Michael M. Parmenter

A remark on k-systems in groups

Czechoslovak Mathematical Journal, Vol. 49 (1999), No. 2, 347–349

Persistent URL: `http://dml.cz/dmlcz/127492`

Czechoslovak Mathematical Journal, 49 (124) (1999), 347–349

A REMARK ON k-SYSTEMS IN GROUPS

M. M. Parmenter,[1] St. John's

(Received August 15, 1996)

If $(G,+)$ is a uniquely 2-divisible Abelian group and * is the usual arithmetic mean value, then $(G,+,*)$ satisfies the identity $x+(y*z)=(x*y)+(x*z)$. Conversely, Kepka and Niemenmaa showed in [3] that if $(G,+)$ is any group supporting a binary operation * which satisfies this identity, then $(G,+)$ is Abelian and 2-divisible. However, G need not be uniquely 2-divisible. To see this, let $Q/\mathbb{Z}$ be the additive group of rational numbers modulo 1 and, for $0<a,b<1$, define $0*0=0$, $a*0=0*a=\frac{a+1}{2}$ and $a*b=\frac{a+b}{2}$ where $\frac{a+1}{2}$ and $\frac{a+b}{2}$ are computed by viewing a,b as elements of Q.

In [3], results are also obtained where a more general identity $x+k(y*z)=(x*y)+(x*z)$ is assumed ($k\in\mathbb{Z}$). Such a system $(G,+,*)$ is called a k-system.

In this brief note, we are interested in determining what additional equations are needed in $(G,+,*)$ to completely characterize the usual arithmetic mean value. Note that if $(G,+)$ is Abelian and uniquely 2-divisible, and * is the mean value, then $x+(y*z)=(x+y)*(x+z)$ also holds. We will show that this identity, together with the earlier one, completes the required characterization. In fact this result holds for all k-systems, and that will be our main result (Theorem 1).

Jakubík [2] also investigated the second identity stated above in a group theoretic setting, while in [1], Gardner and Parmenter (unaware of [2]) studied different aspects of a very similar structure.

Theorem 1. *Let $(G,+,*)$ be a k-system such that for all $x,y,z\in G$, $x+k(y*z)=(x+y)*(x+z)$. Then $(G,+)$ is Abelian and one of the following must occur.*

(i) $|G|=1$.

(ii) $k=1$ *and G is uniquely 2-divisible.*

(iii) $k\neq 1$ *and G is of finite odd exponent dividing $k-1$.*

In all cases, $*$ is the usual arithmetic mean value on G.

[1] This work was supported in part by NSERC grant A8775.

P r o o f. First note that if $k = 0$, then setting $y = x = z$ in the k-system identity gives $x = 2(x * x)$ while setting $y = z = 0$ in the second identity gives $x = (x * x)$. This forces $|G| = 1$, so we assume from now on that $k \neq 0$.

We proceed to make a few basic observations. Setting $x = y = z = 0$ in both identities gives $2(0 * 0) = (0 * 0)$, so $0 * 0 = 0$. Then setting $y = z = 0$ in the second identity gives $x + k(0 * 0) = x * x$, so we have that for all x in G,

$$(x * x) = x.$$

Now putting $y = x = z$ in the first identity gives $x + k(x * x) = 2(x * x)$, so for all x in G,

$$(k - 1)x = 0.$$

If $k \neq 1$, we can now conclude that the exponent of G is finite and divides $k - 1$. Also, we can assume from now on that $(x * y) + (x * z) = x + (y * z) = (x + y) * (x + z)$ for all x, y, z in G.

The next part of the argument follows steps similar to those seen in [3], but we include them for completeness.

Putting $y = z = 0$, we obtain $x = 2(x * 0)$ for all x in G. Note that if $k \neq 1$, we have now proved that the exponent of G is odd (and hence, as remarked in Lemma 1.1 of [3], that G is uniquely 2-divisible).

Next observe that $x + (0 * x) = (x * 0) + (x * x) = (x * 0) + x$ by above. Since $x = 2(x * 0)$, we conclude that $(x * 0) = (0 * x)$ for all x in G. Hence $(x * 0) + (x * y) = x + (0 * y) = x + (y * 0) = (x * y) + (x * 0)$. Thus for all x, y in G,

$$\begin{aligned}
(x * 0) + y &= (x * 0) + (x + (y - x)) \\
&= (x * 0) + (x + (y - x) * (y - x)) \\
&= (x * 0) + 2(x * (y - x)) \\
&= 2(x * (y - x)) + (x * 0) \\
&= y + (x * 0).
\end{aligned}$$

Thus $(x * 0)$ is in the centre of G, and hence $x = 2(x * 0)$ is in the centre of G. We have shown that $(G, +)$ is Abelian.

To show that G is uniquely 2-divisible when $k = 1$, we now only need prove that G has no elements of order 2. So assume that $2x = 0$ for some x in G. Then $(x * 0) + (x * x) = x + (0 * x) = x * (2x) = x * 0$. Hence $x * x = 0$, forcing $x = 0$ and we're done.

Finally, note that setting $y = z$ gives $x + (y * y) = 2(x * y)$, i.e. $x + y = 2(x * y)$. Since G is Abelian and uniquely 2-divisible, $*$ is the usual mean value on G. □

References

[1] *B.J. Gardner and M.M. Parmenter*: Directoids and directoid groups. Algebra Universalis *33* (1995), 254–273.
[2] *Ján Jakubík*: On convexities of d-groups. Czechoslovak Math. J. *44* (1994), 305–314.
[3] *T. Kepka and M. Niemenmaa*: On splitting k-systems. Acta Universitatis Carolinae *27* (1986), 3–10.

Author's address: Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada, A1C 5S7.