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WEAK ORTHOGONALITY AND WEAK PROPERTY (β) IN SOME BANACH SEQUENCE SPACES

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Abstract. It is proved that a Köthe sequence space is weakly orthogonal if and only if it is order continuous. Criteria for weak property (β) in Orlicz sequence spaces in the case of the Luxemburg norm as well as the Orlicz norm are given.

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1. Introduction

Let $(X, \|\cdot\|)$ be a real Banach space and $B(X)$ ($S(X)$) the closed unit ball (the unit sphere) of X, respectively. For any subset A of X, by $\operatorname{conv}(A)$ ($\overline{\operatorname{conv}}(A)$) we denote the convex hull (the closed convex hull) of A. Denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of natural and real numbers, respectively.

Rolewicz [18] introduced the notion of property (β), which can be formulated equivalently as follows:

for every $\varepsilon > 0$ there exists $\delta \in (0,1)$ such that for each element $x \in B(X)$ and each sequence (x_n) in $B(X)$ with $\operatorname{sep}(x_n) \geqslant \varepsilon$ there is an index k for which

$$\left\| \frac{x + x_k}{2} \right\| \leqslant 1 - \delta,$$

where $\operatorname{sep}(x_n) = \inf\{\|x_n - x_m\| : n \neq m\}$ (see [12]).

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We say that a Banach space X has the *weak property* (β) if there is a number $\delta > 0$ such that for any $x \in S(X)$ and any weakly null sequence (x_n) in $B(X)$ there exists $k \in \mathbb{N}$ such that

$$\left\| \frac{x + x_k}{2} \right\| \leqslant 1 - \delta.$$

Let us say that a Banach space X has the *weak Banach-Saks property* whenever, given (x_n) in X such that $x_n \to 0$ weakly, there exists a subsequence (x_{n_k}) of (x_n) such that

$$\sum_{k=1}^{j} \frac{x_{n_k}}{j} \longrightarrow 0$$

in norm.

A Banach space X is said to be *weakly orthogonal* if every weakly null sequence (x_n) in X satisfies

$$\lim_{n\to\infty} \big| \|x_n + x\| - \|x_n - x\| \big| = 0$$

for any $x \in S(X)$.

Recall that the *characteristic of convexity* is the infimum of those $\varepsilon \in (0,2]$ that $\delta_X(\varepsilon) > 0$. Here $\delta_X(\varepsilon)$ denotes the modulus of convexity of X (see [2] and [14]).

Falset [3] showed that if X is weakly orthogonal and its characteristic of convexity is strongly less than 2 (i.e. X is uniformly nonsquare), then X has the fixed point property.

Kottman [10] defined for any Banach space X its *packing constant* $\Lambda(X)$ by

$$\Lambda(X) = \sup\Big\{ r > 0\colon\ \exists (x_n) \subset B(X) \text{ s.t. } \|x_m - x_n\| \geqslant 2r \text{ for } m \neq n \text{ and } \bigcup_{n=1}^{\infty} B_X(x_n, r) \subset B(X) \Big\}$$

under the convention $\sup\{\emptyset\} = 0$, where $B_X(x_n, r) = \{y \in X\colon\ \|x_n - y\| \leqslant r\}$. He also showed that

$$\Lambda(X) = \frac{D(X)}{2 + D(X)},$$

where

$$D(X) = \sup_{(x_n)\subset S(X)} \inf_{m\neq n} \|x_m - x_n\|.$$

Let l^0 be the space of all real sequences. A Banach space $(X, \|\cdot\|)$ is said to be a *Köthe sequence space* (or a *Banach sequence lattice*) if X is a subspace of l^0 that contains an element x with $x(i) \neq 0$ for all $i \in \mathbb{N}$ and it is an ideal, i.e. if $x \in X$,

$y \in l^0$ and $|y(i)| \leqslant |x(i)|$ for every $i \in \mathbb{N}$, then $y \in X$ and $\|y\| \leqslant \|x\|$ (see [9] and [14]).

Recall that an element x of a Köthe sequence space X is said to be *order continuous* if for any sequence (x_n) in X such that $0 \swarrow x_n \leqslant |x|$, we have $\|x_n\| \to 0$.

It is easy to see that an element x of a Köthe sequence space X is order continuous iff

$$\tau(x) = \lim_{n\to\infty} \left\| \sum_{i=n}^{\infty} x(i)e_i \right\| = 0.$$

Denote by X_a the set of all order continuous elements of X. If $X = X_a$, we say that X is *order continuous* (**OC** for short), (see [9] and [14]).

A Köthe sequence space X is said to be *semi-order continuous* (**SOC** for short) if for any sequence (x_n) and x in X we have $\|x_n\| \nearrow \|x\|$ whenever $0 \leqslant x_n \nearrow x$.

It is well known that every linear continuous functional f over a Köthe sequence space X can be uniquely decomposed into the form $f = g + \varphi$, where $g = (g(i))$ belongs to the Köthe dual X' of X, it is identified with the linear functional defined by

$$\langle x, g \rangle = \sum_{i=1}^{\infty} g(i)x(i)$$

for every $x \in X$, and φ is a linear singular functional, i.e. φ vanishes on X_a (see [9]).

A map $\Phi\colon \mathbb{R} \to [0,\infty)$ is said to be an *Orlicz function* if Φ is vanishing only at 0, even and convex. We say an Orlicz function Φ is an *N-function* if

$$\lim_{u\to 0} \frac{\Phi(u)}{u} = 0 \quad \text{and} \quad \lim_{u\to\infty} \frac{\Phi(u)}{u} = \infty.$$

The *Orlicz sequence space* l_Φ is defined by the formula

$$l_\Phi = \left\{ x \in l^0\colon\ I_\Phi(cx) = \sum_{i=1}^{\infty} \Phi(cx(i)) < \infty \ \text{ for some } c > 0 \right\}.$$

We endow this space with the *Luxemburg norm*

$$\|x\| = \inf \left\{ \varepsilon > 0\colon\ I_\Phi\left(\frac{x}{\varepsilon}\right) \leqslant 1 \right\}$$

or with an equivalent one

$$\|x\|_0 = \inf_{k>0} \frac{1}{k} \left(1 + I_\Phi(kx)\right),$$

called the *Orlicz norm* or the *Amemiya norm.*

To simplify notation, we put $l_\Phi = (l_\Phi, \|\cdot\|)$ and $l_\Phi^0 = \left(l_\Phi^0, \|\cdot\|_0\right)$. For every Orlicz function Φ we define a function $\Psi\colon \mathbb{R} \longrightarrow [0,\infty)$, complementary to Φ in the sense of Young, by the formula

$$\Psi(v) = \sup_{u>0} \{u\,|v| - \Phi(u)\}.$$

It is well known that Ψ is also an Orlicz function whenever Φ is an N-function.

We say an Orlicz function Φ *satisfies the δ_2-condition* ($\Phi \in \delta_2$ for short) if there exist constants $k \geqslant 2$ and $u_0 > 0$ such that

$$\Phi(2u) \leqslant k\Phi(u)$$

for every $u \in \mathbb{R}$ with $|u| \leqslant u_0$.

For more details on Orlicz functions and Orlicz spaces we refer to [1], [11], [15], [16] and [17].

2. Results

We begin with some general results.

Theorem 1. *A Köthe sequence space X is weakly orthogonal if and only if it is order continuous.*

Proof. *Necessity.* If X is not order continuous, then X_a is a closed proper subspace of X. By *Riesz's Lemma*, for any $\theta \in (0,1)$ there is $x_\theta \in S(X)$ such that $\|x_\theta - x\| \geqslant \theta$ for any $x \in X_a$. Take a sequence (n_i) of natural numbers such that $n_i \uparrow \infty$ and

$$\left\| \sum_{j=n_i+1}^{n_{i+1}} x_\theta(j) e_j \right\| \geqslant \left(1 - \frac{1}{i}\right)\theta,$$

where $\theta \in (\frac{2}{3}, 1)$. Then, setting

$$x_i = \sum_{j=n_i+1}^{n_{i+1}} x_\theta(j) e_j$$

for $i = 1, 2, \ldots$, we immediately get

$$\left(1 - \frac{1}{i}\right)\theta \leqslant \|x_i\| \leqslant 1 \tag{1}$$

for $i = 1, 2, \ldots$ Moreover,

$$x_i \to 0 \text{ weakly as } i \to \infty. \tag{2}$$

Really, it is easy to see that for any $f = g + \varphi \in X^*$ with $g \in X'$ (the Köthe dual of X) and $\varphi \in (X_a)^\perp$, we have $\langle x_i, f \rangle = \langle x_i, g \rangle$. Since $\sum_{j=1}^{\infty} x_\theta(j)g(j) < \infty$, we get

$$\langle x_i, g \rangle = \sum_{j=n_i+1}^{n_{i+1}} x_\theta(j)g(j) \to 0 \text{ as } i \to \infty.$$

Moreover, by (1) we have

$$\|x_\theta + x_i\| \geqslant 2\,\|x_i\| \geqslant 2\left(1 - \frac{1}{i}\right)\theta$$

for $i = 1, 2, \ldots$. However, $\|x_\theta - x_i\| \leqslant 1$, so by

$$2\left(1 - \frac{1}{i}\right)\theta > \frac{4}{3}\left(1 - \frac{1}{i}\right) \longrightarrow \frac{4}{3}$$

we have

$$\lim_{i \to \infty} |\|x_\theta + x_i\| - \|x_\theta - x_i\|| \geqslant \frac{1}{3},$$

i.e. X is not weakly orthogonal.

Sufficiency. For any $\varepsilon > 0$, any $x \in S(X)$ and any weakly null sequence (x_n) in X, there are i_0 and $n_0 \in \mathbb{N}$ such that

$$\left\|\sum_{i=i_0+1}^{\infty} x(i)e_i\right\| < \frac{\varepsilon}{4} \text{ and } \left\|\sum_{i=1}^{i_0} x_n(i)e_i\right\| < \frac{\varepsilon}{4}$$

for $n \geqslant n_0$. Put

$$\overline{x}_n = \sum_{i=1}^{i_0} x(i)e_i + \sum_{i=i_0+1}^{\infty} x_n(i)e_i \text{ and } \overline{y}_n = \sum_{i=1}^{i_0} x(i)e_i - \sum_{i=i_0+1}^{\infty} x_n(i)e_i$$

for $n = 1, 2, \ldots$ Then $\|\overline{x}_n\| = \|\overline{y}_n\|$ for every $n \in \mathbb{N}$ and

$$\begin{aligned}\|(x + x_n) - \overline{x}_n\| &= \left\|\sum_{i=1}^{i_0} x_n(i)e_i + \sum_{i=i_0+1}^{\infty} x(i)e_i\right\| \\ &\leqslant \left\|\sum_{i=1}^{i_0} x_n(i)e_i\right\| + \left\|\sum_{i=i_0+1}^{\infty} x(i)e_i\right\| \leqslant \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}\end{aligned}$$

for $n \geqslant n_0$. Moreover,

$$\|(x-x_n)-\overline{y}_n\| = \left\| \sum_{i=i_0+1}^{\infty} x(i)e_i - \sum_{i=1}^{i_0} x_n(i)e_i \right\|$$
$$\leqslant \left\| \sum_{i=1}^{i_0} x_n(i)e_i \right\| + \left\| \sum_{i=i_0+1}^{\infty} x(i)e_i \right\| \leqslant \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$$

for $n \geqslant n_0$. Hence, we have

$$\begin{aligned} |\|x+x_n\| - \|x-x_n\|| &= |\|x+x_n\| - \|\overline{x}_n\| + \|x-x_n\| - \|\overline{y}_n\|| \\ &\leqslant |\|x+x_n\| - \|\overline{x}_n\|| + |\|x-x_n\| - \|\overline{y}_n\|| \\ &\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for $n \geqslant n_0$. This means that $\lim_{n\to\infty} |\|x+x_n\| - \|x-x_n\|| = 0$. □

Corollary 1. *Orlicz sequence spaces l_Φ equipped with the Luxemburg norm or with the Orlicz norm are weakly orthogonal if and only if $\Phi \in \delta_2$.*

P r o o f. Since **OC** of l_Φ and l_Φ^0 is equivalent to $\Phi \in \delta_2$, the corollary follows immediately by Theorem 1. □

Theorem 2. *Any Banach lattice that is* **SOC** *and has the weak property* (β) *is* **OC**.

P r o o f. Assume to the contrary that X is not **OC**. Then X contains an almost isometric order copy of l_∞ (see [7]). Therefore, we only need to notice that l_∞ has not the weak property (β). Indeed, define

$$x = (1,\ldots,1,\ldots) \quad \text{and} \quad x_n = (0,\ldots,0,1,0,\ldots).$$

Obviously,

$$\|x\| = \|x_n\| = \left\| \frac{1}{2}(x+x_n) \right\| = 1$$

for any $n \in \mathbb{N}$. So we only need to show that $x_n \to 0$ weakly. Since $\sum_{n=1}^{k} x_n \leqslant x$ for every $k \in \mathbb{N}$, we get for any positive $x^* \in (l_\infty)^*$,

$$\sum_{n=1}^{k} \langle x_n, x^* \rangle = \left\langle \sum_{n=1}^{k} x_n, x^* \right\rangle \leqslant \langle x, x^* \rangle < \infty.$$

Consequently, $\langle x_n, x^* \rangle \to 0$ as $n \to \infty$. Since any $x^* \in (l_\infty)^*$ is a difference of two positive linear continuous functionals, we get that $x_n \to 0$ weakly. □

Corollary 2. *Each Köthe sequence space with the weak property* (β) *is weakly orthogonal.*

P r o o f. This follows by the fact that the weak property (β) implies **OC** and by Theorem 1. □

Proposition 1. *If* $\Phi \in \delta_2$*, then for each* $\varepsilon > 0$*, each* $x \in S(l_\Phi)$ *and each weakly null sequence* (x_n) *in* $B(l_\Phi)$ *there is* $n_0 \in \mathbb{N}$ *such that*

$$\|x + x_n\| < D(l_\Phi) + \varepsilon \quad \text{for} \quad n \geqslant n_0,$$

where

$$D(l_\Phi) = \sup\left\{c_z > 0\colon \sum_{i=1}^{n} \Phi\left(\frac{z(i)}{c_z}\right) = \frac{1}{2},\ \ \sum_{i=1}^{n} \Phi(z(i)) = 1,\ \ n = 1, 2, \ldots\right\}.$$

P r o o f. By $\Phi \in \delta_2$, for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$|I_\Phi(x + y) - I_\Phi(x)| < \varepsilon,$$

whenever $I_\Phi(x) \leqslant 1$ and $I_\Phi(y) \leqslant \delta$ (see [8]).

It is clear that $I_\Phi\left(\frac{x}{D(l_\Phi)+\varepsilon}\right) < \frac{1}{2}$ for any $x \in S(l_\Phi)$ and any $\varepsilon > 0$. So, there is $\varepsilon_1 > 0$ such that

$$I_\Phi\left(\frac{x}{D(l_\Phi) + \varepsilon}\right) + 2\varepsilon_1 < \frac{1}{2}.$$

Next, there is $\delta_1 > 0$ such that

$$|I_\Phi(x + y) - I_\Phi(x)| < \varepsilon_1$$

whenever $I_\Phi(x) \leqslant 1$ and $I_\Phi(y) \leqslant \delta_1$. By $\Phi \in \delta_2$, there is $i_0 \in \mathbb{N}$ such that

$$\sum_{i=i_0+1}^{\infty} \Phi\left(\frac{x(i)}{D(l_\Phi) + \varepsilon}\right) < \delta_1.$$

Since $x_n \to 0$ weakly, so $x_n \to 0$ coordinatewise, whence there is $n_0 \in \mathbb{N}$ such that

$$\sum_{i=1}^{i_0} \Phi\left(\frac{x_n(i)}{D(l_\Phi) + \varepsilon}\right) < \delta_1 \qquad \text{for } n \geqslant n_0.$$

Hence

$$
\begin{aligned}
I_\Phi\left(\frac{x+x_n}{D(l_\Phi)+\varepsilon}\right) &= \sum_{i=1}^{\infty}\Phi\left(\frac{x(i)+x_n(i)}{D(l_\Phi)+\varepsilon}\right)\\
&= \sum_{i=1}^{i_0}\Phi\left(\frac{x(i)+x_n(i)}{D(l_\Phi)+\varepsilon}\right)+\sum_{i=i_0+1}^{\infty}\Phi\left(\frac{x(i)+x_n(i)}{D(l_\Phi)+\varepsilon}\right)\\
&< \sum_{i=1}^{i_0}\Phi\left(\frac{x(i)}{D(l_\Phi)+\varepsilon}\right)+2\varepsilon_1+\sum_{i=i_0+1}^{\infty}\Phi\left(\frac{x_n(i)}{D(l_\Phi)+\varepsilon}\right)<\frac{1}{2}+\frac{1}{2}=1
\end{aligned}
$$

for $n \geqslant n_0$. Thus, $\|x+x_n\| < D(l_\Phi)+\varepsilon$ for $n \geqslant n_0$. □

Remark 1. We do not know whether or not Proposition 1 can be formulated with $\varepsilon = 0$. It is clear that if $c_x < D(l_\Phi)$, we can put $\varepsilon = 0$.

Define for any Orlicz function Φ

$$
p(\Phi) = \sup\left\{\lambda \geqslant 1\colon\ \Phi\left(\frac{u}{2^{1/\lambda}}\right) \leqslant \frac{1}{2}\Phi(u),\ 0<u\leqslant \Phi^{-1}(1)\right\}.
$$

Then $\Psi \in \delta_2$ if and only if $p > 1$ (see [5]).

Theorem 3. *If Φ is an N-function, then l_Φ has the weak property (β) if and only if $\Phi \in \delta_2$ and $\Psi \in \delta_2$.*

Proof. *Sufficiency.* Since $\Psi \in \delta_2$, we get $p := p(\Phi) > 1$. Take $\lambda \in (0, p)$. Then for any $x \in S(l_\Phi)$, we have

$$
I_\Phi\left(\frac{x}{2^{1/\lambda}}\right) = \sum_{i=1}^{\infty}\Phi\left(\frac{x(i)}{2^{1/\lambda}}\right) \leqslant \frac{1}{2}\sum_{i=1}^{\infty}\Phi(x(i)) = \frac{1}{2}.
$$

Hence, $D(l_\Phi) \leqslant 2^{\frac{1}{p}} < 2$. In virtue of Proposition 1 with $\varepsilon > 0$ so small that $D(l_\Phi)+\varepsilon < 2$, we get that l_Φ has the weak property (β).

Necessity. By Corollaries 1 and 2, we only need to prove that $\Psi \in \delta_2$. If $\Psi \notin \delta_2$, there is a sequence $u_n \searrow 0$ such that

$$
\Phi\left(\frac{u_n}{2}\right) \geqslant \frac{1}{2}\left(1-\frac{1}{2^n}\right)\Phi(u_n)
$$

for $n = 1, 2, \ldots$ Passing to a subsequence of (u_n) if necessary, we may assume that there is a sequence (N_n) of natural numbers such that

$$
\left(1-\frac{1}{2^n}\right) \leqslant N_n\Phi(u_n) \leqslant 1
$$

for $n = 1, 2, \ldots$ Put

$$x_{1,n} = (\overbrace{u_n, u_n, \ldots, u_n}^{N_n}, 0, 0, \ldots),$$
$$x_{2,n} = (\overbrace{0, 0, \ldots, 0}^{N_n}, \overbrace{u_n, u_n, \ldots, u_n}^{N_n}, 0, 0, \ldots),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_{m,n} = (\overbrace{0, 0, \ldots, 0}^{(m-1)N_n}, \overbrace{u_n, u_n, \ldots, u_n}^{N_n}, 0, 0, \ldots).$$

Then we can easily prove that

$$\left(1 - \frac{1}{2^m}\right) \leqslant \|x_{m,n}\| \leqslant 1$$

for $m = 1, 2, \ldots$ Moreover, $x_{m,n} \to 0$ weakly as $m \to \infty$.

In fact, we can assume by Corollaries 1 and 2 that $\Phi \in \delta_2$, whence it follows that $(l^0_\Phi)^* = l_\Psi$. Let $y \in l_\Psi$ and $\lambda_0 > 0$ be such that $I_\Psi(\lambda_0 y) < \infty$. Take any $\varepsilon > 0$. Since $I_\Phi(\lambda x_{m,n}) = I_\Phi(\lambda x_{1,n})$ for every $\lambda > 0$ and $m \in \mathbb{N}$, by $(\Phi(u)/u) \to 0$ as $u \to 0$, a positive number λ_1 can be found such that

$$\frac{1}{\lambda_0\lambda_1} I_\Phi(\lambda_1 x_{m,n}) < \frac{\varepsilon}{2}$$

for all $m \in \mathbb{N}$. Let $m_0 \in \mathbb{N}$ be such that

$$\frac{1}{\lambda_0\lambda_1} I_\Psi\left(\lambda_0 \sum_{i>(m-1)N_n} y_i e_i\right) < \frac{\varepsilon}{2}$$

for $m \geqslant m_0$. Then by the Young inequality,

$$\langle x_{m,n}, y\rangle \leqslant \frac{1}{\lambda_0\lambda_1}\left(I_\Phi(\lambda_1 x_{m,n}) + I_\Psi\left(\lambda_0 \sum_{i>(m-1)N_n} y_i e_i\right)\right) < \varepsilon$$

for $m \geqslant m_0$. This shows that $x_{m,n} \to 0$ weakly as $m \to \infty$ for $n = 1, 2, \ldots$

We also have

$$\begin{aligned} I_\Phi\left(\frac{2^n(x_{1,n} + x_{m,n})}{2^{n+1} - 2}\right) &= 2I_\Phi\left(\frac{2^n x_{1,n}}{2^{n+1} - 2}\right) \\ &\geqslant 2\frac{2^n}{2^n - 1} I_\Phi\left(\frac{x_{1,n}}{2}\right) = \frac{2^{n+1}}{2^n - 1} N_n \Phi\left(\frac{u_n}{2}\right) \\ &\geqslant \frac{2^{n+1}}{2^n - 1} \cdot \frac{1}{2}\left(1 - \frac{1}{2^n}\right) N_n \Phi(u_n) \geqslant 1 - \frac{1}{2^n}. \end{aligned}$$

Hence

$$\|x_{1,n} + x_{m,n}\| \geqslant 2\left(1 - \frac{1}{2^n}\right)^2,$$

which means that l_Φ has not the weak property (β). This shows the necessity of $\Psi \in \delta_2$, which completes the proof. □

Theorem 4. *If Φ is an N-function, then l_Φ^0 has the weak property (β) if and only if $\Phi \in \delta_2$ and $\Psi \in \delta_2$.*

P r o o f. *Necessity.* By Corollaries 1 and 2, we have $\Phi \in \delta_2$. So it is enough to prove the necessity of $\Psi \in \delta_2$. Assume to the contrary that $\Psi \notin \delta_2$. Since every non-reflexive Banach sequence lattice has the packing constant equal to $\frac{1}{2}$ (see [6]), we have $D(l_\Phi^0) = 2$, where $D(l_\Phi^0)$ is the constant that defines $\Lambda(l_\Phi^0)$. It is known that

$$D(l_\Phi^0) = \sup\left\{\inf\left\{c_{x,k} > 0\colon\ I_\Phi\left(\frac{kx}{c_{x,k}}\right) = \frac{k-1}{2},\quad k > 1\right\}\colon\ x \in S(l_\Phi^0)\right\}$$

(see [19] and [20]). For any $\varepsilon > 0$ there is $x_0 \in S(l_\Phi^0)$ such that

$$\inf\left\{c_{x_0,k} > 0\colon\ I_\Phi\left(\frac{kx_0}{c_{x_0,k}}\right) = \frac{k-1}{2},\quad k > 1\right\} > D(l_\Phi^0) - \varepsilon.$$

So, for any $k > 1$ we have

$$c_{x_0,k} > D(l_\Phi^0) - \varepsilon \quad \text{if} \quad I_\Phi\left(\frac{kx_0}{c_{x_0,k}}\right) = \frac{k-1}{2}.$$

Take a sequence $(\mathbb{N}_i)$ of subsets of $\mathbb{N}$ such that $\mathrm{Card}(\mathbb{N}_i) = \infty$ $(i = 1, 2, \ldots)$, $\mathbb{N}_k \cap \mathbb{N}_m = \emptyset$ for $k \neq m$, $\inf N_i \to \infty$ as $i \to \infty$ and $\bigcup\limits_{i=1}^{\infty} \mathbb{N}_i = \mathbb{N}$. Let $\mathbb{N}_i = \left\{j_1^i, j_2^i, \ldots, j_k^i, \ldots\right\}$. Define

$$x_i = \sum_{k=1}^{\infty} x_0(k) e_{j_k^i}$$

for $i = 1, 2, \ldots$ Then it is obvious that $\|x_i\|_0 = \|x_0\|_0 = 1$ for $i = 1, 2, \ldots$ Moreover, $x_i \to 0$ weakly as $i \to \infty$.

Really, for any fixed $y \in l_\Psi$ and $\varepsilon > 0$, a positive number λ_0 can be found such that $I_\Psi(\lambda_0 y) < \infty$. By the condition $(\Phi(u)/u) \to 0$ as $u \to 0$, there is $\lambda_1 > 0$ such that

$$\frac{1}{\lambda_0 \lambda_1} I_\Phi(\lambda_1 x_0) < \frac{\varepsilon}{2}.$$

Since $\inf(\operatorname{supp} x_i) \leqslant \inf j_k^i \to \infty$ as $i \to \infty$ and $I_\Phi(\lambda x_i) = I_\Phi(\lambda x_0)$ for all $i \in \mathbb{N}$ and $\lambda > 0$, there is i_0 such that

$$\frac{1}{\lambda_0\lambda_1} I_\Psi\left(\lambda_0 y\, \chi_{\operatorname{supp} x_i}\right) < \frac{\varepsilon}{2}$$

for each $i \geqslant i_0$. Hence

$$\begin{aligned}\langle x_i, y\rangle &= \sum_{k=1}^{\infty} x_i(k) y(k) \\ &\leqslant \frac{1}{\lambda_0\lambda_1}\left(I_\Phi(\lambda_1 x_i) + \frac{1}{\lambda_0\lambda_1} I_\Psi\left(\lambda_0 y\, \chi_{\operatorname{supp} x_i}\right)\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

i.e. $x_i \to 0$ weakly as $i \to \infty$.

Take any $\varepsilon \in (0,1)$. Since Φ is an N-function, for each $i \in \mathbb{N}$ there is $k_i > 1$ such that (see [4])

$$\begin{aligned}\left\|\frac{x_0 + x_i}{D(l_\Phi^0) - \varepsilon}\right\|_0 &= \frac{1}{k_i}\left(1 + I_\Phi\left(\frac{k_i(x_0 + x_i)}{D(l_\Phi^0) - \varepsilon}\right)\right) \\ &= \frac{1}{k_i}\left(1 + 2I_\Phi\left(\frac{k_i x_0}{D(l_\Phi^0) - \varepsilon}\right)\right) \geqslant \frac{1}{k_i}\left(1 + 2I_\Phi\left(\frac{k_i x_0}{c_{x_0,k_i}}\right)\right) = 1.\end{aligned}$$

This means that

$$\|x_0 + x_i\|_0 \geqslant D(l_\Phi^0) - \varepsilon = 2 - \varepsilon$$

for $i = 1, 2, \ldots$, whence it follows that l_Φ^0 has not the weak property (β), completing the proof of necessity of $\Psi \in \delta_2$ for the weak property (β).

Sufficiency. For any $x \in S(l_\Phi^0)$ there is $k_x > 1$ such that

$$\|x\|_0 = \frac{1}{k_x}\left(1 + I_\Phi(k_x x)\right).$$

Since $\Psi \in \delta_2$, the number $\mathbf{M} = \sup\{k_x \colon x \in S(l_\Phi^0)\}$ is finite (see [1]). Put $\mathbf{m} = \inf\{k_x \colon x \in S(l_\Phi^0)\}$. Then $\mathbf{m} > 1$. Indeed, if this is not true, there are a sequence (x_n) in $S(l_\Phi^0)$ and a sequence (k_n) of positive reals such that $k_n \to 1$ as $n \to \infty$ and $\frac{1}{k_n}\left(1 + I_\Phi(k_n x_n)\right) = \|x_n\|_0 = 1$, whence $1 + I_\Phi(k_n x_n) \to 1$ and consequently $\lim_{n\to\infty} I_\Phi(k_n x_n) = 0$. In virtue of $\Phi \in \delta_2$, this means that $\lim_{n\to\infty} \|k_n x_n\|_0 = 0$, i.e. $\lim_{n\to\infty} \|x_n\|_0 = 0$ because $k_n \to 1$, a contradiction.

Using again the fact $\Psi \in \delta_2$, we can conclude (see [4]) that there is $\theta \in (0,1)$ such that

(3) $\qquad \Phi(\lambda u) \leqslant (1-\theta)\lambda\Phi(u)$ whenever $\lambda \in \left[0, \dfrac{\mathbf{M}}{\mathbf{M}+1}\right]$ and $|u| \leqslant \mathbf{M}\Phi^{-1}(1)$.

Since $\Phi \in \delta_2$, for any $\varepsilon \in \left(0, \frac{\theta(\mathbf{m}-1)}{2\mathbf{M}^2}\right)$ and $k > 0$ there is $\delta > 0$ such that $\varepsilon < \frac{\theta(\mathbf{m}-1-\delta)}{2\mathbf{M}^2}$ and $|I_\Phi(x+y) - I_\Phi(x)| < \varepsilon$ whenever $I_\Phi(x) \leqslant k$ and $I_\Phi(y) \leqslant \delta$ (see [8]).

Next, we will show that for such x, y and $\delta > 0$ we have

$$I_\Phi(x+ty) < I_\Phi(x) + t\varepsilon \tag{4}$$

for any $t \in [0,1]$.

Indeed,

$$\begin{aligned} I_\Phi(x+ty) &= I_\Phi(t(x+y) + (1-t)x) \leqslant tI_\Phi(x+y) + (1-t)I_\Phi(x) \\ &\leqslant t(I_\Phi(x)+\varepsilon) + (1-t)I_\Phi(x) = I_\Phi(x) + t\varepsilon. \end{aligned}$$

For any $x_0 \in S(l^0_\Phi)$ and any weakly null sequence (x_n) in $S(l^0_\Phi)$, there is a sequence (k_n) with $k_n > 1$ for $n = 0, 1, 2, \ldots$ such that

$$\|x_n\|_0 = \frac{1}{k_n}\left(1 + I_\Phi(k_n x_n)\right) \tag{5}$$

for $n = 0, 1, 2, \ldots$ Take $i_0 \in \mathbb{N}$ such that

$$\sum_{i=i_0+1}^{\infty} \Phi(k_0 x_0(i)) < \delta. \tag{6}$$

Since $x_n(i) \to 0$ $(i = 1, 2, \ldots)$ as $n \to \infty$, there is $n_0 \in \mathbb{N}$ such that

$$\sum_{i=1}^{i_0} \Phi(k_n x_n(i)) < \delta$$

for $n \geqslant n_0$. Therefore, since $k_0/(k_0+k_n) \leqslant \mathbf{M}/(\mathbf{M}+1)$ and $|x_0(i)| \leqslant \Phi^{-1}(1)$ for each $i \in \mathbb{N}$, in virtue of (3), (4), (5) and (6) we get

$$\begin{aligned}
\|x_0+x_n\|_0 &\leqslant \frac{k_0+k_n}{k_0k_n}\left(1+I_\Phi\left(\frac{k_0k_n}{k_0+k_n}(x_0+x_n)\right)\right) \\
&= \frac{k_0+k_n}{k_0k_n}\left(1+\sum_{i=1}^{i_0}\Phi\left(\frac{k_0k_n}{k_0+k_n}(x_0(i)+x_n(i))\right)\right. \\
&\quad \left.+\sum_{i=i_0+1}^{\infty}\Phi\left(\frac{k_0k_n}{k_0+k_n}(x_0(i)+x_n(i))\right)\right) \\
&\leqslant \frac{k_0+k_n}{k_0k_n}\left(1+\sum_{i=1}^{i_0}\Phi\left(\frac{k_0k_n}{k_0+k_n}x_0(i)\right)+\frac{k_0}{k_0+k_n}\varepsilon\right. \\
&\quad \left.+\sum_{i=i_0+1}^{\infty}\Phi\left(\frac{k_0k_n}{k_0+k_n}x_n(i)\right)+\frac{k_n}{k_0+k_n}\varepsilon\right) \\
&\leqslant \frac{k_0+k_n}{k_0k_n}\left(1+\frac{k_n}{k_0+k_n}\sum_{i=1}^{i_0}\Phi\Big(k_0x_0(i)\Big)\right. \\
&\quad \left.+\frac{k_0}{k_0+k_n}(1-\theta)\sum_{i=i_0+1}^{\infty}\Phi\Big(k_nx_n(i)\Big)+\varepsilon\right) \\
&\leqslant \frac{1}{k_0}\left(1+\sum_{i=1}^{i_0}\Phi\left(k_0x_0(i)\right)\right)+\frac{1}{k_n}\left(1+\sum_{i=i_0+1}^{\infty}\Phi\Big(k_nx_n(i)\Big)\right) \\
&\quad -\frac{\theta}{\mathbf{M}}\sum_{i=i_0+1}^{\infty}\Phi\left(k_nx_n(i)\right)+2\mathbf{M}\varepsilon \\
&\leqslant 1+1-\theta(\mathbf{m}-1-\delta)/\mathbf{M}+2\mathbf{M}\varepsilon =: \sigma < 2
\end{aligned}$$

for $n \geqslant n_0$. Since σ depends neither on x_0 nor on the sequence (x_n), the proof of the theorem is complete. □

Remark 2. Theorem 3 (resp. Theorem 4) states that l_Φ (resp. l_Φ^0) has the weak property (β) iff it is reflexive. On the other hand, by the fact that l^1 has the Schur property, we can conclude that l^1 has the weak property (β). Therefore the assumption that Φ is an N-function is essential in these theorems. Another example of a non-reflexive Köthe sequence space with the weak property (β) is the space c_0. Since the property (β) implies reflexivity (see [18]), these examples show that the weak property (β) does not imply the property (β).

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