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Abstract. We use reflections with respect to submanifolds and related geometric results to develop, inspired by the work of Ferus and other authors, in a unified way a local theory of extrinsic symmetric immersions and submanifolds in a general analytic Riemannian manifold and in locally symmetric spaces. In particular we treat the case of real and complex space forms and study additional relations with holomorphic and symplectic reflections when the ambient space is almost Hermitian. The global case is also taken into consideration and several examples are given.

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1. Introduction

The study of *symmetric immersions* and *submanifolds* in the framework of a study of extrinsic differential geometry has been initiated by Ferus in [5] when the ambient space is a Euclidean space. Moreover, he showed that these manifolds may be characterized as those having parallel second fundamental form and he provided a full classification. (See also [18].) Later on this notion has been extended by several authors to the case of rank one symmetric spaces (see [12] for the detailed references) and more generally to the class of compact symmetric spaces [12]. In these studies the considerations are mostly of a global nature.

Reflections with respect to a linear subspace play a basic role in the treatment of Ferus. This concept has been generalized and the notion of a *local* or *global reflection* with respect to a submanifold of an arbitrary Riemannian manifold has

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been introduced and studied intensively. See, for example, [4], [20], [22], [23] for more information and further references.

In this paper we define and treat *extrinsic locally* and *globally symmetric immersions* and *submanifolds* of a Riemannian manifold in a unified way by considering reflections with respect to submanifolds which are naturally associated to the immersion or submanifold. These reflections are called (*local*) *extrinsic symmetries*. The extrinsic symmetric immersions or submanifolds are the ones which have *isometric* extrinsic symmetries. Results from the study of the geometry related to reflections lead to a series of characterization theorems and provide examples and classifications when the ambient space is locally symmetric, in particular for real and complex space forms. A detailed account of this is given in Section 3 for the general case and in Sections 4 and 5 for the above mentioned special cases. Finally, in Section 6 we consider extrinsic symmetries for the case of almost Hermitian ambient spaces and focus our attention on the study of *holomorphic* and *symplectic* extrinsic symmetries. This leads to the notion of a (*locally*) *Hermitian symmetric immersion* or *submanifold*. Again several examples and characterizations are given.

2. Preliminaries

Let f be an isometric immersion of an n-dimensional Riemannian manifold (M, g) into an $\bar{n}$-dimensional Riemannian manifold $(\overline{M}, \bar{g})$. In what follows and if the argument is local we will sometimes identify M with its image to simplify the notation. Moreover, (M, g), $(\overline{M}, \bar{g})$ and the immersion f are supposed to be analytic and M and $\overline{M}$ connected where necessary. Further, we denote by $\overline{\nabla}$ (resp. ∇) the Levi Civita connection of $\overline{M}$ (resp. M) and the associated Riemannian curvature tensor $\overline{R}$ (resp. R) is taken with the sign convention

$$\overline{R}_{UV} = \overline{\nabla}_{[U,V]} - [\overline{\nabla}_U, \overline{\nabla}_V]$$

for all $U, V \in \mathfrak{X}(\overline{M})$, the Lie algebra of smooth vector fields on $\overline{M}$. Next, we denote by σ the second fundamental form of M, by $\nabla^\perp$ the normal connection in the normal bundle $N(M)$ of M and by $R^\perp$ its curvature tensor. The Gauss and Weingarten formulas are, respectively:

$$\overline{\nabla}_X Y = \nabla_X Y + \sigma(X, Y),$$
$$\overline{\nabla}_X U = -S_U X + \nabla^\perp_X U$$

where $X, Y \in \mathfrak{X}(M)$ and where S_U denotes the shape operator of M corresponding to the local normal vector field U. Here we have $g(S_U X, Y) = \bar{g}(\sigma(X, Y), U)$.

Further, we recall the Codazzi equation

$$(\overline{R}_{XY}Z)^{\perp} = -(\tilde{\nabla}_X\sigma)(Y,Z) + (\tilde{\nabla}_Y\sigma)(X,Z) \tag{2.1}$$

for $X, Y, Z \in \mathfrak{X}(M)$ and where $\tilde{\nabla}\sigma$ is defined by

$$(\tilde{\nabla}_X\sigma)(Y,Z) = \nabla^{\perp}_X(\sigma(Y,Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

f is said to be *totally geodesic* if $\sigma = 0$, *parallel* if $\tilde{\nabla}\sigma = 0$ and *minimal* if $\operatorname{tr}\sigma = 0$. Moreover the immersion f is called a *full* immersion if there does not exist a totally geodesic submanifold N of $\overline{M}$ with $\dim N < \dim \overline{M}$ such that $f(M) \subset N$. The *first normal space* $N^1_m M$ and the *first osculating space* $O^1_m M$ at $m \in M$ are defined by

$$N^1_m M = \operatorname{span}\{\sigma(X,Y) \mid X, Y \in T_m M\}, \quad O^1_m M = T_m M \oplus N^1_m M.$$

When f is parallel, then $\dim N^1_m M$ and $\dim O^1_m M$ are constant and hence, $N^1(M) = \bigcup_{m\in M} N^1_m M$ and $O^1(M) = \bigcup_{m\in M} O^1_m M$ are subbundles of $T\overline{M}_{|M}$, the restriction of the tangent bundle $T\overline{M}$ of $\overline{M}$ to M. Finally, f is said to be *1-full* if $O^1_m M = T_m\overline{M}$ for all $m \in M$. For a parallel immersion this is clearly so if the equality holds at one point m of M.

Now we state a lemma which will be needed later on.

Lemma 2.1. [9], [21] *Let $f\colon M \to \overline{M}$ be a parallel Kähler immersion of a Kähler manifold M into a locally Hermitian symmetric space $\overline{M}$. Then we have*

$$\begin{aligned}
&\overline{R}_m(T_mM, T_mM)T_mM \subset T_mM, && \overline{R}_m(T_mM, T_mM)N^1_mM \subset N^1_mM,\\
&\overline{R}_m(T_mM, N^1_mM)T_mM \subset N^1_mM, && \overline{R}_m(T_mM, N^1_mM)N^1_mM \subset T_mM,\\
&\overline{R}_m(N^1_mM, N^1_mM)T_mM \subset T_mM, && \overline{R}_m(N^1_mM, N^1_mM)N^1_mM \subset N^1_mM
\end{aligned}$$

for each $m \in M$.

3. Local extrinsic symmetries

Let $m \in M$ and denote by $\exp_m^M$ and $\exp_{f(m)}^{\overline{M}}$ the exponential maps of M and $\overline{M}$ at m and $f(m)$, respectively. They induce diffeomorphims of open balls $B_m = B_m(r) \subset T_mM$ and $\overline{B}_{f(m)} = \overline{B}_{f(m)}(r) \subset T_{f(m)}\overline{M}$ onto the open geodesic balls $\mathscr{U}_m \subset M$ and $\overline{\mathscr{U}}_{f(m)} \subset \overline{M}$ of radius r, respectively. Here we suppose that r is smaller than $\min\{i(m), i(f(m))\}$ where $i(m)$ (resp. $i(f(m))$ denotes the injectivity radius of M at m (resp. of $\overline{M}$ at $f(m)$). Since f is an isometry, we have $f(\mathscr{U}_m) \subset \overline{\mathscr{U}}_{f(m)}$. Next, let $B_m^{\perp} = \{u \in N_mM \mid \|u\| < r\}$ denote the $(\bar{n} - n)$-dimensional open ball of radius r in N_mM and let $\mathscr{U}_m^-$ and $\mathscr{U}_m^+$ denote the topologically embedded submanifolds in $\overline{M}$ of dimension n and $\bar{n} - n$, respectively, given by

$$\mathscr{U}_m^- = \exp_{f(m)}^{\overline{M}} f_{*m}B_m, \quad \mathscr{U}_m^+ = \exp_{f(m)}^{\overline{M}} B_m^{\perp}.$$

Clearly, $\mathscr{U}_m^- = f(\mathscr{U}_m)$ for all $m \in M$ when f is totally geodesic.

Next, let $\mathcal{T}_{\mathscr{U}_m^+}(s)$ denote the tubular neighborhood of radius s around $\mathscr{U}_m^+$, that is,

$$\mathcal{T}_{\mathscr{U}_m^+}(s) = \{\exp_{\mathscr{U}_m^+}(p, x) \mid x \in N_p\mathscr{U}_m^+,\ \|x\| < s,\ p \in \mathscr{U}_m^+\}$$

where s is supposed to be smaller than the distance from $\mathscr{U}_m^+$ to its nearest focal point and where $\exp_{\mathscr{U}_m^+}$ denotes the exponential map of the normal bundle $N(\mathscr{U}_m^+)$ of $\mathscr{U}_m^+$, that is,

$$\exp_{\mathscr{U}_m^+}(p, x) = \exp_p^{\overline{M}}(x).$$

Now, we denote by φ_m the local reflection with respect to $\mathscr{U}_m^+$ defined on $\mathcal{T}_{\mathscr{U}_m^+}(s)$ by

$$\varphi_m \colon\ q = \exp_{\mathscr{U}_m^+}(p, x) \mapsto \varphi_m(q) = \exp_{\mathscr{U}_m^+}(p, -x)$$

for all $p \in \mathscr{U}_m^+$ and all $x \in N_p\mathscr{U}_m^+$ such that $\|x\| < s$ [22]. φ_m is said to be the *local extrinsic symmetry* at m for f. It is an involutive local diffeomorphism and $\mathscr{U}_m^+$ belongs to the fixed point set of φ_m. Moreover,

$$\varphi_{m*}x = -x, \quad \varphi_{m*}u = u \tag{3.1}$$

for all tangent x and all normal u of M. For sufficiently small r the restriction of φ_m to $\mathscr{U}_m^-$ coincides with the geodesic symmetry of $\mathscr{U}_m^-$. Further, if f is totally geodesic, then $\varphi_m \circ f = f \circ s_m$ on $\mathscr{U}_m$ where s_m denotes the local geodesic symmetry of M centered at m.

In [4] a criterion is derived for isometric reflections with respect to a submanifold. By using this we obtain the following criterion for isometric local extrinsic symmetries.

Proposition 3.1. *Let $f\colon (M,g)\to(\overline{M},\bar g)$ be an isometric immersion. Then the local extrinsic symmetry φ_m at $m\in M$ for f is an isometry if and only if*

(i) *$\mathscr{U}_m^+$ is totally geodesic;*
(ii) *$(\overline{\nabla}^{2k}_{x\dots x}\overline{R})_{xy}x$ is normal to $\mathscr{U}_m^+$,*
$(\overline{\nabla}^{2k+1}_{x\dots x}\overline{R})_{xy}x$ is tangent to $\mathscr{U}_m^+$,
$(\overline{\nabla}^{2k+1}_{x\dots x}\overline{R})_{xu}x$ is normal to $\mathscr{U}_m^+$

for all $k\in\mathbb{N}$, all normal x,y and all tangent u of $\mathscr{U}_m^+$.

Further, let $\Phi\colon\ m\mapsto\Phi_m$ be the $(1,1)$-tensor field of $\overline{M}$ along the immersion which is defined by

$$\Phi_m x=-x,\quad \Phi_m u=u$$

for all tangent x and all normal u of $f(M)$ at $f(m)$. For each $m\in M$, Φ_m determines the (*local*) *Φ-rotation* ψ_m of $\overline{\mathscr{U}}_{f(m)}$ defined by

$$\psi_m=\exp^{\overline{M}}_{f(m)}\circ\Phi_m\circ(\exp^{\overline{M}}_{f(m)})^{-1}.$$

See [15]. Clearly, ψ_m fixes all points of $\mathscr{U}_m^+$. Further, we have

Lemma 3.1. *On a sufficiently small neighborhood the local extrinsic symmetry φ_m is an isometry if and only if ψ_m is an isometry. In this case we have $\varphi_m=\psi_m$.*

P r o o f. First, let φ_m be an isometry. Then we have, since φ_m fixes $f(m)$,

$$\varphi_m=\exp^{\overline{M}}_{f(m)}\circ(\varphi_{m*})_{f(m)}\circ(\exp^{\overline{M}}_{f(m)})^{-1}.$$

This and (3.1) implies $\varphi_m=\psi_m$ and so, ψ_m is an isometry.

To prove the converse, we first show that

$$(\psi_{m*})_p x=-x \tag{3.2}$$

for all $p\in\mathscr{U}_m^+$ and all $x\in N_p\mathscr{U}_m^+$. Therefore, let α be a curve in $\mathscr{U}_m^+$ from p to $f(m)$. Since ψ_m is an isometry which fixes α we have

$$(\psi_{m*})_p=P_\alpha\circ\Phi_m\circ P_\alpha^{-1}$$

where P_α denotes the parallel translation along α [6]. Then (3.2) follows since $\mathscr{U}_m^+$ is totally geodesic.

Further, since ψ_m is an isometry with $\mathscr{U}_m^+$ belonging to the fixed point set, we have

$$\psi_m=\exp_{\mathscr{U}_m^+}\circ\psi_{m*|\mathscr{U}_m^+}\circ(\exp_{\mathscr{U}_m^+})^{-1}.$$

Using this and (3.2) we see that $\psi_m=\varphi_m$ and hence φ_m is an isometry. □

Using the criterion [8, Theorem 6] we then get

Proposition 3.2. *Let $f\colon (M,g) \to (\overline{M},\bar{g})$ be an isometric immersion. Then the local extrinsic symmetry φ_m at m for f is an isometry if and only if*

$$(\overline{\nabla}^k_{x\dots x}\overline{R})_{xyxy} = (\overline{\nabla}^k_{\Phi x\dots \Phi x}\overline{R})_{\Phi x\Phi y\Phi x\Phi y}$$

for all $k \in \mathbb{N}$ and all $x, y \in T_{f(m)}\overline{M}$.

We finish this section by stating a nice geometric property for parallel immersions which plays a fundamental role in the development of the theory of symmetric submanifolds. It is based on [18, Theorem 2].

Proposition 3.3. *Let $f\colon (M,g) \to (\overline{M},\bar{g})$ be a parallel immersion. If the local extrinsic symmetry φ_m is an isometry, then we have on $\mathscr{U}_m$:*

$$\varphi_m \circ f = f \circ s_m,$$

where s_m denotes the geodesic symmetry of M at m. In particular φ_m maps $f(\mathscr{U}_m)$ onto itself.

4. Locally symmetric immersions

In the rest of this paper we shall consider isometric immersions and submanifolds equipped with isometric local extrinsic symmetries.

We start with

Proposition 4.1. *Let $f\colon (M,g) \to (\overline{M},\bar{g})$ be an isometric immersion. If each local extrinsic symmetry φ_m is an isometry, then f is a parallel immersion and (M,g) is an (intrinsic) locally symmetric space.*

Proof. Since each φ_m is an isometry, we get

$$(\tilde{\nabla}_x\sigma)(y,z) = \varphi_{m*}(\tilde{\nabla}_x\sigma)(y,z) = (\tilde{\nabla}_{\varphi_{m*}x}\sigma)(\varphi_{m*}y,\varphi_{m*}z) = -(\tilde{\nabla}_x\sigma)(y,z)$$

for all x, y, z tangent to $f(M)$ at $f(m)$. Hence $\tilde{\nabla}\sigma = 0$. The rest follows now at once from Proposition 3.3. □

This result motivates the following

Definition 4.1. An isometric immersion f of (M, g) into $(\overline{M}, \bar{g})$ is said to be (extrinsic) locally symmetric if for each $m \in M$ the local extrinsic symmetry φ_m is an isometry. If moreover, M is a submanifold of $\overline{M}$, then M is called an (extrinsic) locally symmetric submanifold.

Next, we derive some characterizations of locally symmetric immersions.

Theorem 4.1. *An isometric immersion of (M, g) into $(\overline{M}, \bar{g})$ is locally symmetric if and only if for each $m \in M$*

(i) $\mathscr{U}_m^+$ *is totally geodesic;*

(ii) $\big(\overline{\nabla}^{k+2l}_{v \overset{k}{\dots} v x \overset{2l}{\dots} x} \overline{R}\big)_{xy} x$ *is tangent,*
$\big(\overline{\nabla}^{k+2l+1}_{v \overset{k}{\dots} v x \overset{2l+1}{\dots} x} \overline{R}\big)_{xy} x$ *is normal,*
$\big(\overline{\nabla}^{k+2l+1}_{v \overset{k}{\dots} v x \overset{2l+1}{\dots} x} \overline{R}\big)_{xu} x$ *is tangent,*

for all $k, l \in \mathbb{N}$ and for all x, y tangent and all u, v normal.

P r o o f. First, let each local extrinsic symmetry φ_m be an isometry. Then $\mathscr{U}_m^+$ is totally geodesic since it belongs to the fixed point set. Further, φ_m preserves $\overline{R}$ and its covariant derivatives. Then (ii) follows by taking into account (3.1).

Next, we prove the converse. Let γ be the geodesic $s \mapsto \gamma(s) = \exp^{\overline{M}}_{f(m)}(sv)$ through $f(m)$ and tangent to a unit vector $v \in N_m M$. Further, let $\{E_j, j = 1, \dots, \bar{n}\}$ be a parallel basis along γ such that, at $f(m)$, $E_1(0), \dots, E_n(0)$ are tangent to $f(M)$ and $E_{n+1}(0), \dots, E_{\bar{n}}(0)$ are normal. For this basis and using the notation

$$\big(\overline{\nabla}^k_{a \dots a} \overline{R}\big)_{bcde} = \big(\overline{\nabla}^k_{E_a \dots E_a} \overline{R}\big)_{E_b E_c E_d E_e}$$

for $a, b, c, d, e \in \{1, \dots, \bar{n}\}$, we may write

$$\begin{aligned}\big(\overline{\nabla}^k_{a \dots a} \overline{R}\big)_{bcde}(\gamma(r)) &= \big(\overline{\nabla}^k_{a \dots a} \overline{R}\big)_{bcde}(f(m)) + r\big(\overline{\nabla}^{k+1}_{va \dots a} \overline{R}\big)_{bcde}(f(m)) \\ &\quad + \frac{1}{2} r^2 \big(\overline{\nabla}^{k+2}_{vva \dots a} \overline{R}\big)_{bcde}(f(m)) + \dots\end{aligned}$$

Then it follows from (ii) that the conditions (ii) in Proposition 3.1 are satisfied because we may, since $\mathscr{U}_m^+$ is totally geodesic, always take the basis such that $x = E_1$, $y = E_2$ and $u = E_{n+1}$ for arbitrary orthogonal tangent x, y and normal u. So, φ_m is an isometry and this completes the proof. □

From this we get at once

Corollary 4.1. *A hypersurface M of a Riemannian manifold $(\overline{M}, \bar{g})$ is (extrinsic) locally symmetric if and only if the conditions* (ii) *from Theorem* 4.1 *hold for M.*

Corollary 4.2. *Each parallel hypersurface of a locally symmetric Riemannian manifold is (extrinsic) locally symmetric.*

The next characterization follows from Proposition 3.2.

Theorem 4.2. *An isometric immersion f of (M, g) into $(\overline{M}, \bar{g})$ is locally symmetric if and only if for each $m \in M$ we have*

$$(\overline{\nabla}^k_{x\ldots x}\overline{R})_{xyxy} = (\overline{\nabla}^k_{\Phi x\ldots\Phi x}\overline{R})_{\Phi x\Phi y\Phi x\Phi y}$$

for all $k \in \mathbb{N}$ and all orthogonal $x, y \in T_{f(m)}\overline{M}$.

Up to now we considered a local treatment of extrinsic symmetry although in the literature mostly the global case is considered. Therefore, we consider now some global aspects. We start with

Definition 4.2. An isometric immersion f of (M, g) into $(\overline{M}, \bar{g})$ is said to be (extrinsic) symmetric if for every $m \in M$ there exist isometries φ_m and s_m of $\overline{M}$ and M, respectively, such that
(i) $s_m(m) = m$;
(ii) $\varphi_m \circ f = f \circ s_m$;
(iii) $\varphi_{m*}a = -a$ for $a \in T_{f(m)}f(M)$ and $\varphi_{m*}a = a$ for $a \in T^{\perp}_{f(m)}f(M)$.

In this case, φ_m is called the (*global*) *extrinsic symmetry* at m for f.

Note that s_m is the (global) intrinsic symmetry of M at m. It follows that then M is a Riemannian symmetric space. In particular, M is complete. Further, if M is an embedded submanifold, then Definition 4.2 is much simpler. In fact, then M is extrinsic symmetric if for every $m \in M$ there exists an isometry φ_m of $\overline{M}$ different from the identity, such that $\varphi_m(f(m)) = f(m)$, $\varphi_m(f(M)) = f(M)$ and (iii) holds.

From the uniqueness of isometries for initial data and taking into account Lemma 3.1, φ_m in Definition 4.2 is the extension to the whole of $\overline{M}$ of the local extrinsic symmetry at m which is an isometry. So, every extrinsic symmetry is extrinsic locally symmetric and every extrinsic symmetric submanifold is extrinsic locally symmetric. The converse does not hold in general. To see this, it is enough the note that the restriction of a locally symmetric immersion to any open submanifold induces a similar immersion. However, we have

Theorem 4.3. *Any complete locally symmetric submanifold M embedded into a simply connected symmetric space $\overline{M}$ is globally symmetric.*

P r o o f. Since $\overline{M}$ is simply connected, each local extrinsic symmetry may be extended to a global isometry which, since M is complete and taking into account Proposition 3.3, maps $f(M)$ onto itself. □

In this section we shall treat locally symmetric immersions into special Riemannian spaces $\overline{M}$. More specifically, we treat locally symmetric, real and complex space forms $\overline{M}$.

A. Locally symmetric immersions into locally symmetric spaces.

The conditions of Theorem 4.1 become much simpler for locally symmetric spaces $\overline{M}$. We have

Corollary 5.1. *An isometric immersion of a Riemannian manifold (M, g) into a locally symmetric Riemannian manifold $(\overline{M}, \bar{g})$ is locally symmetric if and only if*

(i) $\mathscr{U}_m^+$ *is totally geodesic for each $m \in M$;*

(ii) $\overline{R}_{xy}x$ *is tangent to $f(M)$ for all tangent x, y or equivalently, $f(M)$ is curvature invariant.*

Using Proposition 4.1 and the Codazzi equation (2.1) we then have

Proposition 5.1. *Let $(\overline{M}, \bar{g})$ be a locally symmetric Riemannian manifold. An isometric immersion f of (M, g) into $(\overline{M}, \bar{g})$ is locally symmetric if and only if f is parallel and $\mathscr{U}_m^+$ is totally geodesic for each $m \in M$.*

In Corollary 5.1 we may replace (ii) by another condition. Indeed, we have

Proposition 5.2. *Let $(\overline{M}, \bar{g})$ be a locally symmetric Riemannian manifold. An isometric immersion of (M, g) into $(\overline{M}, \bar{g})$ is locally symmetric if and only if $\mathscr{U}_m^+$ and $\mathscr{U}_m^-$ are totally geodesic for each $m \in M$. In this case $\mathscr{U}_m^+$ and $\mathscr{U}_m^-$ are locally symmetric submanifolds.*

P r o o f. The proof follows at once by using Lie triple systems (see [4]). □

Next, we have the following form of Theorem 4.2.

Corollary 5.2. *Let $(\overline{M}, \bar{g})$ be a locally symmetric Riemannian manifold. An isometric immersion f of (M, g) into $(\overline{M}, \bar{g})$ is locally symmetric if and only if*

(i) *f is parallel;*

(ii) $\overline{R}_{uv}u$ *is normal to $f(M)$ for all $u, v \in N(M)$.*

From this and Proposition 5.1 we get

Corollary 5.3. *Let f be a parallel immersion of (M, g) into a locally symmetric Riemannian manifold $(\overline{M}, \bar{g})$. Then $\overline{R}_{uv}u$ is normal to $f(M)$ for all $u, v \in N(M)$ if and only if, for each $m \in M$, $\mathscr{U}_m^+$ is a totally geodesic submanifold of $\overline{M}$.*

B. Locally symmetric immersions into real space forms.

In what follows we shall denote by $\overline{M}(c)$ a Riemannian manifold $(\overline{M}, \bar{g})$ of constant sectional curvature c. Then we have

$$\overline{R}_{XY}Z = c\{\bar{g}(X,Z)Y - \bar{g}(Y,Z)X\}$$

for $X, Y, Z \in \mathfrak{X}(\overline{M})$.

Using Corollary 5.1 and Corollary 5.2 we then obtain

Proposition 5.3. *Let f be an isometric immersion of (M, g) into a real space form $\overline{M}(c)$. Then the following statements are equivalent:*

(i) *f is locally symmetric;*

(ii) *f is parallel;*

(iii) *$\mathscr{U}_m^+$ is totally geodesic for all $m \in M$.*

Note that all complete parallel embedded submanifolds and hence, all symmetric submanifolds of a complete and simply connected $\overline{M}(\bar{c})$ are classified in [1] (see also [5], [19]).

Further, we have

Proposition 5.4. *A Riemannian manifold is a space of constant sectional curvature if and only if every geodesic is a locally symmetric submanifold.*

Proof. For an $M^n(c)$ the result follows from Proposition 5.3.

Conversely, let the geodesic α be a locally symmetric submanifold. Then $\mathscr{U}_m^+$ is totally geodesic. Since α is arbitrary, it follows that the axiom of $(n-1)$-planes is satisfied and hence M has constant curvature [2]. $\square$

C. Locally symmetric immersions into complex space forms.

Now, we suppose that $(\overline{M}, \bar{g}, J) = \overline{M}(h)$ is a Kähler manifold of constant holomorphic sectional curvature h. Then we have

$$\overline{R}_{XY}Z = \frac{h}{4}\Big\{\bar{g}(X,Z)Y - \bar{g}(Y,Z)X + 2\bar{g}(JX,Y)JZ + \bar{g}(JX,Z)JY - \bar{g}(JY,Z)JX\Big\}$$

for $X, Y, Z \in \mathfrak{X}(\overline{M})$ and hence

$$\overline{R}_{uv}u = \frac{h}{4}\Big\{\bar{g}(u,u)v - \bar{g}(u,v)u + 3\bar{g}(Ju,v)Ju\Big\} \tag{5.1}$$

for tangent vectors u, v.

As is well-known, there are two interesting kind of immersions into an almost Hermitian manifold, namely the holomorphic and the totally real ones. More specifically, an isometric immersion f of a Riemannian manifold (M, g) into an almost Hermitian manifold $(\overline{M}, \bar{g}, J)$ is called *holomorphic* (resp. *totally real*) if each tangent space of $f(M)$ is mapped into itself (resp. the normal space) by the almost Hermitian structure J.

These immersions will also play a fundamental role in the study of locally symmetric immersions. Indeed, it follows from [3, Proposition 3.1] that each parallel immersion into a Kähler manifold of constant holomorphic sectional curvature $h \neq 0$ is either holomorphic or totally real. This, together with Corollaries 5.2, 5.3, Theorem 4.1 and (5.1), then yields at once

Proposition 5.5. *Let f be an isometric immersion of (M, g) into a complex space form $\overline{M}(h)$, $h \neq 0$. Then the following statements are equivalent:*

(i) *f is locally symmetric;*

(ii) *f is parallel and holomorphic or totally real with $2 \dim M = \dim \overline{M}$;*

(iii) *for all $m \in M$, $\mathscr{U}_m^+$ is totally geodesic and holomorphic or totally real of dimension $\frac{1}{2} \dim \overline{M}$.*

From now on we will denote by $\overline{M}^{\bar{n}}(h)$ a complete simply connected Kähler manifold of constant holomorphic sectional curvature h and of complex dimension $\bar{n}$. For $h = 0$ this is a Euclidean space $E^{2\bar{n}}$ and since symmetric submanifolds in $E^{2\bar{n}}$ are known [5] we shall suppose further that $h \neq 0$.

Complete parallel Kähler submanifolds embedded in $\overline{M}(h)$ have been classified completely in [14] for $h > 0$ and [7] for $h < 0$. For the case of complete parallel totally real submanifolds and $h \neq 0$ we refer to [10], [11], [13]. The reduction theorem [11, Theorem 2.4] for complete parallel submanifolds in $\overline{M}(h)$, $h \neq 0$, leads to the classification of all these submanifolds and hence, also to that of the (extrinsic) symmetric submanifolds. In fact, using Proposition 5.5 we get

Proposition 5.6. *Let M be an extrinsic symmetric embedded submanifold of $\overline{M}^{\bar{n}}(h)$, $h \neq 0$. Then there exists a unique complete and totally geodesic submanifold N of $\overline{M}^{\bar{n}}(h)$ such that $M \subset N$ and $T_q N = O_q^1 M$ for any $q \in M$. Moreover, the following cases occur:*

(a) *M is a parallel Kähler submanifold and N is $\overline{M}^r(h)$ for some $r \leqslant \bar{n}$;*

(b) *M is a totally geodesic totally real submanifold and so M is $\mathbb{R}P^{\bar{n}}(\frac{h}{4})$ or $\mathbb{R}H^{\bar{n}}(\frac{h}{4})$ for $h > 0$ or $h < 0$, respectively; here $N = M$;*

(c) *M is a totally real but non-totally geodesic submanifold and $N = \overline{M}^{\bar{n}}(h)$.*

Finally, from Proposition 5.5 and using the axioms of holomorphic 2- and $(n-2)$-planes for Kähler manifolds M with $\dim M = n \geqslant 4$ [16], [24], we get

Proposition 5.7. *Let (M, g, J) be a Kähler manifold of $\dim M \geqslant 4$. Then the following statements are equivalent:*

(i) *(M, g, J) is of constant holomorphic sectional curvature;*

(ii) *for each $m \in M$ and each $u \in T_m M$ there exists an extrinsic locally symmetric surface through m and tangent to* span $\{u, Ju\}$;

(iii) *for each $m \in M$ and each $u \in T_m M$ there exists an extrinsic locally symmetric submanifold of codimension 2 through m and orthogonal to* span $\{u, Ju\}$.

6. Locally Hermitian symmetric immersions

In this last section we suppose that $(\overline{M}, \bar{g}, J)$ is an almost Hermitian manifold and f an isometric immersion of an (M, g) into $(\overline{M}, \bar{g})$. We will now focus on the holomorphic or symplectic character of the local extrinsic symmetries φ_m, $m \in M$. Here, φ_m is said to be *holomorphic* if

$$\varphi_{m*} \circ J = J \circ \varphi_{m*} \tag{6.1}$$

and *symplectic* if

$$\varphi_m^* \Omega = \Omega \tag{6.2}$$

where Ω denotes the Kähler form on $(\overline{M}, \bar{g}, J)$ defined by $\Omega(X, Y) = \bar{g}(X, JY)$ for all $X, Y \in \mathfrak{X}(\overline{M})$.

Proposition 6.1. *If the local extrinsic symmetry φ_m is holomorphic for each $m \in M$, then f and each $\mathscr{U}_m^+$ are holomorphic. Moreover, if $(\overline{M}, \bar{g}, J)$ is a Kähler manifold, then $\mathscr{U}_m^+$ is totally geodesic.*

Proof. Let $x \in T_{f(m)} f(M)$. Then

$$\varphi_{m\star} Jx = J \varphi_{m\star} x = -Jx.$$

Hence, from (3.1) it follows that Jx is tangent. In similar way it follows that $\mathscr{U}_m^+$ is holomorphic for each $m \in M$. The last part of the Proposition follows from [4, Corollary 12]. □

Proposition 6.2. *If the local extrinsic symmetry φ_m is symplectic for each $m \in M$, then f and each $\mathscr{U}_m^+$ are holomorphic. Moreover, $\mathscr{U}_m^+$ is minimal.*

Proof. Let $a, b \in T_{f(m)}\overline{M}$, $m \in M$. Then (6.2) implies

$$\bar{g}(\varphi_{m*}a, J\varphi_{m*}b) = \bar{g}(a, Jb). \tag{6.3}$$

Next, let b be normal to $f(M)$. Then $\varphi_{m*}b = b$ and so (6.3) yields that $J\varphi_{m*}a - Ja$ is tangent to $f(M)$. Hence, for a tangent, we get

$$J\varphi_{m*}a - Ja = -2Ja$$

and so, Ja is also tangent. Using a similar procedure we see that also $\mathscr{U}_m^+$ is holomorphic. The minimality of $\mathscr{U}_m^+$ follows from [20, Theorem A] □

Proposition 6.3. *Let f be a totally geodesic immersion of (M, g) into $(\overline{M}, \bar{g}, J)$. If all φ_m, $m \in M$, are holomorphic or symplectic, then $f(M)$ is a locally Hermitian symmetric space.*

Proof. From Proposition 6.1 and Proposition 6.2 it follows that f is holomorphic. Moreover, since f is totally geodesic, the intrinsic local symmetries on $f(M)$ are holomorphic or symplectic. So, the result follows from [17]. □

Next, we give

Definition 6.1. A locally symmetric immersion (resp. submanifold) in an almost Hermitian manifold is said to be a (extrinsic) locally Hermitian symmetric if all its local extrinsic symmetries are holomorphic or, equivalently, symplectic.

Note that in this case the immersion f is parallel and holomorphic. Moreover, with the induced structure (J, g), $f(M)$ is a locally Hermitian symmetric space. In particular, g is a Kähler metric.

Further, using Propositions 6.1 and 6.2 we have

Proposition 6.4. *Let $f\colon (M, g) \to (\overline{M}, \bar{g}, J)$ be an isometric immersion. Then f is locally Hermitian symmetric if and only if all its local extrinsic symmetries are holomorphic and symplectic.*

For Kähler manifolds, locally Hermitian symmetric immersions are precisely Kähler immersions which are locally symmetric. Indeed, we have

Proposition 6.5. *Let $(\overline{M}, \bar{g}, J)$ be a Kähler manifold and $f\colon (M, g) \to (\overline{M}, \bar{g}, J)$ a locally symmetric Kähler immersion. Then f is locally Hermitian symmetric.*

P r o o f. Since f is holomorphic φ_m satisfies $\varphi_{m*} \circ J = J \circ \varphi_{m*}$ at each $f(m)$. Further, for any point y in a normal neighborhood $\overline{\mathscr{U}}$ of $f(m)$ in $\overline{M}$, let α be a geodesic from $f(m)$ to y in $\overline{\mathscr{U}}$. Because φ_m is an isometry, we have $\varphi_{m*} \circ P_\alpha = P_{\varphi_m(\alpha)} \circ \varphi_{m*}$ where P_α and $P_{\varphi_m(\alpha)}$ denote the parallel translations along α and the image curve $\varphi_m(\alpha)$, respectively. Then, using the fact that J is parallel on $\overline{M}$, we conclude that

$$(\varphi_{m*})_y \circ J_y = J_{\varphi_m(y)} \circ (\varphi_{m*})_y.$$

Hence, φ_m preserves J and f is locally Hermitian symmetric. □

From this and Proposition 5.5 we obtain

Corollary 6.1. *An isometric immersion f in a Kähler manifold $\overline{M}(h)$ of constant holomorphic sectional curvature $h \neq 0$ is locally Hermitian symmetric if and only if f is a parallel Kähler immersion.*

Now, we give some results when $(\overline{M}, \bar{g}, J)$ is a locally Hermitian symmetric space. The proofs follow directly by using Propositions 6.1, 6.2 and [4, Theorem 19, Corollaries 15, 16].

Proposition 6.6. *An isometric immersion f of (M, g) into a locally Hermitian symmetric space $(\overline{M}, \bar{g}, J)$ is locally Hermitian symmetric if and only if every local extrinsic symmetry is holomorphic.*

Proposition 6.7. *An isometric immersion f of (M, g) into a locally Hermitian symmetric space $(\overline{M}, \bar{g}, J)$ is locally Hermitian symmetric if and only if for all $m \in M$, $\mathscr{U}_m^+$ is totally geodesic and φ_m is symplectic.*

Proposition 6.8. *Let $(\overline{M}, \bar{g}, J)$ be a locally Hermitian symmetric space and let $f\colon (M, g) \to (\overline{M}, \bar{g})$ be locally symmetric. Then f is locally Hermitian symmetric if and only if $\mathscr{U}_m^+$ is holomorphic.*

Further, using Lemma 2.1, Corollary 5.2 and Proposition 6.5, we get

Proposition 6.9. *A* 1*-full, parallel Kähler immersion of a Kähler manifold into a locally Hermitian symmetric space is locally Hermitian symmetric.*

Finally, based on Proposition 6.9 and [21, Corollary 4.3] we have

Proposition 6.10. *Let $(\overline{M}, \bar{g}, J)$ be a simply connected Hermitian symmetric space and f a parallel Kähler immersion of a Kähler manifold M into $\overline{M}$. Then we have:*

(i) *There exists a unique complete totally geodesic Kähler submanifold N of $\overline{M}$ such that $f(M) \subset N$ and $T_qN = O^1_qM$ for any point $q \in M$.*
(ii) *$f\colon M \to N$ is a locally Hermitian symmetric immersion.*

Remark 6.1. All 1-full immersions $f\colon M \to N$ as in Proposition 6.10 have been classified for M complete in [21]. In that paper it is proved that M and N are holomorphically isometric to Riemannian products $\mathbb{C}^m \times \overline{M}_1 \times \ldots \times \overline{M}_s \times M_1 \times \ldots \times M_t$ and $\mathbb{C}^m \times \overline{M}_1 \times \ldots \times \overline{M}_s \times \mathbb{C}P^{n_1} \times \ldots \times \mathbb{C}P^{n_t}$, respectively. Here $\mathbb{C}^m$ is an m-dimensional complex Euclidean space, $\overline{M}_i$, $1 \leqslant i \leqslant s$, and M_j, $1 \leqslant j \leqslant t$, are semisimple Hermitian symmetric spaces. The immersion f is, up to congruences, the product immersion $f = \mathrm{id}_0 \times \mathrm{id}_1 \times \ldots \times \mathrm{id}_s \times f_1 \times \ldots \times f_t$ where $f_j\colon M_j \to \mathbb{C}P^{n_j}$ are full parallel Kähler embeddings. Following [14] f_j is one of the following embeddings: a Veronese embedding of degree 2, a Segre embedding or a canonical embedding of a compact irreducible Hermitian symmetric space of rank 2.

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