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PATHS WITH RESTRICTED DEGREES OF THEIR VERTICES IN PLANAR GRAPHS

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Dedicated to Professor Miroslav Fiedler on the occasion of his seventieth birthday

Abstract. In this paper it is proved that every 3-connected planar graph contains a path on 3 vertices each of which is of degree at most 15 and a path on 4 vertices each of which has degree at most 23. Analogous results are stated for 3-connected planar graphs of minimum degree 4 and 5. Moreover, for every pair of integers $n \geqslant 3$, $k \geqslant 4$ there is a 2-connected planar graph such that every path on n vertices in it has a vertex of degree k.

1. Introduction and results

Throughout this paper, by a plane graph we understand an embedding of a connected planar graph into the sphere.

We deal only with plane graphs which have minimum degree at least 3 and which have no face with at most two edges on its boundary. We use the standard terminology and notation of the graph theory. We recall, however, more specialized notions. Let us call a path (a cycle) on n vertices to be the *n-path* (the *n-cycle*, respectively). Let $\mathscr{G}_c(n; \delta, \varrho)$ be the family of c-connected plane graphs containing a subgraph isomorphic to an n-path and having only vertices of degrees $\geqslant \delta$ and faces of degrees $\geqslant \varrho$. Note that in this case $c \leqslant 5$, $\max\{\delta, \varrho\} \leqslant 5$ and $\min\{\delta, \varrho\} = 3$.

It is an old classical consequence of the famous Euler's polyhedral formula that any plane graph contains a vertex of degree at most 5. A beautiful Kotzig's theorem [17] (see also [18, 19]), popularized in the West by Grünbaum [7, 8, 9], states that every 3-connected plane graph contains an edge with degree-sum of endvertices being at most 13 in the general case and at most 11 in the absence of degree-3 vertices. These bounds are the best possible as can be seen from the graph obtained by placing 20 or 12 small pyramids on the faces of the icosahedron or the dodecahedron graphs, respectively, as well as for infinitely many other 3-connected plane graphs.

This Kotzig's result was further developed in various directions and has served as a starting point for discovering many structural properties of plane graphs, see e.g. Borodin [1, 2, 3, 4, 5], Borodin and Sanders [6], Grünbaum [7, 8, 9], Grünbaum and Shephard [10], Horňák and Jendroľ [12], Ivančo [13], Ivančo and Jendroľ [14], Jendroľ and Skupien [15], Jucovič [16], Zaks [22].

The following problem seems to be of interest.

Problem. *What is the minimum number* $r = r_c(n; \delta, \varrho)$ *such that every graph* $G \in \mathscr{G}_c(n; \delta, \varrho)$ *contains an* n*-path with all vertices of degree* $\leqslant r$*?*

The Kotzig's results state that $r_3(2; 3, 3) = 10$, $r_3(3; 4, 3) = 7$. Recently Borodin [1] proved that $r_2(2; 3, 3) = 10$. By Lebesgue's results [20], (see also Ore [21]) we have $r_2(3; 3, 4) \leqslant 5, r_2(4; 3, 5) = 3$ and $r_2(5; 3, 5) \leqslant 5$. In the recent paper [11] the authors have proved that $r_2(4; 3, 4) = 10$.

In this paper we consider the above mentioned problem for 3-paths and 4-paths. Our main results read as follows:

Theorem 1.

(i) $$r_3(3; 3, 3) = 15,$$

(ii) $$r_3(3; 4, 3) = 9,$$

(iii) $$r_3(3; 5, 3) = 6.$$

Theorem 2.

(i) $$20 \leqslant r_3(4; 3, 3) \leqslant 23,$$

(ii) $$15 \leqslant r_3(4; 4, 3) \leqslant 17,$$

(iii) $$6 \leqslant r_3(4; 5, 3) \leqslant 7.$$

The requirement of 3-connectivity is substantial. Namely, for 2-connected graphs we have

Theorem 3. *Let* $n \geqslant 3$, $k \geqslant 3$ *be integers. There is a graph* $G \in \mathscr{G}_2(n; 3, 3)$ *such that every* n*-path in* G *contains a vertex of degree* $\geqslant k$*.*

2. Terminology and notation

The degree of a face α is the number of edges incident to α where each cut-edge is counted twice. Vertices and faces of degree i are called i-vertices and i-faces, respectively. For a plane graph G let $V(G), E(G)$ and $F(G)$ be the vertex set, the edge set and the face set of G, respectively. We use the notation $v(G), e(G)$ and $f(G)$ for the number of vertices, edges and faces of G, respectively. If G is known, these numbers are simply written as v, e and f.

By a *block decomposition* $\mathscr{R} = \{R_i, i \in J\}$ of a plane graph G we mean a family of submaps R_i of G, called the *blocks* of $\mathscr{R}$, such that

(i) each block R_i is a union of faces of G,
(ii) the union of the blocks R_i is the whole sphere,
(iii) the interiors of the blocks are pairwise disjoint, and
(iv) each block is a closed topological disk.

We define the vertices (or the edges) of a block R as those belonging to some face of R. An edge of R is *interior* if it belongs to two faces of R. The boundary of R is the set of noninterior edges in R. The subgraph induced by these edges of R forms a cycle. The vertices of R lying on the boundary cycle are called *boundary vertices*, otherwise they are *interior*.

Two edges with a common vertex A lying on the boundary of a face α define an angle at the vertex A on α. The angle at a boundary vertex A of a block R is called the *block angle of* R if it is the angle of a face belonging to the block R.

For the purposes of this paper a vertex A is said to be m-*minor* (or m-*major*) if its *degree* $d(A)$ $d(A) \leqslant m$ (or $d(A) > m$, respectively). For a graph G let $M(G)$ denote the set of all m-major vertices of G.

Denote by $I(R)$ the set of all interior vertices of R, by $F(R)$ the set of faces of R, by $a(R)$ the number of block angles of R and let $f(R) = |F(R)|$. The charge of the block R, $c(R)$, is defined by

$$c(R) = \sum_{A \in I(R)} (6 - 2d(A)) + \sum_{\alpha \in F(R)} (6 - \deg\, \alpha). \tag{1}$$

3. Proof of the upper bounds in Theorem 2

The upper bound in Theorem 2 states that every plane 3-connected graph contains a 4-path P_4 all vertices of which are m-minor with $m = 23, 17$ and 7 in the case (i), (ii) and (iii), respectively. Suppose the upper bound is not true.

Let G be a counterexample to the theorem having the minimum number of vertices, say n, among all counterexamples and the maximum number of edges among all counterexample on n vertices. Now we shall investigate properties of G.

Property 1. *Each face of G is a 3-face (i.e. G is a triangulation).*

Proof. If not, then G has a k-face α, $k \geqslant 4$. The face α contains at least one m-major vertex A and at least one vertex, say B, not adjacent with A. Inserting a new edge (diagonal) AB into α we obtain a graph G', which is also 3-connected. It has no 4-path on m-minor vertices but has one edge more than G, a contradiction. □

Property 2. *There is no separating 3-cycle in G having only m-minor vertices.*

Proof. Assume such a cycle $C = [A_0 A_1 A_2]$ does exist. Remove the interior of C from G; the resulting graph G^0 is also a triangulation without 4-paths on m-minor vertices, a contradiction because $v(G^0) < v(G)$. □

Let $\mathscr{B}(G)$ be the subgraph of G induced on the set $M(G)$ of m-major vertices of G. Simple observations yield

Property 3. *$\mathscr{B}(G)$ induces a block-decomposition $\mathscr{R}(G) = (R_i, i \in J)$ of G such that every block R_i is either a face or contains in its interior only m-minor vertices which form connected subgraphs of G. These subgraphs are either 3-cycles or stars $K_{1,s}$, $s \geqslant 0$. The latter follows by the fact that G does not contain any 4-path on m-minor vertices.*

As G is a 3-connected graph we can find out

Property 4. *G has no two adjacent 3-vertices, no 3-face $[A_0 A_1 A_2]$ with $d(A_0) = 3$, $d(A_1) = d(A_2) = 4$ and no 3-path $[A_0 A_1 A_2]$ with $d(A_0) = d(A_2) = 3$ and $d(A_1) = 4$.*

Property 5. *For a block R let the subgraph induced on the set $I(R)$ be a 3-cycle $[A_0 A_1 A_2]$ with $d(A_i) = k_i$, $i = 0, 1, 2$. Then, because of Property 3,*

$$f(R) = k_0 + k_1 + k_2 - 5, \quad c(R) = k_0 + k_1 + k_2 + 3,$$
$$a(R) = 2(k_0 + k_1 + k_2) - 15.$$

Property 6. *Let a subgraph of a block R induced on the set $I(R)$ be a star $K_{1,s}$, $s \geqslant 1$, with a central k-vertex A_0. Let $A_1, A_2, \ldots, A_s$ be m-minor neighbours of A_0 and let $B_1, B_2, \ldots, B_{k-s}$ be m-major neighbours of A_0. Let $d(A_i) = k_i$, $i = 0, 1, \ldots, s$ and $k_0 = k$; clearly $k_i \leqslant m$. Since the set $\{A_1, A_2, \ldots, A_s\}$ is independent in R we have $s \leqslant \lfloor \frac{k}{2} \rfloor$. Then for the block R we have*

$$f(R) = \sum_{i=0}^{s} k_i - 2s, \quad a(R) = 2\sum_{i=0}^{s} k_i - 6s$$

and

$$c(R) = \sum_{i=0}^{s} k_i + 6.$$

Property 7. *For a block R let $g(R) = \frac{c(R)}{a(R)}$. If R is a block of a graph $G \in \mathcal{G}_3(3;\delta,3)$ then $g(R) \leqslant \frac{7}{4}$ if $\delta = 3$, $g(R) \leqslant \frac{5}{3}$ if $\delta = 4$ and $g(R) \leqslant \frac{6}{5}$ if $\delta = 5$.*

P r o o f. If we put $x = \sum\limits_{i=0}^{s} k_i$, then $g(R)$ has one of the form

$$g(R) = \begin{cases} 1 & \text{if } R \text{ is a 3-face,} \\ \dfrac{x+3}{2x-15} \text{ and } s = 3 & \text{if } R \text{ is as in Property 5,} \\ \dfrac{x+6}{2x-6s} \text{ and } s \geqslant 0 & \text{in the other cases.} \end{cases}$$

Put $\gamma(x,s) = \frac{x+6}{2x-6s}$. If s is fixed, then the function $\gamma(x,s)$ is decreasing. In the case $\delta = 3$, taking into consideration Properties 4 and 5, we have $\gamma(x,0) \leqslant \frac{3}{2}$, $\gamma(x,1) \leqslant \frac{13}{8}$, $\gamma(x,2) \leqslant \frac{17}{10}$ and $\gamma(x,s) \leqslant \frac{5s+6}{4s} \leqslant \frac{7}{4}$ for $s \geqslant 3$. Therefore $\gamma(x,s) \leqslant \frac{7}{4}$ for all pairs (x,s).

Analogously, if $\delta = 4$ we get $\gamma(x,0) \leqslant \frac{5}{4}$, $\gamma(x,1) \leqslant \frac{7}{5}$, $\gamma(x,2) \leqslant \frac{3}{2}$ and $\gamma(x,s) \leqslant \frac{4}{3}$ for $s \geqslant 3$; hence $\gamma(x,s) \leqslant \frac{3}{2}$ for all pairs (x,s). If $\delta = 5$ we can easily calculate that $\gamma(x,0) \leqslant \frac{11}{10}$, $\gamma(x,1) \leqslant \frac{8}{7}$, $\gamma(x,2) \leqslant \frac{7}{6}$, $\gamma(x,s) \leqslant \frac{7s+6}{8s} \leqslant \frac{9}{8}$ for $s \geqslant 3$. Hence $\gamma(x,s) \leqslant \frac{7}{6}$ for all pairs (x,s) in the case $\delta = 5$.

Similarly we can easily check, using Properties 4 and 5, that the function $\frac{x+3}{2x-15}$ is bounded from above by the value $\frac{5}{3}$ if $\delta = 3$ or 4 and by $\frac{6}{5}$ if $\delta = 5$. From these observations the assertion of Property 7 follows immediately. □

Euler's formula $v - e + f = 2$ for G may be rewritten using the relations

$$2e = \sum_{A \in V(G)} d(A) = 3f$$

as

$$(2) \qquad \sum_{A\in V(G)} (6-2d(A)) + 3f = 12.$$

To get a contradiction we use in the rest of the proof the Discharging Method. We assign to each vertex A a charge $c(A) = 6 - 2d(A)$ and to each face α, a charge $c(\alpha) = 3$. These charges will now be locally redistributed, keeping their sum constant according to the following rules.

First we use

Rule 1. Each face and each m-minor vertex transfers all its charge to the block in which it is contained. (Note that every m-minor vertex is in the interior of a block).

By Property 3 using (1), (2) and Rule 1 we have

$$(3) \qquad \sum (6-2d(A)) + \sum c(R) = 12$$

where the first sum is taken over all m-major vertices and the second over all blocks of $\mathscr{R}(G)$.

Then Rule 2 follows.

Rule 2. Each block R transfers to each of its boundary vertex (which is always an m-major vertex) via each of its block angles the charge

$$g(R) = \frac{c(R)}{a(R)}.$$

New charges of the vertices and faces of G are denoted by a function h. Then, using (1), (2), (3) and Rules 1, 2 we have

$$(4) \qquad \sum_{X\in V(G)\cup F(G)} h(X) = 12.$$

The rest of our proof consists in verifying $h(X) \leqslant 0$ for each $X \in V(G) \cup F(G)$; this will yield an obvious contradiction with (4).

Because each face and each m-minor vertex transfers all its charge to the block R in which it is contained, we have $h(\alpha) = 0$ for each face α and $h(A) = 0$ for each m-minor vertex A of G. Since the block R has $a(R)$ block angles, all the charge of R, $c(R)$, is sent (by Rule 2) to the major vertices on the boundary of R. To complete the proof it is enough to verify that $h(A) \leqslant 0$ for each major vertex A of G. Each such vertex A receives an additional charge from blocks having A as a boundary vertex via block angles. Therefore

$$h(A) \leqslant 6 - 2d(A) + d(A)p(G) \tag{5}$$

where $p(G) = \max\{g(R) | R \in \mathscr{R}(G)\}$. In the case (i) or (ii) or (iii) of Theorem 2 we have $\delta = 3$, $m = 23$ or $\delta = 4$, $m = 17$ or $\delta = 5$, $m = 7$, respectively. By Property 7 in these cases $p(G) = \frac{7}{4}$ or $\frac{5}{3}$ or $\frac{6}{5}$, respectively. In all these case (5) is nonpositive.

4. Proof of the upper bounds from Theorem 1

The proof is a simple analogue of the proof of Theorem 2. We suppose that there is a counterexample G. First we show that G is a triangulation. Then we partition the vertices of G into two subsets, the set of m-minor vertices and the set of m-major vertices where $m = 15$ if $\delta = 3$, $m = 9$ if $\delta = 4$ and $m = 6$ if $\delta = 5$. The graph induced in G on the set of m-major vertices induces a block-decomposition $\mathscr{R}(G) = \{R_i, i \in J\}$ in which any block is a 3-face or a block as in Property 6 but only with one or two adjacent m-minor vertices in its interior, that is with $s = 0$ or 1. The rest of the proof uses the same idea of discharging. Then the inequality (5) is used with $p(G) = \frac{13}{8}$, $\frac{7}{5}$ or $\frac{8}{7}$ for $\delta = 3, 4$ or 5, respectively.

5. Proof of lower bounds

In Theorem 1(i). If we place the configuration of Fig. 1 into every face of the dodecahedron graph (dashed lines indicate the dodecahedron edges) we obtain a graph in which every 3-path contains an x-vertex, $x \geqslant 15$. It is easy to see that this graph is 3-connected.

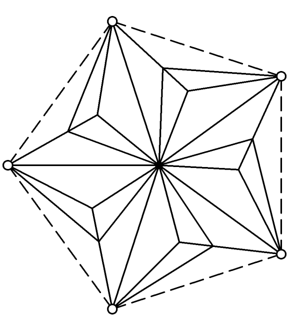

Fig. 1

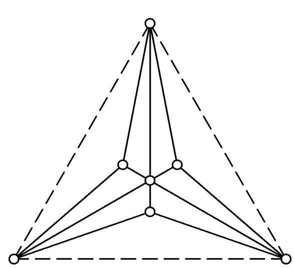

Fig. 2

In Theorem 2(i). If we insert into each 3-face of the icosahedron graph the configuration in Fig. 2, we obtain a 3-connected graph with the property that every 4-path in this graph contains a 20-vertex.

In Theorem 1(ii). Every face α of the rhombic triacontahedron in Fig. 3a is a 4-face $[A, B, C, D]$ with $d(A) = d(C) = 5$, $d(B) = d(D) = 3$. If we insert a configuration of Fig. 3b into each face α of the graph so that the vertex X of Fig. 3b is placed into the vertex X of α, $X \in \{A, B, C, D\}$, we obtain a graph in which every 3-path contains an x-vertex, $x \geqslant 9$.

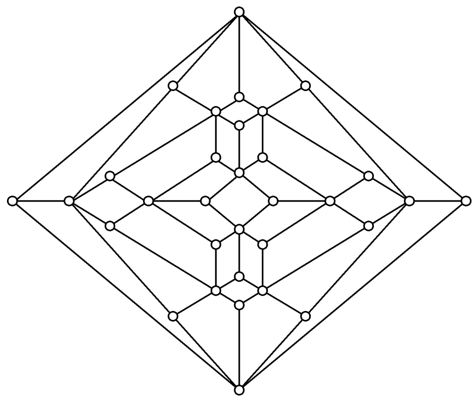

Fig. 3a

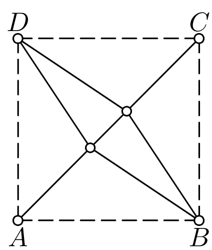

Fig. 3b

In Theorem 2(ii). Analogously as above we insert into each 3-face of the icosahedron graph the configuration in Fig. 4. We obtain a 3-connected graph of minimum degree 4 in which every 4-path has a 15-vertex.

In Theorems 1(iii) and 2(iii). Placing 12 small pyramids on the faces of the dodecahedron graph we obtain a triangulation which has the property that every 3-path and every 4-path contains a 6-vertex and each vertex of this graph has degree at least 5.

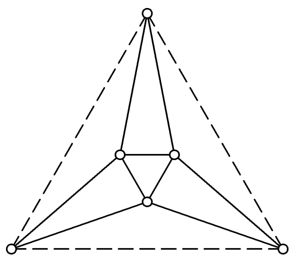

Fig. 4

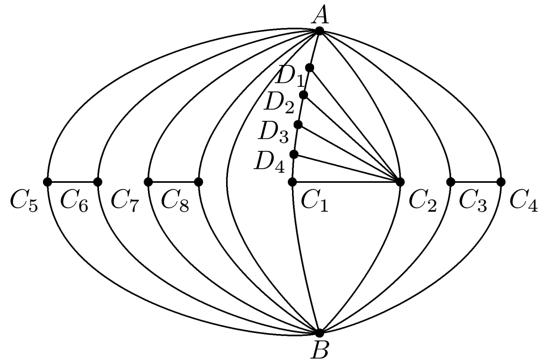

Fig. 5

6. Proof of Theorem 3

For every k, $k \geqslant 3$, we have to find a 2-connected plane graph $G(n,k)$ with the property that every n-path, $n \geqslant 3$, in it contains a k-vertex. To do it consider a plane multigraph consisting of two k-vertices A and B joined by k parallel edges $e_1, e_2, \ldots, e_k$. For any i, $i = 1, \ldots, 2\lfloor \frac{k}{2} \rfloor$, we insert a new vertex C_i on the edge e_i. Then we add a new edge $C_{2j-1}C_{2j}$ for any j, $j = 1, 2, \ldots, \lfloor \frac{k}{2} \rfloor$. If we insert $n-3$ new vertices $D_1, D_2, \ldots, D_{n-3}$ into the edge AC_1 and join them with the vertex C_2 we obtain a required graph $G(n,k)$. In Fig. 5 there is a graph $G(7,9)$. It is easy to check that the graph obtained has the required property.

7. Remark

The method used in the proof of Theorem 2 (Section 3) can be used for finding results also for cases $r_3(n;\delta,\varrho)$ with $n \geqslant 5$. Together with my student I. Fabrici, using the same methods, we have proved

Theorem 3.

$$24 \leqslant r_3(5;3,3) \leqslant 27,$$
$$15 \leqslant r_3(5;4,3) \leqslant 19,$$
$$6 \leqslant r_3(5;5,3) \leqslant 7.$$

The problem in proving the results for $n \geqslant 5$ consists in the fact that more types of blocks must be investigated than in the proof of Theorem 2.

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