James D. Stein

Fixed points of inequality-preserving maps in complete lattices

Czechoslovak Mathematical Journal, Vol. 49 (1999), No. 1, 35–43

Persistent URL: `http://dml.cz/dmlcz/127464`

Czechoslovak Mathematical Journal, 49 (124) (1999), 35–43

FIXED POINTS OF INEQUALITY-PRESERVING MAPS IN COMPLETE LATTICES

J.D. Stein Jr., Long Beach

(Received November 16, 1995)

Introduction

Easily the best-known fixed-point theorem for partially-ordered sets is Tarski's result that every isotone map T of a complete lattice L into itself has a fixed point ([1], p. 113). This theorem is noteworthy not only for the attractiveness and utility of the conclusion but for the brevity of the proof. Because the proof is so brief, we give it here, as the central ideas will appear repeatedly throughout this paper.

Let $H = \{x \in L\colon\ Tx \leqslant x\}$. H is non-empty since $1 \in H$, so let $h \in \Lambda\{x\colon\ x \in H\}$. If $x \in H$, $h \leqslant x$, and so $Th \leqslant Tx$ by the isotony of T. Since $x \in H \Longrightarrow Tx \leqslant x$, we see that $Th \leqslant x$. Taking the greatest lower bound of H shows that $Th \leqslant h$, and so $h \in H$. Since T is isotone, $Th \leqslant h \Longrightarrow T(Th) \leqslant Th$, and so $Th \in H$. Therefore $h \leqslant Th$, and so $Th = h$. $\square$

The proof revolves around the definition of the set H and the inequality-preserving property of T. Other than the crucial fact that the domain is a complete lattice, the hypotheses of Tarski's Theorem involve: (1) the set of pairs (x, y) satisfying $x \leqslant y$ for which the map preserves the order relation, and (2) the power (or powers) of T involved in preserving the order relation. In Tarski's Theorem, hypothesis (1) uses the entire set of pairs (x, y) satisfying $x \leqslant y$, and hypothesis (2) uses the first power of T, that is, $T^1x \leqslant T^1y$. The purpose of this paper is to investigate generalizations of Tarski's Theorem in which one or both of these hypotheses are weakened.

Throughout this paper, T will denote a map of a complete lattice L into itself.

Combining the basic proof of the Tarski Theorem with an elementary result from number theory enables us to prove the following result.

Theorem 1. *Let p and k be positive integers. Let $m = GCD(p,k)$, and suppose that $x \leqslant y \Longrightarrow T^p x \leqslant T^k y$. Then T^m has a fixed point.*

Proof. If $p = k$, the desired conclusion is simply the Tarski Theorem applied to the isotone map T^p. Assume that $p < k$. After the result has been proved in this case, the argument used will enable us to prove the result for the case $p > k$ by duality.

Let $H = \{x \in L\colon\ T^k x \leqslant x\}$. Obviously, $1 \in H$, and so we can define $h = \Lambda\{x\colon\ x \in H\}$. $x \in H \Longrightarrow h \leqslant x$, and so $T^p h \leqslant T^k x \leqslant x$. Taking the greatest lower bound of H shows that $T^p h \leqslant h$. Assume inductively that $T^{np} h \leqslant h$. Then $x \in H \Longrightarrow T^{np} h \leqslant x$. As before, $T^{(n+1)p} h \leqslant T^k x \leqslant x$, and so $T^{(n+1)p} h \leqslant h$. Therefore, $T^{np} h \leqslant h$ for all positive n. Consequently, $T^{(n+1)p} h \leqslant T^k h$, and continuing this process we deduce that $T^{(n+j)p} h \leqslant T^{jk} h$ for all positive n and j. Let $j = p - 1$ and $n = k - p + 1$. Then $(n + j)p = kp$, $jk = k(p - 1) = kp - k$. Therefore, $T^{kp} h \leqslant T^{kp-k} h$, and so $T^{kp-k} h \in H \Longrightarrow h \leqslant T^{kp-k} h$.

Observe that $T^{kp-k-p} h \leqslant T^{kp-k-p} h \Longrightarrow T^{kp-k} h \leqslant T^{kp-p} h$. We conclude that $T^{(k-1)p} h \leqslant h \leqslant T^{kp-k} h \leqslant T^{kp-p} h = T^{(k-1)p} h$, and so $h = T^{kp-k} h = T^{kp-p} h$. Therefore $T^{k-p} h = T^{k-p}(T^{kp-k} h) = T^{kp-p} h = h$, and so $T^p h = T^p(T^{k-p} h) = T^k h$.

Recall that $T^p h \leqslant h$. If $T^{jp} h \leqslant T^{(j-1)p} h$ for a positive integer j, then $T^{(j+1)p} h \leqslant T^{(j-1)p+k} h = T^{(j-1)p}(T^k h) = T^{(j-1)p}(T^p h) = T^{jp} h$. Therefore, $h \geqslant T^p h \geqslant \ldots \geqslant T^{(k-1)p} h = h$, and so $T^p h = T^k h = h$.

Since $m = GCD(p,k)$, there are integers a and b such that $m = ap + bk$. Note that if $j > 0$ then $T^{jk} h = T^{k+\cdots+k} h = h$, and similarly $T^{jp} h = h$. One of the integers a and b is positive and the other negative; we shall assume that $a > 0$ and $b < 0$ (the proof is similar in the other case). Now $T^m h = T^{ap+bk} h = T^{ap+bk}(T^{-bk} h) = T^{ap} h = h$, and the proof is complete. □

The following elementary example indicates that the above theorem may well be the best possible result. Let L be the two-point lattice $\{0,1\}$, $T0 = 1$, $T1 = 0$. Then $x \leqslant y \Longrightarrow T^2 x \leqslant T^2 y$, but T has no fixed point. The same map also satisfies the hypotheses $x \leqslant y \Longrightarrow T^2 x \leqslant T^4 y$ or $x \leqslant y \Longrightarrow T^4 x \leqslant T^2 y$.

It should also be noted that Theorem 1 cannot simply be proved as a consequence of Tarski's Theorem by showing that if $m = GCD(p,k)$, then T^m is isotone. Let L be the lattice of subsets of $\{0,1\}$, and define T by $T(\{0,1\}) = \{0\}$, $T(\{0\}) = \{1\}$, and $T(\{1\}) = T(\emptyset) = \emptyset$. If x is any element of L, $T^3 x = \emptyset$, so $x \leqslant y \Longrightarrow T^3 x \leqslant$

Ty. $GCD(3,1) = 1$, but T is not isotone, as $\{0\} \leqslant \{0,1\}$ and $T(\{0\}) = \{1\}$, $T(\{0,1\}) = \{0\}$.

A relation is a set of ordered pairs. Let R be the set of all ordered pairs (x, y) such that $x \leqslant y$. If $T\colon L \longrightarrow L$, define $T^*\colon L \times L \longrightarrow L \times L$ by $T^*(x,y) = (Tx, Ty)$. T is isotone iff $T^*(R)$ is a subset of R. We now investigate the question of whether we can obtain a fixed-point theorem if we require $T^*(E)$ to be a subset of R for some proper subset E of R. The next theorem not only answers this question in the affirmative, but can be shown to be sharp.

Theorem 2. *Let k and n be positive integers. Assume that for x, $y \in X$, $T^n x \leqslant y \Longrightarrow T^k(T^n x) \leqslant T^k y$ and $x \leqslant T^n y \Longrightarrow T^k x \Longrightarrow T^k x \leqslant T^k(T^n x)$. Then T^k has a fixed point.*

P r o o f. Let $H = \{x \in L\colon x \geqslant T^{nk}x\}$. Clearly, $1 \in H$. If $x \in H$, then $T^{nk}x \leqslant x$, and so $T^n(T^{(k-1)n}x) \leqslant x$. Therefore $T^k(T^{nk}x) \leqslant T^k x$, and so $T^{nk}(T^k x) \leqslant T^k x$. This implies that $T^k x \in H$. So $x \in H \Longrightarrow T^{jk}x \in H$ for $j = 1, 2, \ldots$. Notice that if $x \in H$, $x \geqslant T^{nk}x \geqslant T^{nk}(T^{nk}x) = T^{2nk}x \geqslant \ldots$.

As usual, let $h = \Lambda\{x\colon x \in H\}$. Let q be so large that $(qn-1)k - n \geqslant 0$. Let $x \in H$. Then $T^{(qn-1)k}x \in H$, and so $h \leqslant T^{(qn-1)k}x = T^n T^{(qn-1)k-n}x$. Consequently, $T^k h \leqslant T^k(T^{(qn-1)k}x) = T^{qnk}x \leqslant x$. Therefore $T^k h \leqslant h$.

Assume inductively that $T^{pk}h \leqslant h$ for a positive integer p. If we repeat the argument of the above paragraph, using $T^{pk}h$ in place of h on the lower end of the inequalities, we can show that $T^k(T^{pk}h) = T^{(p+1)k} \leqslant h$. We can therefore conclude that $T^{nk}h \leqslant h$, and so $h \in H$. We know from earlier work that $h \in H \Longrightarrow T^k h \in H$, and so $h \leqslant T^k h$. However, we also have shown that $T^k h \leqslant h$, and so $T^k h = h$. $\square$

We now present three examples to illustrate ways in which the above theorem cannot be strengthened.

Example 1. Let $L = \{0,1\}$ and define $T0 = 1$, $T1 = 0$. Since T^2 is the identity, $x \leqslant Ty \Longrightarrow T^2 x \leqslant T^2(Ty)$ and $Tx \leqslant y \Longrightarrow T^2(Tx) \leqslant T^2 y$, but although T^2 has fixed points, T does not.

The subset E of R referred to in the discussion prior to Theorem 2 is the collection of all pairs (u, v) such that one of the two elements belongs to L, the other to $T^n(L)$. The following example shows that one of the elements must be allowed to range over all of L.

Example 2. Let $L = [0,1]$, and define $Tx = \frac{1}{2}(1+x)$ if $0 \leqslant x < 1$ and $T1 = 0$. T obviously has no fixed points, and is monotone increasing on $[0,1)$. So, if x, $y \in [0,1]$, Tx and $Ty \in [0,1)$. If $Tx \leqslant Ty$, then $T(Tx) \leqslant T(Ty)$. This example illustrates that the domain of both of the variable in the definition of the subset E of R cannot be restricted to the range of T.

One might inquire as to whether it is necessary to have both $T^n x \leqslant y \Longrightarrow T^k(T^n x) \leqslant T^k y$ and $x \leqslant T^n y \Longrightarrow T^k x \leqslant T^k(T^n x)$; perhaps one of these implications suffices to guarantee the existence of a fixed point. The previous example shows that this cannot be done. Since T is monotone on $[0,1)$, and this set also contains the range of T, $x \leqslant Ty \Longrightarrow Tx \leqslant T(Ty)$. Of course, T does not satisfy $Tx \leqslant y \Longrightarrow T(Tx) \leqslant Ty$; simply let $y = 1$.

Example 3. Let L be the lattice of all subsets of the positive integers. If $F \in L$ is either empty or finite, define $T(F)$ to be $F \cup \{f\}$, where f is the smallest integer not in F. If F is infinite, define $T(F)$ to be the result of removing the smallest integer in F from F. Then if $F \leqslant T(F)$, $T(F) \leqslant T\big(T(F)\big)$, and if $T(F) \leqslant F$, $T\big(T(F)\big) \leqslant T(F)$, but T has no fixed or periodic points. This example shows that requiring T^* to map the set $E = \big\{(x,y)\colon\ y = Tx \text{ or } x = Ty\big\}$ into R is insufficient to insure the existence of a fixed point.

It is of interest to discover situations in which families of maps have simultaneous fixed points.

Suppose that T has a fixed point x and S commutes with T. Then $Sx = S(Tx) = T(Sx)$, so Sx is a fixed point of T. If the fixed point of T is unique (as, for example, is the case in the Banach Contraction Principle in a complete metric space), then $Sx = x$, and so x is also a fixed point of S.

This need not be the case if T has multiple fixed points. Let $X = \{1,2,3,4\}$, and define operators S and T on X as follows:

$$\begin{array}{cccc} S1 = 1 & S2 = 3 & S3 = 2 & S4 = 4 \\ T1 = 4 & T2 = 2 & T3 = 3 & T4 = 1 \end{array}$$

It is easy to verify that S and T commute, but they do not have a simultaneous fixed point.

This situation changes if we have a family of commuting maps on a complete lattice which satisfy a condition similar to that imposed in Theorem 2. The theorem that results is analogous to the Markov-Kakutani fixed point theorem ([2], p. 456) for commuting continuous linear maps on a compact convex subset of a linear topological space.

Theorem 3. *Assume that $\mathscr{F}$ is a commuting family of maps on L satisfying the following hypothesis: if S, $T \in \mathscr{F}$, then $x \leqslant Sy \Longrightarrow Tx \leqslant T(Sy)$ and $Sx \leqslant y \Longrightarrow T(Sx) \leqslant Ty$. Then $\mathscr{F}$ has a simultaneous fixed point.*

P r o o f. Let $H = \{x \in L\colon Tx \leqslant x \text{ for all } T \in \mathscr{F}\}$. Since $1 \in H$, let $h = \Lambda\{x\colon x \in H\}$.

We first show that if $x \in H$ and $T \in \mathscr{F}$, then $Tx \in H$. Let $S \in \mathscr{F}$. If $x \in H$, $Sx \leqslant x$, and so $T(Sx) \leqslant Tx$. Since S and T commute, $S(Tx) \leqslant Tx$, and so $Tx \in H$.

We now show that $h \in H$. If $x \in H$ and $T \in \mathscr{F}$, then $Tx \in H$, and so $h \leqslant Tx$. So $Th \leqslant T(Tx)$. Since $Tx \in H$, it follows that $T(Tx) \leqslant Tx$. Since $x \in H$, we see that $Tx \leqslant x$. Therefore $Th \leqslant x$ for all $x \in H$, and so $Th \leqslant h$.

Let $T \in \mathscr{F}$. Since $h \in H \Longrightarrow Th \in H$, we see that $h \leqslant Th$. Combining this with the previous result that $Th \leqslant h$, we see that h is a simultaneous fixed point for all $T \in \mathscr{F}$. □

The same conclusion can be reached if the family of maps $\mathscr{F}$ forms a group, although the hypothesis in this situation is no longer as restrictive. Because the identity $I \in \mathscr{F}$, the hypothesis of Theorem 3 is equivalent to requiring that all $T \in \mathscr{F}$ be isotone. If we let $H = \{x \in L\colon Tx \leqslant x \text{ for all } T \in \mathscr{F}\}$ and $h = \Lambda\{x\colon x \in H\}$ as usual, $x \in H$ and $T \in \mathscr{F} \Longrightarrow h \leqslant x$, and so $Th \leqslant Tx \leqslant x$. So $Th \leqslant h$ for all $T \in \mathscr{F}$. If $T \in \mathscr{F}$, because $\mathscr{F}$ is a group, $T^{-1} \in \mathscr{F}$, and so $T^{-1}h \leqslant h$. Therefore $h = T(T^{-1}h) \leqslant Th$ for all $T \in \mathscr{F}$, and so h is a simultaneous fixed point of $\mathscr{F}$.

Non-commuting families of isotone maps need not have a simultaneous fixed point. If $L = \{0, 1\}$ and $Sx = 0$, $Tx = 1$ for all $x \in L$, then even though both S and T are isotone (and thus have fixed points), they do not commute, and have different fixed points.

We conclude this section with a result that includes aspects of both Theorems 1 and 2.

Theorem 4. *Let n and k be positive integers, and let $p = GCD(n, k)$. Assume that $x \leqslant Ty \Longrightarrow T^n x \leqslant T^n(Ty)$ and $Tx \leqslant y \Longrightarrow T^k(Tx) \leqslant T^k y$. Then T^p has a fixed point.*

P r o o f. Let $H = \{x \in L\colon T^{nk}x \leqslant x\}$; $1 \in H$, so we can define $h = \Lambda\{x\colon x \in H\}$.

Observe that $x \in H \Longrightarrow T^{nk}x \leqslant x$, and so $T^{nk}(T^k x) = T^k(T^{nk}x) \leqslant T^k x$. Therefore $T^k x \in H$. We can continue this procedure to show that if $x \in H$, then $T^{jk}x \in H$ for any positive integer j. We also have $x \in H \Longrightarrow x \geqslant T^{nk}x \geqslant T^{2nk}x \geqslant \ldots$.

Let $x \in H$. Since $T^{nk}x \in H$, $h \leqslant T^{nk}x \Longrightarrow T^n h \leqslant T^n(T^{nk}x) = T^{nk+n}x$. Continuing to apply T^n to both sides of this inequality, we obtain $T^{nk}h \leqslant T^{2nk}x \leqslant x$. Taking the greatest lower bound, $T^{nk}h \leqslant h$. We can apply T^k to both sides of this inequality to obtain $T^{nk}(T^k h) = T^k(T^{nk}h) \leqslant T^k h$, so $T^k h \in H$.

We now continue to apply T^k to both sides of the last inequality, eventually obtaining $T^{nk}(T^{nk}h) \leqslant T^{nk}h$. Therefore, $T^{nk}h \in H$, and so $h \leqslant T^{nk}h$. Since we already have $T^{nk}h \leqslant h$, we see that $T^{nk}h = h$.

Recall that $T^k h \in H$, so $h = T^{nk}h \leqslant T^k h$. This latter form of the inequality allows us to apply T^k repeatedly to both sides of the inequality $h \leqslant T^k h$, eventually obtaining $h \leqslant T^k h \leqslant T^{2k}h \leqslant \ldots \leqslant T^{nk}h = h$. Therefore $T^k h = h$.

Again, since $h = T^{nk}h$, $T^n h = T^n(T^{nk}h)$, and so $T^n h \geqslant T^{nk}(T^n h)$. Therefore $T^n h \in H$, and so $h \leqslant T^n h$. Since $h = T^k h$ is in the range of T, we can apply T^n repeatedly to both sides of the inequality $h \leqslant T^n h$, obtaining $h \leqslant T^n h \leqslant T^{2n}h \leqslant \ldots \leqslant T^{nk}h = h$. Therefore $T^n h = h$.

Since $T^k h = T^n h = h$, the argument given at the end of Theorem 1 enables us to conclude that $T^p h = h$, where $p = GCD(n, k)$. □

The same example given after Theorem 1 shows that, in general, this result cannot be improved. Let $L = \{0, 1\}$, and define $T0 = 1$, $T1 = 0$. Then T^2 is the identity, so letting $n = k = 2$ in Theorem 4, we see that $Tx \leqslant y \Longrightarrow T^2(Tx) \leqslant T^2 y$ and $x \leqslant Ty \Longrightarrow T^2 x \leqslant T^2(Ty)$. So $GCD(n, k) = 2$, but T has no fixed points.

2. Banach limits, Fatou's lemma, and fixed points

Much of analysis revolves around sequences. Many ideas in analysis have the form " …there is an integer N such that $n \geqslant N \Longrightarrow \ldots$". This idea, applied to powers of the map T, motivates the results in this section.

An important result in functional analysis is the Hahn-Banach Theorem, which can be used to demonstrate the existence of a linear functional on 1^∞ which assigns to a bounded sequence of real numbers $\{s_n\colon n = 1, 2, \ldots\}$ a real number which in some sense generalizes the idea of the limit of a sequence. Adopting the notation of ([2], p. 73), we denote the value of this functional by $\operatorname*{LIM}_{n\to\infty} s_n$. The relevant properties of $\operatorname*{LIM}_{n\to\infty}$ are

(1) $\operatorname*{LIM}_{n\to\infty} s_n = \operatorname*{LIM}_{n\to\infty} s_{n+1}$,
(2) $\liminf_{n\to\infty} s_n \leqslant \operatorname*{LIM}_{n\to\infty} s_n \leqslant \limsup_{n\to\infty} s_n$,
(3) $\operatorname*{LIM}_{n\to\infty} s_n \geqslant 0$ if $s_n \geqslant 0$ for $n = 1, 2, \ldots$

This functional is called a Banach limit. These properties are reflected in the following definition.

Definition 1. Let $1^\infty(L)$ denote the space of all sequences from L. We say that a map LIM from $1^\infty(L)$ to L is a lower Banach limit if for any sequence $\{x_n\colon n = 1, 2, \ldots\}$ of elements in L,

(1) $\mathrm{LIM}\, x_{n+1} \leqslant \mathrm{LIM}\, x_n$,

(2) if $x_n = a$ for all n, then $\operatorname{LIM} x_n = a$,
(3) $x_n \leqslant y_n$ for all but finitely many $n \Longrightarrow \operatorname{LIM} x_n \leqslant \operatorname{LIM} y_n$.

Two useful examples of lower Banach limits that can be defined in any complete (or countably complete) lattice are

$$\lim_{n\to\infty} \inf a_n = \bigvee_{N=1}^{\infty} \bigwedge_{n=N}^{\infty} a_n,$$
$$\lim_{n\to\infty} \sup a_n = \bigwedge_{N=1}^{\infty} \bigvee_{n=N}^{\infty} a_n.$$

Theorem 5. *Let* LIM *be a lower Banach limit on L. Consider the following hypotheses.*
(1) $x \leqslant y \Longrightarrow \exists N = N(x,y)$ *such that* $n \geqslant N \Longrightarrow T^n x \leqslant T^n y$,
(2) $\operatorname{LIM} a_{n+1} = \operatorname{LIM} a_n$.
If $x \in L$, let $Qx = \operatorname{LIM} T^n x$. Then
(a) (1) $\Longrightarrow Q$ *has a fixed point,*
(b) (1) & (2) $\Longrightarrow QT$ *has a fixed point,*
(c) (1), (2) & $QT = TQ \Longrightarrow T$ *has a fixed point.*

Proof. If (1) holds, property (3) of lower Banach limits shows that Q is isotone, and so has a fixed point by Tarski's theorem. If (2) also holds, let x be a fixed point of Q. Then we have $QTx = \operatorname{LIM} T^n(Tx) = \operatorname{LIM} T^{n+1}x = \operatorname{LIM} T^n x = Qx = x$. If Q and T also commute, again let x be a fixed point of Q. Then $QTx = x$ as before, and $Tx = TQx = QTx = x$. □

A basic result from integration theory is the following inequality, which is known as Fatou's Lemma ([3], p. 22).

$$\int \Big(\lim_{n\to\infty} \inf f_n\Big)\, \mathrm{d}\mu \leqslant \lim_{n\to\infty} \inf \int f_n \,\mathrm{d}\mu.$$

Definition 2. Let LIM be a lower Banach limit on a complete lattice L. A map $T\colon L \longrightarrow L$ is said to satisfy Fatou's Condition if for any sequence $\{x_n\colon n = 1, 2, \ldots\}$ in L,

$$T(\operatorname{LIM} x_n) \leqslant \operatorname{LIM} T x_n.$$

With these definitions in hand, we can now prove a fixed point theorem rooted in analytical concepts.

Theorem 6. *Let* LIM *be a lower Banach limit on L, and assume that T satisfies Fatou's Condition. Assume further that for each pair x, $y \in L$ satisfying $x \leqslant y$,* *there exists an integer $N = N(x, y)$ such that $n \geqslant N \Longrightarrow T^n x \leqslant T^n y$. Then T has a fixed point.*

P r o o f. Let $H = \{x \in L\colon \operatorname{LIM} T^n x \leqslant x\}$. Since $1 \in H$, H is non-empty. Since L is complete, let $h = \Lambda\{x\colon x \in H\}$.

Let $x \in H$. Since $h \leqslant x$, there is an integer N such that $n \geqslant N \Longrightarrow T^n h \leqslant T^n x$. By property (3) of lower Banach limits, $\operatorname{LIM} T^n h \leqslant \operatorname{LIM} T^n x \leqslant x$. Let $b = \operatorname{LIM} T^n h$. Taking the greatest lower bound over all $x \in H$, we see that $b \leqslant h$.

Since T satisfies Fatou's Condition, and using property (1) of lower Banach limits, $Tb = T(\operatorname{LIM} T^n h) \leqslant \operatorname{LIM} T^{n+1} h \leqslant \operatorname{LIM} T^n h = b$. We can therefore find an integer I such that $n \geqslant I \Longrightarrow T^n T b \leqslant T^n b$. Therefore, $n \geqslant I \Longrightarrow T^{n+1} b \leqslant T^n b$. The sequence $\{T^n b\colon n \geqslant I\}$ forms a decreasing chain, and using properties (2) and (3) of lower Banach limits we see that $\operatorname{LIM} T^{I+n} b \leqslant T^I b$, and so $T^I b \in H$, as does $T^j b \in H$ for $j \geqslant I$. So $b \leqslant h \leqslant T^I b$.

Choose an integer J such that $n \geqslant J \Longrightarrow T^n b \leqslant T^n T^I b = T^{n+I} b$. If $p = \max(I, J)$, then $T^p b \leqslant T^{p+I} b \leqslant \ldots \leqslant T^p b$, since $p \geqslant I$ and the sequence $\{T^n b\colon n \geqslant I\}$ forms a decreasing chain.

Therefore, $T^p b = T^{p+1} b = T(T^p b)$, and T has a fixed point. $\square$

The above theorem holds under the assumptions that LIM is an upper Banach limit ($\operatorname{LIM} x_n \leqslant \operatorname{LIM} x_{n+1}$) an the reverse of Fatou's Condition applies; i.e. $T(\operatorname{LIM} x_n) \leqslant \operatorname{LIM} T x_n$, by simply dualizing the proof.

It is possible to prove a fixed-point theorem assuming only (1) of Theorem 5 if L is a complete chain.

Theorem 7. *Assume that L is a complete chain, and that T satisfies $x \leqslant y \Longrightarrow \exists N = N(x, y)$ such that $n \geqslant N \Longrightarrow T^n x \leqslant T^n y$. Then T has a fixed point.*

P r o o f. Suppose that T has no fixed point. Let $A = \{x \in L\colon x \geqslant Tx\}$, $B = \{x \in L\colon x \leqslant Tx\}$. Since T does not have a fixed point, A and B are disjoint; clearly $0 \in B$, $1 \in A$.

Suppose that $x \in A$. If $\exists p$ such that $T^p x \leqslant T^{p+1} x$, then $\exists N_1$ such that $n \geqslant N_1 \Longrightarrow T^{p+n} x \leqslant T^{p+n+1} x$. Since $x \geqslant Tx$, $\exists N_2$ such that $n \geqslant N_2 \Longrightarrow T^n x \geqslant T^{n+1} x$. If $n \geqslant \max(N_1 + p, N_2)$, then $T^{n+1} x \leqslant T^n x \leqslant T^{n+1} x$, and so T has a fixed point. Therefore, $x \in A \Longrightarrow T^p x \geqslant T^{p+1} x$, and so $\lim_{n\to\infty} \inf T^n x \leqslant x$. Similarly, $\lim_{n\to\infty} \inf T^n x \geqslant x$ for $x \in B$.

Let $a = \Lambda\{x\colon x \in A\}$. $x \in A \Longrightarrow a \leqslant x$, and so $\exists N$ such that $n \geqslant N \Longrightarrow T^n a \leqslant T^n x$. Therefore $\lim_{n\to\infty} \inf T^n a \leqslant \lim_{n\to\infty} \inf T^n x \leqslant x$. So $\lim_{n\to\infty} \inf T^n a \leqslant a$. If $a \in B$, then $\lim_{n\to\infty} \inf T^n a = a \Longrightarrow a = Ta = T^2 a = \ldots$, and T has a fixed point. Therefore $a \in A$.

If $Ta < a$, then $Ta \in A$, since $a > Ta \geqslant T^2 a = T(Ta) \geqslant \ldots$), which contradicts the fact that a is a lower bound for A. So $Ta = a$, and T has a fixed point. $\square$

The condition $x \leqslant y \Longrightarrow \lim_{n\to\infty} \inf T^n x \leqslant \lim_{n\to\infty} \inf T^n y$ is not sufficient to guarantee the existence of a fixed point. Let $L = [0,1]$ with the usual order. Define $Tx = \frac{1}{2}x$ if $x > 0$ and $T0 = 1$. For each $x \in [0,1]$, $T^n x \longrightarrow 0$, and so the above condition is trivially satisfied, but T has no fixed point.

Conditions involving powers of T are lattice-theoretic in nature, but lower Banach limits and Fatou's Condition have obvious analytic antecedents. In a sense, the previous example points out a deficiency that the hypotheses of Theorem 5 remedy. Let $a_n = 1/n$. Then $\lim_{n\to\infty} \inf Ta_n = 0$, but $T(\lim_{n\to\infty} \inf a_n) = 1$.

If this example is modified by defining $Tx = \frac{1}{2}(1+x)$ if $x < 1$ and $T1 = 0$, then T satisfies the following hypothesis: $x \leqslant y \Longrightarrow \exists N = N(x,y)$ such that $n \geqslant N \Longrightarrow Tx \leqslant T^n y$, but T does not have a fixed point. The failure of this hypothesis to result in a fixed point is reminiscent of a similar situation in an example following Theorem 2.

References

[1] *G. Birkhoff*: Lattice Theory. American Mathematical Society, Providence R.I., 1967.
[2] *N. Dunford, J. Schwartz*: Linear Operators, Part I. Interscience, New York N.Y., 1958.
[3] *W. Rudin*: Real and Complex Analysis. McGraw-Hill, New York, 1966.

Author's address: Department of Mathematics, California State University, Long Beach, 1250 Bellflower Blvd., Long Beach, Ca. 90840, U.S.A.