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ALGEBRAIC DUALITY OF CONSTANT ALGEBRAS

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Let p, q be terms of the same similarity type. An identity $p = q$ is called *normal* if it is either of the form $x = x$ or none of p, q is equal to a single variable. For a variety $\mathscr{V}$, denote by $\operatorname{Id}\mathscr{V}$ or $\operatorname{Id}_N \mathscr{V}$ the set of all identities or the set of all normal identities of $\mathscr{V}$, respectively. Of course, $\operatorname{Id}_N \mathscr{V} \subseteq \operatorname{Id}\mathscr{V}$ and hence $\mathscr{V}$ is a subvariety of the variety $\mathscr{N}(\mathscr{V})$ defined by $\operatorname{Id}_N \mathscr{V}$. $\mathscr{V}$ is called *normally presented*, [1] if $\mathscr{V} = \mathscr{N}(\mathscr{V})$. If $\mathscr{V} \neq \mathscr{N}(\mathscr{V})$ then $\mathscr{N}(\mathscr{V})$ covers $\mathscr{V}$ in the lattice of all varieties of a given type, see [3], [5].

An algebra $\mathscr{A} = (A, F)$ is called a *constant algebra* if there exists an element $0 \in A$ (the so called *constant* of $\mathscr{A}$) such that

$$f(a_1, \ldots, a_n) = 0$$

for each n-ary $f \in F$ and all $a_1, \ldots, a_n \in A$. It is an easy exercise to show that for a given similarity type σ, a constant algebra $\mathscr{A}$ of type σ satisfies exactly all normal identities of type σ, i.e. if p, q are two arbitrary terms of type σ not identically equal to a single variable then $p = q$ holds in $\mathscr{A}$. So the class $\mathscr{C}$ of all constant algebras of type σ forms a variety. As mentioned previously, $\mathscr{C}$ is the least normally presented variety of type σ and it covers the trivial variety $\mathscr{T}$ in the lattice of all varieties of type σ. Hence, we have (see also [1])

Lemma 1. *Let σ be a given similarity type.*

(i) *The class $\mathscr{C}$ of all constant algebras of type σ is a variety and $\operatorname{Id}\mathscr{C}$ is the set of all normal identities of type σ;*

(ii) *$\mathscr{C} = \mathscr{N}(\mathscr{T})$ and it is the atom of the lattice of all varieties of type σ.*

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Given a similarity type σ, denote by $\underline{B}$ the two-element constant algebra of type σ. The elements of $\underline{B}$ will be denoted by 0_B and b throughout the paper, 0_B is the constant of $\underline{B}$. Let us note that if σ contains a nullary operation c then $c = 0_B$ since $f(a_1, \ldots, a_n) = 0_B$, thus $c = 0_B$ is a normal identity of type σ. If $\mathscr{A} = (A, F)$ is a constant algebra of type σ, we denote by 0_A its constant. Hence, 0_A can be considered as a nullary term operation of type σ (it is determined by the identity

$$f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$$

where $f \in \sigma$ is an n-ary operation of $\mathscr{A}$).

Lemma 2. *Let $\mathscr{C}$ be the variety of all constant algebras of type σ. Then*
(i) *$\underline{B}$ is the only subdirectly irreducible member of $\mathscr{C}$;*
(ii) *$\mathscr{C} = \mathbb{ISP}(\underline{B})$.*

Proof. It is easy to check that $\operatorname{Con} \mathscr{A} = \operatorname{Eq} A$ for each $\mathscr{A} \in \mathscr{C}$, where $\operatorname{Eq} A$ denotes the lattice of all equivalences on the set A. Hence, $\mathscr{A} \in \mathscr{C}$ is subdirectly irreducible if and only if $\operatorname{Eq} A$ has exactly one atom. This is the case if $\operatorname{card} A = 2$, i.e. if $\mathscr{A} = \underline{B}$. Since every algebra $\mathscr{A} \in \mathscr{C}$ is a subdirect product of $\underline{B}$, we conclude $\mathscr{C} = \mathbb{ISP}(\underline{B})$. □

Let us recall a necessary concept of algebraic duality in the sense of B. Davey [2]. Let $\mathscr{V} = \mathbb{ISP}(\underline{P})$ where $\underline{P} = (P, F)$ and $\underset{\sim}{P} = (P; G, H, R, \tau)$ where:

- G is a set of finitary operations on P such that if $g \in G$ is nullary then $\{g\}$ is a subalgebra of $\underline{P}$ and if g is n-ary for $n \geqslant 1$ then $g\colon\ \underline{P}^n \to \underline{P}$ is a homomorphism;
- H is a set of partial operations on P of arity at least 1 such that if $h \in H$ is n-ary then its domain $\operatorname{dom} h$ is a subalgebra of $\underline{P}^n$ and $h\colon \operatorname{dom} h \to \underline{P}$ is a homomorphism;
- R is a set of finitary relations on P such that if $r \in R$ is n-ary then r is a subalgebra of $\underline{P}^n$;
- τ is the discrete topology on P.

In this case we say that $\underset{\sim}{P}$ is *algebraic* over $\underline{P}$.

Let $\mathscr{W} = \mathbb{IS}_C\mathbb{P}(\underset{\sim}{P})$ be the class of all topological structures of the same type as $\underset{\sim}{P}$ which are isomorphic (i.e. simultaneously isomorphic and homeomorphic) to a closed substructure of a power of $\underset{\sim}{P}$ (with the product topology). If $X, Y \in \mathscr{W}$ then a continuous map $\varphi\colon\ X \to Y$ preserving G, H, R is called a *morphism*.

Denote by $\operatorname{Hom}(X, Y)$ the set of all morphisms from X to Y. If $\mathscr{A}, \mathscr{B} \in \mathscr{V}$, denote by $\operatorname{Hom}(\mathscr{A}, \mathscr{B})$ the set of all homomorphisms from $\mathscr{A}$ into $\mathscr{B}$. For $\mathscr{A} \in \mathscr{V}$ define its *dual* $D(\mathscr{A}) = \operatorname{Hom}(\mathscr{A}, \underline{P})$ endowed by G, H, R, τ as given before such that $(D(\mathscr{A}); G, H, R, \tau)$ is a closed substructure of a direct power $\underline{P}^A$, and hence $D(\mathscr{A}) \in \mathscr{W}$, see e.g. Lemma 1.1. in [2]. For $\mathscr{A} \in \mathscr{V}$, $E(D(\mathscr{A}))$ is denoted briefly by $ED(\mathscr{A})$ and called the *second dual* of $\mathscr{A}$.

The maps $e_a\colon D(\mathscr{A}) \to \underset{\sim}{P}$ given by

$$e_a(x) = x(a) \quad \text{for each} \quad a \in A \quad \text{and every} \quad x \in D(\mathscr{A})$$

and $\varepsilon_x\colon E(X) \to \underline{P}$ given by

$$\varepsilon_x(\alpha) = \alpha(x) \quad \text{for each} \quad x \in X \quad \text{and every} \quad \alpha \in E(X)$$

are called *evaluation maps.*

Now, we are able to give the following concept of [2]:

Definition. If for $\mathscr{V} = \mathbb{ISP}(\underline{P})$ there exists $\underset{\sim}{P}$ algebraic over $\underline{P}$ such that the evaluation maps e_a $(a \in A)$ are the only morphisms of $D(\mathscr{A})$ into $\underset{\sim}{P}$ for each $\mathscr{A} \in \mathscr{V}$, we say that $\underset{\sim}{P}$ *yields a duality* on $\mathscr{V}$. If, moreover, the evaluation maps ε_x are the only homomorphisms of $E(X)$ into $\underline{P}$ for each $X \in \mathbb{IS}_C\mathbb{P}(\underset{\sim}{P})$ then the duality is called *full.*

The aim of our paper is to determine $\underset{\sim}{B}$ which yields a duality for the variety $\mathscr{C}$ of all constant algebras of type σ.

Lemma 3. *Let $\mathscr{C}$ be the variety of all constant algebras of type σ and $\mathscr{A} \in \mathscr{C}$. A mapping $h\colon A \to B$ is a homomorphism of $\mathscr{A}$ into $\underline{B}$ if and only if $h(0_A) = 0_B$.*

The proof is evident. □

Let us introduce two binary and one nullary operations on the support $\{0_B, b\}$ of the algebra $\underline{B}$ as follows:

- 0_B is the nullary operation;
- the binary operations $\vee$, $\wedge$ are given by setting

$$0_B \vee b = b \vee 0_B = b \vee b = b \wedge b = b,$$
$$0_B \vee b = b \vee 0_B = 0_B \wedge 0_B = 0_B \vee 0_B = 0_B.$$

Let $\mathscr{L} = (L, \wedge, \vee, 0)$ be a relatively complemented lattice. Introduce a new binary operation $p(x, y)$ on L, where for $a, b \in L$, $p(a, b)$ is the relative complement of a in the interval $[0, a \vee b]$. Let τ be the discrete topology on $\{0_B, b\}$.

Lemma 4. $\underset{\sim}{B} = (\{0_B, b\}; \vee, \wedge, p(x, y), 0_B, \tau)$ *is a topological Boolean lattice with the least element 0_B and it is algebraic over $\underline{B}$.*

P r o o f. Evidently, $\{0_B\}$ is a subalgebra of $\underline{B}$. Moreover, for each n-ary $f \in \sigma$ and arbitrary $a_1, \ldots, a_n, c_1, \ldots, c_n \in \{0_B, b\}$ we have

$$f(a_1, \ldots, a_n) \vee f(c_1, \ldots, c_n) = 0_B \vee 0_B = 0_B = f(a_1 \vee c_1, \ldots, a_n \vee c_n),$$
$$p(f(a_1, \ldots, a_n), f(c_1, \ldots, c_n)) = p(0_B, 0_B) = 0_B = f(p(a_1, c_1), \ldots, p(a_n, c_n)),$$

analogously for $\wedge$, thus $\underset{\sim}{B}$ is algebraic over $\underline{B}$. □

Theorem 1. *For each $\mathscr{A} \in \mathscr{C}$ its dual $D(\mathscr{A})$ is an atomic Boolean lattice. Moreover, there is a bijection between the set of all atoms of $D(\mathscr{A})$ and the set of all non-zero elements of $\mathscr{A}$.*

P r o o f. $D(\mathscr{A})$ is a closed substructure of the direct power $\underset{\sim}{B}^A$. By Lemma 4, $D(\mathscr{A})$ is a Boolean lattice. Further, for each $a \in A$, $a \neq 0_A$ we define a mapping $h_a \colon A \to B$ by setting

$$h_a(a) = b \quad \text{and} \quad h_a(c) = 0_B \quad \text{for} \quad c \neq a.$$

By Lemma 3, $h_a \in D(\mathscr{A})$ for each $a \neq 0_A$. Of course, h_a is an atom of $D(\mathscr{A})$ and for $a_1 \neq a_2$ also $h_{a_1} \neq h_{a_2}$, thus the mapping $a \mapsto h_a$ is a bijection of $A \setminus \{0\}$ onto the set of all atoms of $D(\mathscr{A})$. Evidently, for each $h \in D(\mathscr{A})$ we have

$$h = \vee\{h_a; a \in A \text{ and } h(a) \neq 0_B\},$$

so every element of $D(\mathscr{A})$ is a join of atoms. □

Corollary. *The class $\mathscr{W} = \mathbb{I}\mathbb{S}_C\mathbb{P}(\underset{\sim}{B})$ is the class of all atomic Boolean lattices with the least element endowed with the product topology.*

P r o o f. Of course, every $\mathscr{L} \in \mathscr{W}$ is a Boolean lattice. However, the product topology is compact and, by using Lemma 4.1. in [6], $\mathscr{L}$ is atomic. □

For a lattice L, we denote by $\mathrm{At}(L)$ the set of all atoms of L.

Lemma 5. *Let $L \in \mathscr{W}$. Then there exists a one-to-one correspondence between the set $\{0\} \cup \mathrm{At}(L)$ and $E(L) = \mathrm{Hom}(L, \underset{\sim}{B})$.*

P r o o f. Since $L \in \mathscr{W}$, it can be considered as a sublattice of the lattice $\underset{\sim}{B}^{\mathrm{At}(L)}$ (where the space $\underset{\sim}{B}^{\mathrm{At}(L)}$ is endowed with the product topology). Every nonzero homomorphism of L onto $\underset{\sim}{B}$ is induced by an ultrafilter, i.e. for each $f \in E(L)$ there exists an ultrafilter U of L with $f(U) = \{b\}$ and $f(L \setminus U) = \{0_B\}$. Prove that U is principal.

Suppose that U is not principal. Let $\mathscr{S}$ be a basis of the product topology on $\underset{\sim}{B}^{\mathrm{At}(L)}$. Since f is non-zero, there exists $h \in U$ with $f(h) = b$. For every neighbourhood $V \in \mathscr{S}$ of h there exists a *finite* subset $T_V \subseteq \mathrm{At}(L)$ and a function $\psi_V \colon T_V \to \{0_B, b\}$ such that for any $\varphi \in L$ the conditions $\varphi \in V$ and $\varphi|_{T_V} = \psi_V$ are equivalent. Consider an element $h_{T_V} \in L$ such that $h_{T_V}|_{T_V} = \psi_V$ and $h_{T_V}(\mathrm{At}(L) \setminus T_V) = \{0_B\}$. Since $h_{T_V}|_{T_V} = \psi_V$, we have $h_{T_V} \in V$. We show that $h_{T_V} \notin U$. If $\psi_V = 0_B$ for each $x \in T_V$ then $h_{T_V} = 0 \notin U$, a contradiction. Thus $X = \{x \in T_V; \psi(x) = b\} \neq \emptyset$. Consider the elements y_x of L defined as follows:

$$y_x|_{\{x\}} = b \quad \text{and} \quad y_x|_{(\mathrm{At}(L)\setminus\{x\})} = 0_B$$

for each $x \in X$. Evidently, $\{y_x; x \in X\} \subseteq \mathrm{At}(L)$ and $h_{T_V} = \vee\{y_x; x \in X\}$. Since X is finite and U is an ultrafilter, we conclude $y_x \in U$ for some $x \in X$. Hence, U contains an atom y_x, i.e. U is principal, a contradiction. We have shown that for every neighbourhood $V \in \mathscr{S}$ and each $h \in V$ there exists $h_{T_V} \in V$ such that $h_{T_V} \notin U$. However, every neighbourhood W of an element h is the union of elements of the basis $\mathscr{S}$, thus the same is valid also for W.

Evidently, $f^{-1}(0_B)$ is closed and $h \notin f^{-1}(0_B)$. As was shown, h is a limit point of $f^{-1}(0_B)$, thus $h \in f^{-1}(0_B)$, a contradiction. Thus U is a principal ultrafilter.

However, every principal ultrafilter in an atomic Boolean lattice is generated by an atom, i.e. the continuous non-zero homomorphisms of L onto $\underset{\sim}{B}$ are in a one-to-one correspondence with $\mathrm{At}(L)$. If we add 0 to $\mathrm{At}(L)$ and the zero homomorphism, we are done. $\square$

Theorem 2. *The structure $\underset{\sim}{B} = (\{0_b, b\}; \vee, \wedge, p(x,y), 0_B, \tau)$ yields a full duality of the variety $\mathscr{C}$ of all constant algebras of a given type.*

P r o o f. Of course, for each $\mathscr{A} \in \mathscr{C}$ and every $a \in A$, the evaluation map $e_a \in ED(\mathscr{A})$. Since $e_a \neq e_b$ for $a \neq b$, $a, b \in A$, the set of all evaluation maps has the same cardinality as the algebra $\mathscr{A}$. By Theorem 1, $D(\mathscr{A})$ is an atomic Boolean lattice and the set of all atoms of $D(\mathscr{A})$ corresponds to the set of all non-zero elements of $\mathscr{A}$. Finally, Lemma 5 yields that there is a bijection between $\mathscr{A}$ and $ED(\mathscr{A})$. $\square$

Remark 1. The variety $\mathscr{C}$ is equivalent to the category of sets with one fixed element. Hence $\mathscr{C}$ does not essentially depend on the given similarity type σ. Indeed, we can consider σ containing just one operation which is nullary. Then $\underline{B}$ is the algebra $\underline{B} = (\{0_B, b\}; 0_B)$. It is well known that for a Boolean algebra $\underline{P} = (\{0,1\}, \vee, \wedge, ', 0, 1)$ the structure $\underset{\sim}{P} = (\{0,1\}, \emptyset, \tau)$ yields a duality (which is the well-known Stone duality). If $\underline{P} = (\{0,1\}, \vee, \wedge, 0, 1)$ is a bounded distributive lattice then $\underset{\sim}{P} = (\{0,1\}, \leqslant, \tau)$ yields a duality (which is the well-known Priestly duality). Recently it was shown by M. Haviar [4] that for $\underline{P} = (\{0,1\}, \leqslant)$ the structure

$\underset{\sim}{P} = (\{0,1\}, \vee, \wedge, 0, 1, \tau)$ yields a duality, i.e. the role of topology in $\underline{P}$ and $\underset{\sim}{P}$ is transfered to the other side of Priestley duality. From this point of view, the duality of constant algebras derived here can be also viewed as the transfering of the topology in $\underline{P}$, $\underset{\sim}{P}$ to the other side of the Stone duality.

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