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ON POSETS WITH ISOMORPHIC INTERVAL POSETS

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Let $\mathbb{A} = (A, \leqslant)$ be a partially ordered set, Int $\mathbb{A}$ the system of all (nonempty) intervals of $\mathbb{A}$, partially ordered by the set-theoretical inclusion $\subseteq$. We are interested in partially ordered sets $\mathbb{B} = (B, \leqslant)$ with Int $\mathbb{B}$ isomorphic to Int $\mathbb{A}$. We are going to show that they correspond to couples of binary relations on A satisfying some conditions. If $\mathbb{A}$ is a directed partially ordered set, the only $\mathbb{B}$ with Int $\mathbb{B}$ isomorphic to Int $\mathbb{A}$ are $\mathbb{A}_1^\delta \times \mathbb{A}_2$ corresponding to direct decompositions $\mathbb{A}_1 \times \mathbb{A}_2$ of $\mathbb{A}$ ($\mathbb{A}_1^\delta$ denotes the dual of $\mathbb{A}_1$). The present results include those presented in the paper [11] by V. Slavík. Systems of intervals, particularly of lattices, have been investigated by many authors, cf. [1]–[11].

1.

By an interval of a partially ordered set $\mathbb{A} = (A, \leqslant)$ a set $\langle a, b\rangle = \{x \in A\colon\ a \leqslant x \leqslant b\}$ with $a, b \in A, a \leqslant b$ is meant. If $a = b$, we use the notation $\langle a\rangle$ instead of $\langle a, a\rangle$. The system of all intervals of $\mathbb{A}$ is denoted by Int $\mathbb{A}$. Consider the set-theoretical inclusion on Int $\mathbb{A}$. The following lemma is easy to verify:

1.1. Lemma. a) $\langle a, b\rangle = \inf\{\langle a_1, b_1\rangle, \langle a_2, b_2\rangle\}$ *if and only if* $\langle a, b\rangle = \langle a_1, b_1\rangle \cap \langle a_2, b_2\rangle$;

b) $\langle a, b\rangle = \sup\{\langle a_1, b_1\rangle, \langle a_2, b_2\rangle\}$ *if and only if* $a = \inf\{a_1, a_2\}$, $b = \sup\{b_1, b_2\}$.

Let U, V be binary relations on A. Consider the following conditions:

(P1) $U, V \subseteq \{(x, y) \in A \times A\colon\ x \nparallel y\}$;

(P2) $x, y \in A$, $x \leqslant y \Longrightarrow$ there exists a unique couple of elements $p, q \in \langle x, y\rangle$ satisfying $pVxUqVyUp$;

(P3) $u \leqslant x, y$, $xVuUy \Longrightarrow u = \inf\{x, y\}$, $v = \sup\{x, y\}$ exists and $yVvUx$ holds;

(P3′) $v \geqslant x, y$, $yVvUx \Longrightarrow v = \sup\{x, y\}$, $u = \inf\{x, y\}$ exists and $xVuUy$ holds;

(P4) $a = a_1Ua_2U\dots Ua_n = a'$, $a = a'_1Va'_2V\dots Va'_m = a'$ $(n, m \in N) \Longrightarrow a = a'$;

(P5) for every $a, a' \in A$ there exist $n, m \in N$, $a_1, \dots, a_n, a'_1, \dots, a'_m \in A$ satisfying $a = a_1Ua_2U\dots Ua_n = a'_1Va'_2V\dots Va'_m = a'$.

We are going to prove the following theorem:

1.2. Theorem. *Let $\mathbb{A}$ be a connected partially ordered set. Then there exists a mapping Φ of the system of all couples of binary relations U, V on A satisfying the conditions* (P1)–(P3′) *onto the system of all isomorphism classes of partially ordered sets $\mathbb{B}$ with* Int $\mathbb{B}$ *isomorphic to* Int $\mathbb{A}$. *If a couple (U, V) satisfies* (P1)–(P5), *then the class $\Phi((U, V))$ consists of all partially ordered sets isomorphic to $\mathbb{A}_1^\delta \times \mathbb{A}_2$ for a direct decomposition $\mathbb{A}_1 \times \mathbb{A}_2$ of $\mathbb{A}$. Conversely, the class of all partially ordered sets isomorphic to $\mathbb{A}_1^\delta \times \mathbb{A}_2$ for a direct decomposition $\mathbb{A}_1 \times \mathbb{A}_2$ of $\mathbb{A}$ is $\Phi((U, V))$ for a couple (U, V) satisfying* (P1)–(P5).

Let us remark that the connectivity of $\mathbb{A}$ is not a limiting assumption. Namely, if $\mathbb{P}$ is any partially ordered set, P can be decomposed into maximal connected subsets P_i $(i \in I)$ and the system Int $\mathbb{P}$ is the cardinal sum of the interval posets Int $\mathbb{P}_i$ of these subsets. Now a partially ordered set $\mathbb{Q}$ satisfies the condition Int $\mathbb{Q} \cong$ Int $\mathbb{P}$ if and only if $\mathbb{Q}$ is the cardinal sum of some $\mathbb{Q}_i$ $(i \in I)$ with Int $\mathbb{Q}_i \cong$ Int $\mathbb{P}_i$.

Further let us notice that if partially ordered sets $\mathbb{A}$, $\mathbb{B}$ have isomorphic interval posets, then they are of the same cardinality; so we may assume, without loss of generality, that $\mathbb{A}$, $\mathbb{B}$ have the same underlying set.

2.

Let $\mathbb{A} = (A, \leqslant)$ be a connected partially ordered set, U, V binary relations on A satisfying (P1)–(P3′). First we will show some properties of U, V following from the conditions (P1)–(P3′).

The following is obtained immediately, using (P2).

2.1. Lemma. *The relations U, V are reflexive.*

2.2. Lemma. *The relations U, V are symmetric.*

Proof. Let xUy. By (P1) x, y are comparable. Suppose, e.g., that $x \leqslant y$. We have $x \leqslant x, y$, $xVxUy$ and since $y = \sup\{x, y\}$, using (P3) we obtain yUx. To prove yUx for $x \geqslant y$, we use (P3′). □

2.3. Lemma. *If $x, y \in A$ and one of these elements covers the other, then $(x, y) \in U \cup V$.*

This follows immediately from (P2).

2.4. Lemma. *If* $(x,y) \in U \cap V$, *then* $x = y$.

P r o o f. Let $(x,y) \in U \cap V$. Without loss of generality we can suppose $x \leqslant y$. Then both $xUyVy$ and $xUxVy$ hold, so $x = y$ by (P2). $\square$

2.5. Lemma. *If* $x \leqslant y \leqslant z$, *then* $xUyUz$ *implies* xUz *and* $xVyVz$ *implies* xVz.

P r o o f. We are going to prove, e.g., the part concerning U. Hence let $x \leqslant y \leqslant z$, $xUyUz$. (P2) ensures the existence of an element $p \in \langle x,z\rangle$ with $zUpVx$. Now $x \leqslant p,y$, $pVxUy$, so that $\sup\{p,y\} = v$ exists and satisfies $yVvUp$ by (P3). Evidently $v \leqslant z$. We have $y \leqslant v,z$, $vVyUz$, so in view of (P3) we obtain $y = \inf\{v,z\} = v$. But then $x = \inf\{p,y\} = p$ and consequently xUz. $\square$

2.6. Lemma. *Let* $x,y \in A$, $x \leqslant y$, p, q *be as in* (P2). *If* $a \in \langle x,y\rangle$, *there exists a unique quadruple of elements* $p_1 \in \langle x,p\rangle$, $q_1 \in \langle x,q\rangle$, $p_2 \in \langle p,y\rangle$, $q_2 \in \langle q,y\rangle$ *satisfying* $aUp_1VxUq_1VaUq_2VyUp_2Va$, p_1VpUp_2, q_1UqVq_2.

P r o o f. Let $a \in \langle x,y\rangle$. Then $x \leqslant a$ implies the existence of $p_1,q_1 \in \langle x,a\rangle$ satisfying $p_1VxUq_1VaUp_1$ and $a \leqslant y$ implies that $p_2VaUq_2VyUp_2$ for some $p_2,q_2 \in \langle a,y\rangle$, by (P2). Using again (P2) we obtain that there exist $p' \in \langle p_1,p_2\rangle$, $q' \in \langle q_1,q_2\rangle$ such that $p_1Vp'Up_2$, $q_1Uq'Vq_2$. But then 2.5 yields $p'VxUq'VyUp'$. The uniqueness of p, q in (P2) implies $p' = p$, $q' = q$. The uniqueness of p_1, q_1, p_2, q_2 follows from (P3) and (P3$'$). Namely, $p_1 = \inf\{p,a\}$, $q_1 = \inf\{a,q\}$, $p_2 = \sup\{p,a\}$, $q_2 = \sup\{a,q\}$. $\square$

2.7. Lemma. *If* $x \leqslant a \leqslant y$, *then* xUy *implies* $xUaUy$ *and* xVy *implies* $xVaVy$.

P r o o f. Let $x \leqslant a \leqslant y$, xUy. Using the notation as in 2.6, we have $p = x$, $q = y$, $p_1 = x$, $q_1 = a$, $p_2 = a$, $q_2 = y$. By 2.6 we have pUp_2Uy, hence $xUaUy$. The part concerning V can be shown analogously. $\square$

2.8. Lemma. *Let* $x,y \in A$, $x \leqslant y$, p, q *be as in* (P2). *Then for each* $a \in \langle x,y\rangle$, $\inf\{p,a\}$, $\inf\{a,q\}$ *exist and they satisfy* $pV\inf\{p,a\}UaV\inf\{a,q\}Uq$. *The mapping* $\alpha\colon a \mapsto (\inf\{p,a\},\inf\{a,q\})$ *is an isomorphism of* $\langle x,y\rangle$ *onto* $\langle x,p\rangle \times \langle x,q\rangle$.

P r o o f. Let $a \in \langle x,y\rangle$, p_1, q_1 be as in 2.6. Then $p_1 = \inf\{p,a\}$, $q_1 = \inf\{a,q\}$ by (P3). Further, 2.6 ensures that pVp_1UaVq_1Uq holds. Now using (P3$'$) and 2.6 we obtain $a = \sup\{p_1,q_1\}$. Let $p_1' \in \langle x,p\rangle$, $q_1' \in \langle x,q\rangle$. Since $x \leqslant p_1',q_1'$ and $p_1'VxUq_1'$ holds, by 2.7, the condition (P3) yields that $\sup\{p_1',q_1'\} = a'$ exists and we have $q_1'Va'Up_1'$. But then $p_1' = \inf\{p,a'\}$, $q_1' = \inf\{a',q\}$, so that $\alpha(a') = (p_1',q_1')$. We have proved that α is onto.

Let $a, a' \in \langle x, y\rangle$, $a \leqslant a'$. Then evidently $(\inf\{p, a\}, \inf\{a, q\}) \leqslant (\inf\{p, a'\}, \inf\{a', q\})$. Hence α preserves the order.

Finally, let $a, a' \in \langle x, y\rangle$, $(\inf\{p, a\}, \inf\{a, q\}) \leqslant (\inf\{p, a'\}, \inf\{a', q\})$. Then $a = \sup\{\inf\{p, a\}, \inf\{a, q\}\} \leqslant \sup\{\inf\{p, a'\}, \inf\{a', q\}\} = a'$, completing the proof. □

2.9. Lemma. *Let $x, y \in A$, $x \leqslant y$, p, q be as in (P2). If $x \leqslant a \leqslant a' \leqslant y$ and $aUa'(aVa')$, then $\inf\{p, a\} = \inf\{p, a'\}(\inf\{a, q\} = \inf\{a', q\})$.*

Proof. Suppose that $x \leqslant a \leqslant a' \leqslant y$ and, e.g., aUa'. Using 2.8 we get $\inf\{p, a\} \leqslant \inf\{p, a'\}$, $pV \inf\{p, a\}UaUa'$. Now $\inf\{p, a'\} \in \langle \inf\{p, a\}, a'\rangle$, so that $\inf\{p, a\}U \inf\{p, a'\}$. But simultaneously $\inf\{p, a\}V \inf\{p, a'\}$ by 2.7. Hence $\inf\{p, a\} = \inf\{p, a'\}$ by 2.4. □

Now we are going to introduce a "new" order on A, corresponding to a couple of U, V satisfying (P1)–(P3′).

2.10. Definition. For $x, y \in A$ set $x \leqslant_1 y$ if there exists $u \in A$, $u \leqslant x, y$, satisfying $xVuUy$.

2.11. Lemma. *The above defined relation $\leqslant_1$ is a partial order.*

Proof. The reflexivity of U, V ensures that $x \leqslant_1 x$ for each $x \in A$. Let $x \leqslant_1 y$, $y \leqslant_1 x$. Then there exist u_1, u_2 such that $u_1 \leqslant x, y$, xVu_1Uy, $u_2 \leqslant y, x$, yVu_2Ux. Using (P3) we obtain $u_1 = \inf\{x, y\} = u_2$. Hence $(u_1, x) \in U \cap V$ and also $(u_1, y) \in U \cap V$ and consequently $x = u_1 = y$ by 2.4. Let $x \leqslant_1 y$, $y \leqslant_1 z$. Then there exist $u_1, u_2 \in A$ satisfying $u_1 \leqslant x, y$, xVu_1Uy, $u_2 \leqslant y, z$, yVu_2Uz. Using (P3′) we obtain that $\inf\{u_1, u_2\} = u$ exists and u_1VuUu_2 holds. But then $u \leqslant x, z$ and $xVuUz$ by 2.5, so that $x \leqslant_1 z$. □

The aim is to prove that $\mathrm{Int}\ (A, \leqslant) \cong \mathrm{Int}\ (A, \leqslant_1)$. Let $x, y \in A, x \leqslant y$, p, q be as in (P2). Then evidently $p \leqslant_1 q$. Set $f(\langle x, y\rangle) = \langle p, q\rangle_1$, where $\langle p, q\rangle_1 = \{t \in A\colon\ p \leqslant_1 t \leqslant_1 q\}$. Recall that $\langle x, y\rangle$ is isomorphic to $\langle x, p\rangle \times \langle x, q\rangle$. Now we have:

2.12. Lemma. *The mapping α defined in 2.8 is an isomorphism of $\langle p, q\rangle_1$ onto $\langle x, p\rangle^\delta \times \langle x, q\rangle$.*

Proof. Evidently $a \in \langle x, y\rangle$ if and only if $a \in \langle p, q\rangle_1$ and α is onto. Further let us suppose that $a, a' \in \langle p, q\rangle_1$, $a \leqslant_1 a'$. We have to prove $\inf\{p, a\} \geqslant \inf\{p, a'\}$, $\inf\{a, q\} \leqslant \inf\{a', q\}$. Take $p_1 = \inf\{p, a\}$, $q_1' = \inf\{a', q\}$ and $u \leqslant a, a'$ satisfying $aVuUa'$. In view of (P3′), $r = \inf\{p_1, u\}$, $s = \inf\{u, q_1'\}$ exist such that $p_1VrUuVsUq_1'$. But then $pVrUa'$, $aVsUq$, so that $r = \inf\{p, a'\}$, $s = \inf\{a, q\}$

and we have $r \leqslant p_1, s \leqslant q_1'$. Conversely let $a, a' \in \langle p, q\rangle_1$, $p_1 \geqslant p_1', q_1 \leqslant q_1'$, where $p_1 = \inf\{p, a\}$, $q_1 = \inf\{a, q\}$, $p_1' = \inf\{p, a'\}$, $q_1' = \inf\{a', q\}$. Since $x \leqslant p_1', q_1$ and $p_1' V x U q_1$, $\sup\{p_1', q_1\} = t$ exists and $q_1 V t U p_1'$. Obviously $t \leqslant a, a'$. Moreover, aVq_1 yields aVt and $p_1' U a'$ implies tUa'. Thus $a \leqslant_1 a'$. The proof is complete. □

2.13. Lemma. *The mapping f assigning to $\langle x, y\rangle$ the interval $\langle p, q\rangle_1$ is an isomorphism of* $\mathrm{Int}(A, \leqslant)$ *onto* $\mathrm{Int}(A, \leqslant_1)$.

Proof. Let $r \leqslant_1 s$. Then there exists $u \leqslant r, s$ such that $rVuUs$. By (P3), $v = \sup\{r, s\}$ exists and $sVvUr$ holds. Evidently $f(\langle u, v\rangle) = \langle r, s\rangle_1$. The mapping f is onto.

Now let $\langle x, y\rangle \subseteq \langle x_1, y_1\rangle$, $f(\langle x, y\rangle) = \langle p, q\rangle_1$, $f(\langle x_1, y_1\rangle) = \langle p_1, q_1\rangle_1$. Take $\inf\{p_1, p\} = p_1'$, $\inf\{q, q_1\} = q_1'$. We have $p_1' \leqslant p_1, p$, $p_1 V p_1' U p$, so $p_1 \leqslant_1 p$. Analogously $q_1' \leqslant q, q_1$, $qVq_1'Uq_1$ ensures $q \leqslant_1 q_1$. Hence $\langle p, q\rangle_1 \subseteq \langle p_1, q_1\rangle_1$.

Next suppose that $f(\langle x, y\rangle) = \langle p, q\rangle_1 \subseteq \langle p_1, q_1\rangle_1 = f(\langle x_1, y_1\rangle)$. We have to show $\langle x, y\rangle \subseteq \langle x_1, y_1\rangle$. Let $u \leqslant p_1, p$, $p_1 V u U p$ and $v \leqslant q, q_1$, $qVvUq_1$. Since $p \geqslant u, x$, $xVpUu$ and $q \geqslant x, v$, $vVqUx$, there exist $a \leqslant u, x$, $b \leqslant x, v$ satisfying $uVaUxVbUv$, by (P3′). Finally consider $c = \inf\{a, b\}$, whose existence follows from (P3′). We have $p_1 VuVaVcUbUvUq_1$, hence $c = \inf\{p_1, q_1\} = x_1$ by (P3) and (P2). Now obviously $x_1 \leqslant x$. The relation $y \leqslant y_1$ can be proved analogously. □

Summarizing, we have:

2.14. Theorem. *Let $\mathbb{A} = (A, \leqslant)$ be a connected partially ordered set, U, V binary relations on A satisfying* (P1)–(P3′). *If $\leqslant_1$ is the relation on A defined as in* 2.10 *with the aid of U, V, then $(A, \leqslant_1)$ is a partially ordered set with* $\mathrm{Int}(A, \leqslant_1)$ *isomorphic to* $\mathrm{Int}(A, \leqslant)$.

It is easy to see that the couples $U_1 = \{(x, y) \in A \times A\colon x \nparallel y\}$, $V_1 = \{(x, x)\colon x \in A\}$ and $U_2 = \{(x, x)\colon x \in A\}$, $V_2 = \{(x, y) \in A \times A\colon x \nparallel y\}$ satisfy the conditions (P1)–(P3′). The corresponding orders $\leqslant_1$, $\leqslant_2$ are $\leqslant$ and $\leqslant^\delta$, respectively. Some partially ordered sets $\mathbb{A} = (A, \leqslant)$ have no other orders $\leqslant_1$ besides $\leqslant$ and $\leqslant^\delta$, satisfying $\mathrm{Int}\ (A, \leqslant_1) \cong \mathrm{Int}\ (A, \leqslant)$. This is the case e.g. for $\mathbb{A}$ in Fig. 1. On the other hand, it is easy to see that the partially ordered sets in Fig. 2 and Fig. 3 have isomorphic interval systems, but they are neither isomorphic nor dually isomorphic. In fact, the first is the direct product of two copies of $\mathbb{A}$ in Fig. 1, while the other is isomorphic to $\mathbb{A}^\delta \times \mathbb{A}$.

Further assume that U, V satisfy also the conditions (P4), (P5). Define binary relations $\overline{U}$, $\overline{V}$ on A as follows:

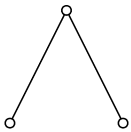

Fig. 1

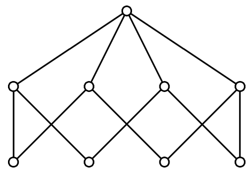

Fig. 2

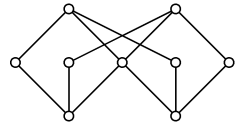

Fig. 3

2.15. Definition. For $x, y \in A$ set $x\overline{U}y$ ($x\overline{V}y$) if there exists a finite sequence $x_1, x_2, \ldots, x_n$ of elements of A such that $x_1 = x$, $x_n = y$ and every two adjoining elements are in the relation $U(V)$.

The following statement is evident.

2.16. Lemma. *The relations $\overline{U}$, $\overline{V}$ are equivalence relations.*

Consider the decompositions $A/\overline{U}$, $A/\overline{V}$. Denote by $[a]\overline{U}$, $[a]\overline{V}$ the equivalence classes containing the element a.

2.17. Definition. Set $[a]\overline{U} \leqslant [b]\overline{U}$ ($[a]\overline{V} \leqslant [b]\overline{V}$) if and only if there exist $a_1 \in [a]\overline{U}$, $b_1 \in [b]\overline{U}$ ($a_1 \in [a]\overline{V}$, $b_1 \in [b]\overline{V}$) satisfying $a_1 \leqslant b_1$.

2.18. Lemma. *For any $a, b \in A$ the following conditions are equivalent:*

(1) *$[a]\overline{U} \leqslant [b]\overline{U}$;*
(2) *for each $a_1 \in [a]\overline{U}$ there exists $b_1 \in [b]\overline{U}$ with $a_1 \leqslant b_1$;*
(3) *for each $b_1 \in [b]\overline{U}$ there exists $a_1 \in [a]\overline{U}$ with $a_1 \leqslant b_1$.*

Proof. The implications (2) $\Longrightarrow$ (1), (3) $\Longrightarrow$ (1) are evident. We are going to prove (1) $\Longrightarrow$ (2). The proof of (1) $\Longrightarrow$ (3) would be analogous. So let $[a]\overline{U} \leqslant [b]\overline{U}$. We can suppose that $a \leqslant b$. Take any $a_1 \in [a]\overline{U}$. Then there exist $x_1, \ldots, x_n$ such that $a = x_1$, $a_1 = x_n$, $x_1 \leqslant x_2$, $x_2 \geqslant x_3, \ldots, x_{n-1} \geqslant x_n$, $x_1 U x_2 U \ldots U x_n$. Using the conditions (P2), (P3) we can construct elements $y_1, y_2, \ldots, y_n$ such that $y_1 \in \langle x_1, b\rangle$, $x_1 V y_1 U b$, $y_2 \geqslant y_1, x_2$, $x_2 V y_2 U y_1$, $y_3 \in \langle x_3, y_2\rangle$, $x_3 V y_3 U y_2$, …, $y_n \in \langle x_n, y_{n-1}\rangle$, $x_n V y_n U y_{n-1}$. We have $a_1 \leqslant y_n$, $y_n \in [b]\overline{U}$. $\square$

Obviously the same holds for $\overline{V}$.

2.19. Lemma. *The above defined relation $\leqslant$ on $A/\overline{U}$ is a partial order.*

Proof. The reflexivity is trivial. Further let $[a]\overline{U} \leqslant [b]\overline{U}$, $[b]\overline{U} \leqslant [a]\overline{U}$. Then there exist $a_1, a_2 \in [a]\overline{U}$ satisfying $a_1 \leqslant b \leqslant a_2$. Take $z \in \langle a_1, a_2\rangle$ such that $a_1 U z V a_2$. We have $z\overline{U}a_2$ and simultaneously zVa_2. Using (P4) we obtain $z = a_2$ and consequently $a_1 \leqslant b \leqslant z$, which implies $a_1 U b$ by 2.7. Hence $[b]\overline{U} = [a_1]\overline{U} = [a]\overline{U}$. Finally, let $[a]\overline{U} \leqslant [b]\overline{U}$, $[b]\overline{U} \leqslant [c]\overline{U}$. Then there exist $a_1 \in [a]\overline{U}$, $c_1 \in [c]\overline{U}$ such that $a_1 \leqslant b \leqslant c_1$ and this implies $[a]\overline{U} \leqslant [c]\overline{U}$. $\square$

Evidently the same holds for $\overline{V}$. The symbol $\mathbb{A}/\overline{U}$ ($\mathbb{A}/\overline{V}$) will be used for $A/\overline{U}$ ($A/\overline{V}$) with the order $\leqslant$ as above.

2.20. Theorem. *Let $\mathbb{A} = (A, \leqslant)$ be a connected partially ordered set, U, V binary relations on A satisfying* (P1)–(P5). *If $\leqslant_1$ is as in* 2.10, *then $\mathbb{A}$ is isomorphic to $\mathbb{A}/\overline{U} \times \mathbb{A}/\overline{V}$, while $\mathbb{A}_1 = (A, \leqslant_1)$ is isomorphic to $(\mathbb{A}/\overline{U})^\delta \times \mathbb{A}/\overline{V}$.*

P r o o f. Define $\alpha\colon A \to A/\overline{U} \times A/\overline{V}$ by $\alpha(a) = ([a]\overline{U}, [a]\overline{V})$. α is onto: Take $([a_1]\overline{U}, [a_2]\overline{V}) \in A/\overline{U} \times A/\overline{V}$. By (P5) there exists $x \in A$ satisfying $a_1\overline{U}x\overline{V}a_2$. Then $\alpha(x) = ([a_1]\overline{U}, [a_2]\overline{V})$.

The implication $a \leqslant b \Longrightarrow \alpha(a) \leqslant \alpha(b)$ is evident. Conversely, let $\alpha(a) \leqslant \alpha(b)$. Then $[a]\overline{U} \leqslant [b]\overline{U}$, $[a]\overline{V} \leqslant [b]\overline{V}$ and consequently $a \leqslant b_1, b_2$ for some $b_1 \in [b]\overline{U}$, $b_2 \in [b]\overline{V}$. Take $b_1' \in \langle a, b_1\rangle$, $b_2' \in \langle a, b_2\rangle$ such that $aVb_1'Ub_1, aUb_2'Vb_2$. The condition (P3) yields the existence of $t \geqslant b_1', b_2'$ with $b_2'VtUb_1'$. Now $t\overline{U}b$, $t\overline{V}b$, hence $t = b$ by (P4). We have $b \geqslant a$.

Suppose $a \leqslant_1 b$. Then there exists $u \leqslant a, b$ satisfying $aVuUb$ and this implies that $[a]\overline{U} \geqslant [u]\overline{U} = [b]\overline{U}$, $[a]\overline{V} = [u]\overline{V} \leqslant [b]\overline{V}$.

Finally, let $[a]\overline{U} \geqslant [b]\overline{U}$, $[a]\overline{V} \leqslant [b]\overline{V}$. We have to show $a \leqslant_1 b$. The assumptions yield the existence of $a_1 \in [a]\overline{U}$, $a_2 \in [a]\overline{V}$ with $a_2 \leqslant b \leqslant a_1$. Take $c \in \langle a_2, a_1\rangle$ satisfying a_2VcUa_1. Then $c = a$ by (P4). In view of 2.8 $u = \inf\{a, b\}$ exists and $aVuUb$. The proof is complete. □

3.

Let $\mathbb{A} = (A, \leqslant)$ be a connected partially ordered set, $\mathbb{A}' = (A, \leqslant')$ another partially ordered set with the same underlying set and let f be an isomorphism of Int $\mathbb{A}$ onto Int $\mathbb{A}'$. The aim is to prove that $\mathbb{A}'$ can be obtained in the way described in the preceding section. Define $f'\colon A \to A$ by

$$f'(a) = b \Longleftrightarrow f(\langle a\rangle) = \langle b\rangle' = \langle b\rangle.$$

($\langle x, y\rangle'$ will mean the set $\{t \in A\colon\ x \leqslant' t \leqslant' y\}$). Evidently f' is a bijective mapping of A onto A. Consider the following binary relations on A: $U = \{(x, y) \in A \times A\colon\ x \leqslant y$ and $f'(x) \leqslant' f'(y)\} \cup \{(x, y) \in A \times A\colon\ x \geqslant y$ and $f'(x) \geqslant' f'(y)\}$, $V = \{(x, y) \in A \times A\colon\ x \leqslant y$ and $f'(x) \geqslant' f'(y)\} \cup \{(x, y) \in A \times A\colon\ x \geqslant y$ and $f'(x) \leqslant' f'(y)\}$. Evidently U, V satisfy the condition (P1).

3.1. Lemma. *Let $x, y \in A$, $x \leqslant y$, $f(\langle x, y\rangle) = \langle r, s\rangle'$. Then $r = \inf\{f'(x), f'(y)\}$, $s = \sup\{f'(x), f'(y)\}$ in $\mathbb{A}'$.*

P r o o f. Since $\langle x,y\rangle = \sup\{\langle x\rangle, \langle y\rangle\}$, we have $\langle r,s\rangle' = \sup\{f(\langle x\rangle), f(\langle y\rangle\}$. But $f(\langle x\rangle) = \langle f'(x)\rangle$, $f(\langle y\rangle) = \langle f'(y)\rangle$ so that $r = \inf\{f'(x), f'(y)\}$, $s = \sup\{f'(x), f'(y)\}$ in $\mathbb{A}'$ by 1.1. $\square$

Taking into account that f^{-1} is also an isomorphism and $(f^{-1})' = (f')^{-1}$, we obtain:

3.2. Lemma. *If $x,y \in A$, $x \leqslant y$, $f(\langle x,y\rangle) = \langle r,s\rangle'$, $r = f'(p)$, $s = f'(q)$, then $x = \inf\{p,q\}$, $y = \sup\{p,q\}$ in $\mathbb{A}$.*

3.3. Lemma. *The above defined U, V fulfil* (P2).

P r o o f. Let $x,y \in A$, $x \leqslant y$. The previous lemma guarantees the existence of such p, q as we need. Now let $p_1, q_1 \in \langle x,y\rangle$ also satisfy $p_1 V x U q_1 V y U p_1$. The relations $p_1 V x$, $x \leqslant p_1$ imply $f'(x) \geqslant' f'(p_1)$ while yUp_1, $p_1 \leqslant y$ imply $f'(p_1) \leqslant' f'(y)$. Hence $f'(p_1) \leqslant' r$ by 3.1. Analogously $f'(q_1) \geqslant' s$. On the other hand $\langle p_1\rangle, \langle q_1\rangle \subseteq \langle x,y\rangle$ yields $\langle f'(p_1)\rangle, \langle f'(q_1)\rangle \subseteq \langle r,s\rangle'$ and consequently $r \leqslant' f'(p_1), f'(q_1) \leqslant' s$. So we have $f'(p_1) = r = f'(p)$, $f'(q_1) = s = f'(q)$, which implies $p_1 = p$, $q_1 = q$. $\square$

3.4. Lemma. *Let $x,y \in A$, $x \leqslant y$, xUy (xVy). Then for each $t \in \langle x,y\rangle$ we have $xUtUy$ ($xVtVy$).*

P r o o f. We will prove, e.g., the part concerning U. Take any $t \in \langle x,y\rangle$. We have $\langle x,t\rangle \subseteq \langle x,y\rangle$, hence $f(\langle x,t\rangle) \subseteq f(\langle x,y\rangle)$. By the assumption xUy we have $f'(x) \leqslant' f'(y)$ and using 3.1 we obtain $f(\langle x,y\rangle) = \langle f'(x), f'(y)\rangle'$. Let $f(\langle x,t\rangle) = \langle a,b\rangle'$. Then $\langle a,b\rangle' \subseteq \langle f'(x), f'(y)\rangle'$, which implies $f'(x) \leqslant' a$. On the other hand $a = \inf\{f'(x), f'(t)\}$ in $\mathbb{A}'$ by 3.1, so that $a \leqslant' f'(t)$. Summarizing we obtain $f'(x) \leqslant' f'(t)$. We have xUt. The relation tUy can be shown analogously. $\square$

3.5. Lemma. *The above defined U, V fulfil* (P3) *and* (P3$'$).

P r o o f. We are going to show that (P3) holds. The condition (P3$'$) can be verified analogously. Let $u \leqslant x,y$, $xVuUy$. Then $f'(x) \leqslant' f'(u) \leqslant' f'(y)$. Let $\langle f'(x), f'(y)\rangle' = f(\langle a,b\rangle$. Using 3.1 and 3.2 we obtain $a = \inf\{x,y\}$, $b = \sup\{x,y\}$ in $\mathbb{A}$, $f'(x) = \inf\{f'(a), f'(b)\}$, $f'(y) = \sup\{f'(a), f'(b)\}$ in $\mathbb{A}'$. Since u is a lower bound of $\{x,y\}$, we infer $u \leqslant a$. Now $a \in \langle u,x\rangle \cap \langle u,y\rangle$, hence uUa and simultaneously uVa by 3.4. Since $u \leqslant a$, we have $f'(u) \leqslant' f'(a)$ and simultaneously $f'(u) \geqslant' f'(a)$. Then $f'(u) = f'(a)$ and consequently $u = a$. We have proved $u = \inf\{x,y\}$. It remains to show $yVbUx$. Since $b \geqslant x,y$, we have to verify $f'(x) \leqslant' f'(b) \leqslant' f'(y)$, but this is evident. $\square$

Summarizing, having an isomorphism $f\colon \operatorname{Int}\mathbb{A} \to \operatorname{Int}\mathbb{A}'$, we can construct binary relations U, V on A satisfying (P1)–(P3′). Further, using 2.14, we obtain a partially ordered set $\mathbb{A}_1$ such that Int $\mathbb{A}$ and Int $\mathbb{A}_1$ are isomorphic. The following theorem makes clear the relation between $\mathbb{A}'$ and $\mathbb{A}_1$.

3.6. Theorem. *Let $\mathbb{A} = (A, \leqslant)$ be a connected partially ordered set, $\mathbb{A}' = (A, \leqslant')$ any partially ordered set such that* $\operatorname{Int}\mathbb{A}'$ *is isomorphic to* $\operatorname{Int}\mathbb{A}$. *If U, V are defined with the aid of an isomorphism $f\colon \operatorname{Int}\mathbb{A} \to \operatorname{Int}\mathbb{A}'$ as above, then the partially ordered set $\mathbb{A}_1$ corresponding to U, V in the sense of* 2.14 *is isomorphic to $\mathbb{A}'$.*

Proof. We are going to show that the mapping f' belonging to the isomorphism $f\colon \operatorname{Int}\mathbb{A} \to \operatorname{Int}\mathbb{A}'$ is an isomorphism of $\mathbb{A}_1$ onto $\mathbb{A}'$. It is sufficient to prove that $x \leqslant_1 y$ if and only if $f'(x) \leqslant' f'(y)$. Let $x \leqslant_1 y$. Then there exist $u \leqslant x, y$ satisfying $xVuUy$. Using the definition of U, V we obtain $f'(x) \leqslant' f'(u) \leqslant' f'(y)$. Conversely, let $f'(x) \leqslant' f'(y)$. Considering $\langle a, b\rangle = f^{-1}(\langle f'(x), f'(y)\rangle')$ and using 3.2, 3.1, we obtain $a = \inf\{x, y\}$, $xVaUy$. Hence $x \leqslant_1 y$. The proof is complete. □

Notice that if $(A, \leqslant)$, $(A, \leqslant_1)$ are as in 2.14, then Int $(A, \leqslant)$, Int $(A, \leqslant_1)$ are not only isomorphic, but even identical as systems of subsets of A. Moreover, every $(A, \leqslant')$ satisfying that Int $(A, \leqslant')$ is identical with Int $(A, \leqslant)$ can be obtained in this way by 3.6.

To have 1.2 completely proved, we add:

3.7. Theorem. *Let $\mathbb{A} = \mathbb{C} \times \mathbb{D}$ be a connected partially ordered set, $\mathbb{A}' = \mathbb{C}^\delta \times \mathbb{D}$, let $f\colon \operatorname{Int}\mathbb{A} \to \operatorname{Int}\mathbb{A}'$ be defined by*

$$f(\langle (c_1, d_1), (c_2, d_2)\rangle) = \langle (c_2, d_1), (c_1, d_2)\rangle'.$$

Then f is an isomorphism and if U, V are defined with the aid of f as at the beginning of this section, they satisfy the conditions (P4) *and* (P5).

Proof. The assertion that f is an isomorphism is evident. Obviously, f' is the identity mapping, so that $(c_1, d_1)U(c_2, d_2)$ means that $c_1 = c_2$ and simultaneously d_1, d_2 are comparable while $(c_1, d_1)V(c_2, d_2)$ means that c_1, c_2 are comparable and $d_1 = d_2$.

Now let

$$(c, d) = (c_1, d_1)U(c_2, d_2)U \ldots U(c_n, d_n) = (c', d'),$$
$$(c, d) = (c_1', d_1')V(c_2', d_2')V \ldots V(c_m', d_m') = (c', d').$$

Then $c = c_1 = c_2 = \ldots = c'$ and $d = d'_1 = d'_2 = \ldots = d'$, so that $(c,d) = (c',d')$.

Finally, taking any $(c,d),(c',d') \in C \times D$ and using the fact that $\mathbb{A}$ is connected so that $\mathbb{C},\mathbb{D}$ are connected, too, we can find $c_1,\ldots,c_n \in C$, $d_1,\ldots,d_m \in D$ such that $c_1 = c$, $c_n = c'$, $d_1 = d$, $d_m = d'$, c_i is comparable with c_{i+1} for each $i \in \{1,\ldots,n-1\}$ and d_j is comparable with d_{j+1} for each $j \in \{1,\ldots,m-1\}$. Then we have $(c,d) = (c_1,d_1)U(c_1,d_2)U\ldots U(c_1,d_m) = (c_1,d')V(c_2,d')V\ldots V(c_n,d') = (c',d')$. The proof is complete. $\square$

Notice that the mapping Φ in Theorem 1.2 is not one-to-one, in general. For example, if $\mathbb{A}$ is a selfdual partially ordered set, then both $U_1 = \{(x,y) \in A \times A\colon\ x \nparallel y\}$, $V_1 = \{(x,x)\colon\ x \in A\}$ and $U_2 = \{(x,x)\colon\ x \in A\}$, $V_2 = \{(x,y) \in A \times A\colon\ x \nparallel y\}$ lead to the same isomorphism class of partially ordered sets. In this connection, a natural question arises: under what conditions two couples U_1, V_1 and U_2, V_2 of binary relations on A satisfying (P1)–(P3′) give the same isomorphism class of partially ordered sets.

Having a bijection α of A onto A, binary relations U_1, V_1 on A satisfying (P1)–(P3′) and the corresponding partial order $\leqslant_1$ (in the sense of 2.10), consider the following conditions:

(C1) $\alpha(x) \leqslant \alpha(y) \Longrightarrow$ there exists a unique couple of elements $p,q \in A$ satisfying $\alpha(x) \leqslant \alpha(p)$, $\alpha(q) \leqslant \alpha(y)$, $p \leqslant_1 x$, $y \leqslant_1 q$;

(C2) $p \leqslant_1 q \Longrightarrow \inf\{\alpha(p),\alpha(q)\}$, $\sup\{\alpha(p),\alpha(q)\}$ exist and

$$x = \alpha^{-1}(\inf\{\alpha(p),\alpha(q)\}),$$
$$y = \alpha^{-1}(\sup\{\alpha(p),\alpha(q)\})$$

are the only elements of A satisfying $\alpha(x) \leqslant \alpha(p)$, $\alpha(q) \leqslant \alpha(y)$, $p \leqslant_1 x$, $y \leqslant_1 q$.

3.8. Theorem. *Let U_1, V_1 and U_2, V_2 be two couples of binary relations on A satisfying* (P1)–(P3′), *let $\leqslant_1$ and $\leqslant_2$ be the corresponding partial orders (in the sense of* 2.10*). If α is an isomorphism of $(A,\leqslant_1)$ onto $(A,\leqslant_2)$, then*

(1) *α fulfils* (C1), (C2) *and*

(2) $U_2 = \{(x,y) \in A\times A\colon\ x \leqslant y$ *and* $\alpha^{-1}(x) \leqslant_1 \alpha^{-1}(y)\} \cup \{(x,y) \in A\times A\colon\ x \geqslant y$ *and* $\alpha^{-1}(x) \geqslant_1 \alpha^{-1}(y)\}$, $V_2 = \{(x,y) \in A \times A\colon\ x \leqslant y$ *and* $\alpha^{-1}(x) \geqslant_1 \alpha^{-1}(y)\} \cup \{(x,y) \in A \times A\colon\ x \geqslant y$ *and* $\alpha^{-1}(x) \leqslant_1 \alpha^{-1}(y)\}$.

Proof. We start with (2). Obviously $U_2 = \{(x,y) \in A \times A\colon\ x \leqslant y$ and $x \leqslant_2 y\} \cup \{(x,y) \in A \times A\colon\ x \geqslant y$ and $x \geqslant_2 y\}$. Since $r \leqslant_2 s$ $(r,s \in A)$ is equivalent to $\alpha^{-1}(r) \leqslant_1 \alpha^{-1}(s)$, we have what we need for U_2. As to V_2, we proceed analogously.

Now let $\alpha(x) \leqslant \alpha(y)$. Then there exists a unique couple of $\alpha(p), \alpha(q) \in \langle \alpha(x), \alpha(y) \rangle$ with $\alpha(p) V_2 \alpha(x) U_2 \alpha(q) V_2 \alpha(y) U_2 \alpha(p)$ by (P2). The latter is equivalent to $\alpha(p) \leqslant_2 \alpha(x)$, $\alpha(y) \leqslant_2 \alpha(q)$ and this holds if and only if $p \leqslant_1, x, y \leqslant_1 q$.

To prove (C2), take $p \leqslant_1 q$. Then $\alpha(p) \leqslant_2 \alpha(q)$ and there exists $\alpha(x) \leqslant \alpha(p), \alpha(q)$ satisfying $\alpha(p) V_2 \alpha(x) U_2 \alpha(q)$. Using (P3) we obtain that $\alpha(x) = \inf\{\alpha(p), \alpha(q)\}$, $\alpha(y) = \sup\{\alpha(p), \alpha(q)\}$ exists and we have $\alpha(q) V_2 \alpha(y) U_2 \alpha(p)$. The latter means $\alpha(p) \leqslant_2 \alpha(y) \leqslant_2 \alpha(q)$, which is equivalent to $p \leqslant_1 y \leqslant_1 q$. The relation $p \leqslant_1 x \leqslant_1 q$ follows from $\alpha(p) V_2 \alpha(x) U_2 \alpha(q)$. Now having x_1, y_1 satisfying $\alpha(x_1) \leqslant \alpha(p), \alpha(q) \leqslant \alpha(y_1)$, $p \leqslant_1 x_1$, $y_1 \leqslant_1 q$, it is easy to see that $\alpha(p) V_2 \alpha(x_1) U_2 \alpha(q) V_2 \alpha(y_1) U_2 \alpha(p)$, which yields $\alpha(x_1) = \inf\{\alpha(p), \alpha(q)\}$, $\alpha(y_1) = \sup\{\alpha(p), \alpha(q)\}$ by (P3) and (P3$'$). But then $\alpha(x_1) = \alpha(x)$, $\alpha(y_1) = \alpha(y)$ and consequently $x_1 = x$, $y_1 = y$.

The proof is complete. $\square$

Conversely, we have:

3.9. Theorem. *Let U_1, V_1 be a couple of binary relations on A satisfying* (P1)–(P3$'$) *and let α be any bijection of A onto A satisfying* (C1), (C2). *Taking U_2, V_2 as in* (2) *of* 3.8, *they satisfy the conditions* (P1)–(P3$'$) *and α is an isomorphism of $(A, \leqslant_1)$ onto $(A, \leqslant_2)$ ($\leqslant_1$ and $\leqslant_2$ are the partial orders corresponding to U_1, V_1 and U_2, V_2, respectively, in the sense of* 2.10*).*

P r o o f. The relations U_2, V_2 satisfy (P1) trivially. To prove (P2), let $x', y' \in A$, $x' \leqslant y'$. Take $x, y \in A$ with $\alpha(x) = x'$, $\alpha(y) = y'$. The condition (C1) yields the existence of a unique couple of elements $p, q \in A$ satisfying $\alpha(x) \leqslant \alpha(p)$, $\alpha(q) \leqslant \alpha(y)$, $p \leqslant_1 x, y \leqslant_1 q$. Set $\alpha(p) = p'$, $\alpha(q) = q'$. Then p', q' are the only elements of the interval $\langle x', y' \rangle$ with $p' V_2 x' U_2 q' V_2 y' U_2 p'$.

We are going to prove (P3). The proof of (P3$'$) would be analogous. Let $u' \leqslant p', q'$, $p' V_2 u' U_2 q'$. Take u, p, q with $\alpha(u) = u'$, $\alpha(p) = p'$, $\alpha(q) = q'$. Using the definition of U_2, V_2 we obtain $p \leqslant_1 u \leqslant_1 q$. Now (C2) ensures the existence of $\inf\{\alpha(p), \alpha(q)\}$ and $\sup\{\alpha(p), \alpha(q)\}$, together with $u' = \inf\{\alpha(p), \alpha(q)\}$. It remains to show that $q' V_2 \sup\{p', q'\} U_2 p'$, which is equivalent to $p \leqslant_1 \alpha^{-1}(\sup\{p', q'\}) \leqslant_1 q$. But this holds by (C2).

Finally, we have to prove that having any $x, y \in A$, $x \leqslant_1 y$ is equivalent to $\alpha(x) \leqslant_2 \alpha(y)$. Let $x \leqslant_1 y$. Then there exists $u \in \langle x, y \rangle_1$ such that $\alpha(u) = \inf\{\alpha(x), \alpha(y)\}$ by (C2). Now $\alpha(u) \leqslant \alpha(x)$ together with $u \geqslant_1 x$ yields $\alpha(u) V_2 \alpha(x)$, while $\alpha(u) \leqslant \alpha(y)$, $u \leqslant_1 y$ implies $\alpha(u) U_2 \alpha(y)$. Consequently $\alpha(x) \leqslant_2 \alpha(y)$. Conversely, let $\alpha(x) \leqslant_2 \alpha(y)$. Then there exists $\alpha(t) \leqslant \alpha(x), \alpha(y)$ with $\alpha(x) V_2 \alpha(t) U_2 \alpha(y)$. So we have $x \leqslant_1 t \leqslant_1 y$ and the proof is complete. $\square$

It is easy to see that if α is an automorphism or a dual automorphism of $\mathbb{A}$, then α satisfies the conditions (C1), (C2) for any U_1, V_1 fulfilling (P1)–(P3$'$) and the corresponding order $\leqslant_1$. But a bijection α fulfilling (C1), (C2) need not be an isomorphism or a dual isomorphism, as the following example shows:

3.10. Example. Let $\mathbb{A}$ be as in Fig. 4. Let $U_1 = \{(a,b) \in A \times A\colon\ a \not\parallel b\}$, $V_1 = \{(t,t)\colon\ t \in A\}$, $U_2 = V_1$, $V_2 = U_1$, $U_3 = \{(u,y),(y,u),(x,v),(v,x)\} \cup \{(t,t)\colon\ t \in A\}$, $V_3 = \{(x,u),(u,x),(y,v),(v,y)\} \cup \{(t,t)\colon\ t \in A\}$, $U_4 = V_3$, $V_4 = U_3$. It is easy to see that all couples satisfying (P1)–(P3′) are those of U_1, V_1, U_2, V_2, U_3, V_3 and U_4, V_4. But each of them yields the same isomorphism class of partially ordered sets with the interval system isomorphic to Int $\mathbb{A}$. E.g., the mapping α such that $\alpha(u) = v$, $\alpha(v) = u$, $\alpha(x) = x$, $\alpha(y) = y$ mediates the transition from U_1, V_1 to U_2, V_2, while Ψ defined by $\Psi(u) = y$, $\Psi(y) = v$, $\Psi(v) = x$, $\Psi(x) = u$ is an intermediary between U_1, V_1 and U_4, V_4.

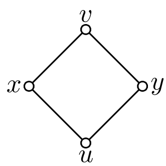

Fig. 4

The following question remains open:

Let $\mathbb{A}_1 \times \mathbb{A}_2$ be a direct decompositon of $\mathbb{A}$. Consider the class of all partially ordered sets isomorphic to $\mathbb{A}_1^\delta \times \mathbb{A}_2$. Does every pre-image (U,V) of this class under Φ satisfy (P4), (P5)? In particular, if the class $\Phi((U,V))$ consists of all partially ordered sets isomorphic to $\mathbb{A}$ (or $\mathbb{A}^\delta$), does (U,V) satisfy (P4), (P5)?

4.

In this section we will apply the foregoing results to the case of a directed partially ordered set.

4.1. Lemma. *Let $\mathbb{A} = (A, \leqslant)$ be a directed partially ordered set, U, V binary relations on A satisfying* (P1)–(P3′). *Then U, V satisfy also* (P4) *and* (P5).

P r o o f. Let $a = a_1Ua_2U\ldots Ua_n = a'$, $a = a'_1Va'_2V\ldots Va'_m = a'$. Take a lower bound x and an upper bound y of the set $\{a_1,\ldots,a_n,a'_1\ldots,a'_m\}$ and elements p, q as in (P2). Using 2.9 we get $\inf\{p,a\} = \inf\{p,a_1\} = \inf\{p,a_2\} = \ldots = \inf\{p,a'\}$ and analogously $\inf\{a,q\} = \inf\{a',q\}$. But then $a = a'$ by 2.8.

Further let $a, a' \in A$. Take a lower bound x and an upper bound y of $\{a,a'\}$ and p, q as in (P2). Then $p_1 = \inf\{p,a\}$, $q'_1 = \inf\{a',q\}$ satisfy pVp_1Ua, $a'Vq'_1Uq$, $p_1VxUq'_1$. (P3) ensures the existence of $t = \sup\{p_1,q'_1\}$ with q'_1VtUp_1. Hence $aUp_1UtVq'_1Va'$, completing the proof. $\square$

Using 1.2 we immediately get:

4.2. Theorem. *Let $\mathbb{A}$ be a directed partially ordered set. If $\mathbb{B}$ is a partially ordered set with* $\operatorname{Int}\mathbb{B}$ *isomorphic to* $\operatorname{Int}\mathbb{A}$*, then there exist partially ordered sets $\mathbb{C}$, $\mathbb{D}$ such that $\mathbb{A}$ is isomorphic to $\mathbb{C}\times\mathbb{D}$ and $\mathbb{B}$ is isomorphic to $\mathbb{C}^{\delta}\times\mathbb{D}$.*

The converse is evident, so we have:

4.3. Corollary. *Let $\mathbb{A}$ be a directed partially ordered set, $\mathbb{B}$ any partially ordered set. The following conditions are equivalent:*

(i) $\operatorname{Int}\mathbb{B}$ *is isomorphic to* $\operatorname{Int}\mathbb{A}$,

(ii) *there exist partially ordered sets $\mathbb{C}$, $\mathbb{D}$ such that $\mathbb{A}$ is isomorphic to $\mathbb{C}\times\mathbb{D}$ and $\mathbb{B}$ is isomorphic to $\mathbb{C}^{\delta}\times\mathbb{D}$.*

Since lattices are directed partially ordered sets, we obtain Theorem 1 of [11] as a consequence of 4.3. Let us notice that if $\mathbb{A}$ is a lattice and $\mathbb{B}$ is a partially ordered set with $\operatorname{Int}\mathbb{B}$ isomorphic to $\operatorname{Int}\mathbb{A}$, then $\mathbb{B}$ is also a lattice, as 4.3 shows.

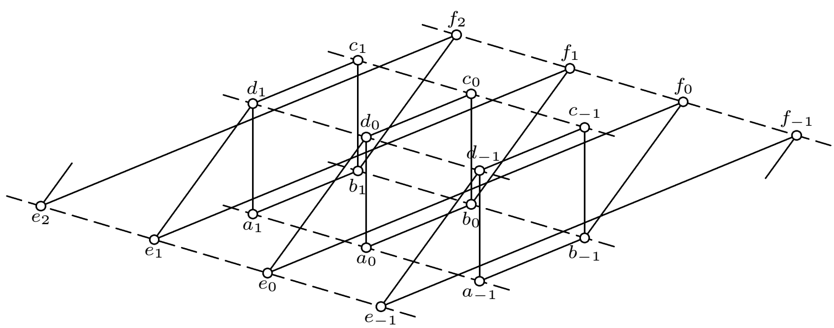

Fig. 5

Without the assumption that $\mathbb{A}$ is directed, the assertion of 4.3 is false. To show this, consider $\mathbb{A}$ as in Fig. 5. Let U and V be the relations marked out by the full and dashed lines, respectively. It is easy to see that U, V fulfil the conditions (P1)–(P3′) and (P5), but (P4) is not satisfied (e.g. e_0Ve_1 and simultaneously $e_0Ud_0Ua_0Ub_0Uf_1Ue_1$ holds). Taking the corresponding $\mathbb{A}_1$ (in the sense of 2.10) as $\mathbb{B}$, depicted in Fig. 6, it fulfils (i) of 4.3, while it fails to satisfy (ii) of 4.3.

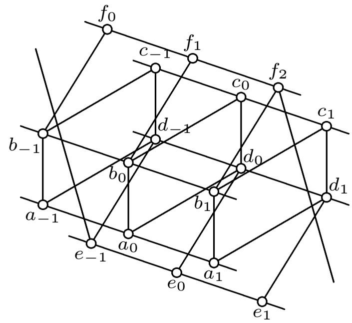

Fig. 6

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