Czechoslovak Mathematical Journal

Mukul Pal; Mrityunjoy Nath

Certain transformations T_ω and Lebesgue measurable sets of positive measure

Czechoslovak Mathematical Journal, Vol. 49 (1999), No. 1, 1–12

Persistent URL: http://dml.cz/dmlcz/127461

Terms of use:

© Institute of Mathematics AS CR, 1999

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://dml.cz

Czechoslovak Mathematical Journal, 49 (124) (1999), 1–12

CERTAIN TRANSFORMATIONS T_ω AND LEBESGUE MEASURABLE SETS OF POSITIVE MEASURE

Mukul Pal and Mrityunjoy Nath, Kalyani

(Received October 30, 1995)

Suppose that for every ω belonging to a metric space Ω, there is a certain transformation T_ω transforming a Lebesgue measurable set in $\mathbb{R}_N$ (N-dimensional Euclidean space) into a Lebesgue measurable set in $\mathbb{R}_N$.

In [1] T. Neubrunn and T. Šalát introduced this type of transformations T_ω for measurable sets of the real line satisfying certain conditions.

In [2] M. Pal considered such transformations T_ω for measurable sets in $\mathbb{R}_N$ satisfying the following conditions which are equivalent to the conditions as introduced by T. Neubrunn and T. Šalát for transformations T_ω transforming a measurable set of the real line into a measurable set of the real line provided $N = 1$.

(I) There exists $\omega_0 \in \Omega$ such that for every closed ball $K = B[a, r] \subset \mathbb{R}_N$ with centre a and radius r and for every sequence $\{\omega_n\}$ ($\omega_n \in \Omega$) converging to ω_0,

$$\lim_{n\to\infty} \big[\sup \big\{ |a - T_{\omega_n}(K)| \big\} \big] = r$$

holds where the symbol $\big\{|a - A|\big\}$, $a \in A$, $A \in \mathbb{R}_N$ denotes the set of all numbers $|a - x|$, $x \in A$.

(II) If E and F are measurable sets in $\mathbb{R}_N$ with $F \subset E$ then $T_\omega(F) \subset T_\omega(E)$ for every $\omega \in \Omega$.

(III) For $\omega_0 \in \Omega$ as in (I), for every sequence $\{\omega_n\}$ ($\omega_n \in \Omega$) converging to ω_0 and for every measurable set E,

$$\lim_{n\to\infty} \big|T_{\omega_n}(E)\big| = \big|T_{\omega_0}(E)\big| = |E|$$

Keywords: N-dimensioanal Euclidean space, metric space, Lebesque measure, T_ω transformation

MSC 1991: 28A05

where $|E|$ denotes the Lebesgue measure of the set E. Then among other results, M. Pal in [2] proved the following theorem which extends a theorem [Theorem 1.1] proved by T. Neubrunn and T. Šalát in [1].

Theorem 1 [2]. *Let $T_{\omega_n}(\omega_n \in \Omega)$ be transformations satisfying the conditions* (I), (II), (III) *and let the sequence $\{\omega_n\}$ converge to ω_0 (in Ω). Let A be a set of positive measure in $\mathbb{R}_N$. Then there exists a natural number N_0 such that for $n \geqslant N_0$, $A \cap T_{\omega_n}(A)$ is a set of positive measure.*

In [4] N.G. Saha and K.C. Ray also extended the above theorem considering a family of transformations like T_ω transforming a set in α^N (the collection of all measurable subsets of $\mathbb{R}_N$) into a set α^N and satisfying (II) and (III) mentioned above and the following condition stated in the form we consider here:

(I′) Let a, $b \in \mathbb{R}_N$ and let there exist a point $\omega_0 \in \Omega$ such that for every sequence $\{\omega_n\}$ $(\omega_n \in \Omega)$ converging to ω_0,

$$\lim_{n\to\infty} \left[\sup \left\{|b - T_{\omega_n}(K)|\right\}\right] = r$$

holds for every ball $K = B[a, r]$, $r > 0$.

The purpose of the paper is to study some properties of sets in $\mathbb{R}_N$ under transformations like T_ω which transform a measurable set in $\mathbb{R}_N$ into a measurable set in $\mathbb{R}_N$.

In this paper we relax the conditions (I) and (II) as considered by M. Pal in [2] and extend the result of Theorem 1 of [2] as stated earlier. Theorem 2 of this paper gives an extension of the result of a theorem in [4] by relaxing the condition (I′) as considered by N.G. Saha and K.C. Ray. We prove Theorem 3 in which a further extension of Theorem 2 is achieved. Before going into details we explain some notation used in the sequel.

Notation.

1) $B[c, \varrho]$ stands for the closed ball with centre c and radius ϱ while $B(c, \varrho)$ denotes the open ball with the same centre and radius.
2) $|x|$ denotes the norm of the vector $x \in \mathbb{R}_N$, while $|x|$ stands for the absolute value of the real number x.
3) $A \setminus B$ denotes the set of all points of the set A which do not belong to the set B.
4) For a set $A \subset \mathbb{R}_N$ and a non-zero real number α, αA is the set $\{\alpha x\colon\ x \in A\}$.
5) For sets A and B, $A - B$ denotes the difference set

$$\{a - b\colon\ a \in A, b \in B\}.$$

6) For a ball $K = B[a, r] \subset \mathbb{R}_N$, d_n stands for $\sup \left\{\left|a - T_{\omega_n}(K)\right|\right\}$.

Throughout the paper Ω is a metric space and the set under consideration is a set in $\mathbb{R}_N$. Now we introduce the conditions (i) and (iii) which are less restrictive than the conditions (I) and (III) as considered by M. Pal in [2].

(i) Let there exist a point $\omega_0 \in \Omega$ such that for every sequence $\{\omega_n\}$ ($\omega_n \in \Omega$) converging to ω_0 and for every sequence $\{\alpha_n\}$ of non-zero real numbers converging to a non-zero real number α_0 and for every ball $K = B[a, r]$,

$$\lim_{n\to\infty} \big[\sup\big\{|\alpha_n a - T_{\omega_n}(\alpha_n K)|\big\}\big] = |\alpha_0| r.$$

(ii) For every $\omega \in \Omega$ and for sets E and F with $F \subset E$, let

$$T_\omega(F) \subset T_\omega(E).$$

(iii) For $\omega_0 \in \Omega$ as in (i), for every sequence $\{\omega_n\}$ ($\omega_n \in \Omega$) converging to ω_0, for every sequence $\{\alpha_n\}$ of non-zero real numbers converging to a non-zero real number α_0 and for any measurable set E,

$$\lim_{n\to\infty} \big|T_{\omega_n}(\alpha_n E)\big| = \big|T_{\omega_0}(\alpha_0 E)\big| - |\alpha_0 E|.$$

If we take $\alpha_n = i$, $n = 1, 2, \ldots$, then (i) and (iii) reduce to the conditions (I) and (II).

In proving the results we follow the method as adopted by K. C. Ray in [3] with the necessary modifications. In this connection we note a well known result [5], viz. if T is a linear transformation in $\mathbb{R}_N$ given by $x'_i = \sum_{j=1}^{N} a_{ij} x_j$, $i = 1, 2, \ldots, N$ and a_{ij}'s are real numbers and if E is a measurable set in $\mathbb{R}_N$, then $\big|T(E)\big| = \delta |E|$ where δ is the absolute value of the determinant of T.

As a Corollary of this result, it can be easily deduced that if α is a real number and E is a measurable set in $\mathbb{R}_N$, then $|\alpha E| = |\alpha|^N E$.

To substantiate our conditions (i) and (ii) as introduced above we present the following examples:

Examples. Let $\mathbb{R}$ be the real line. Then $\mathbb{R}$ is a metric space with the usual metric.

(i) Let $T_{\omega_n}(E) = E + \frac{1}{2n}$, E being a measurable subset of $\mathbb{R}$ and let $\alpha_n = \big(2 - \frac{1}{2n}\big)$. Then $\lim_{n\to\infty} \alpha_n = 2$. So, for a closed interval $[1, 5]$

$$K = \sup_{x\in K} \left|3(\alpha_n) - T_{\omega_n}\left[\left(2 - \frac{1}{2n}\right)x\right]\right|,$$

where 3 is the middle point of the interval $[1, 5]$.

$$= \sup_{x\in K} \left|3\left(2 - \frac{1}{2n}\right) - \left[\left(2 - \frac{1}{2n}\right)x + \frac{1}{2n}\right]\right| = \left|6 - \left[\left(2 - \frac{1}{2n}\right)5 + \frac{1}{2n} + \frac{3}{2n}\right]\right|$$

$$= \left|6 - 10 + \frac{1}{n}\right| = \left| - 4 + \frac{1}{n}\right| = \left|4 - \frac{1}{n}\right|.$$

So, $\lim\limits_{n\to\infty} d_n = 4 = 2\cdot 2 = |\alpha_0|\cdot r$, where $r = \sup\limits_{x\in[1,5]} \big\{|3-x|\big\}$.

(ii) Let $E=[0,1]$ and let $\{\alpha_n\}$ be a sequence of non-zero real numbers converging to a non-zero real number α_0. Then $\lim\limits_{n\to\infty} \big|T_{\omega_n}\big(\alpha_n[0,1]\big)\big|$, where

$$T_{\omega_n}(E) = E + \frac{1}{n} = \lim_{n\to\infty}\Big|[0,\alpha_n] + \frac{1}{n}\Big| \text{ or } \lim_{n\to\infty}\Big|[\alpha_n,0] + \frac{1}{n}\Big|$$
$$\text{provided } \alpha_n > 0 \text{ or } \alpha_n < 0$$
$$= \lim_{n\to\infty} |\alpha_n| = |\alpha_0|.$$

Theorem 1. *Let there exist an element $\omega_0 \in \Omega$ such that for a sequence $\{\omega_n\}$ ($\omega_n \in \Omega$) converging to ω_0 and for a sequence $\{\alpha_n\}$ of non-zero real numbers converging to a non-zero real number α_0 such that the sequence $\{T_{\omega_n}\}$ of transformations satisfies the conditions* (I), (II), (III).

Let A be a set of positive Lebesgue measure in $\mathbb{R}_N$. Then there exists a positive integer N_0 such that for a system of p positive integers $N_1, N_2, \dots, N_p$ with $N_i > N_0$, the set

$$\frac{1}{\alpha_0}A \cap T_{\omega_{N_1}}\Big(\frac{1}{\alpha_{N_1}}A\Big) \cap T_{\omega_{N_2}}\Big(\frac{1}{\alpha_{N_2}}A\Big) \cap \ldots \cap T_{\omega_{N_p}}\Big(\frac{1}{\alpha_{N_p}}A\Big)$$

is a set of positive measure.

P r o o f. Since A is a set of positive Lebesgue measure, there exists a ball $K_1 = B[a,r]$, $a \neq 0$ such that

$$|K_1 \setminus A| < \varepsilon|K_1| \text{ where } 0 < \varepsilon < \frac{1}{(1+p)(1+2p)}.$$

Let

$$d_n = \sup\Big\{\Big|\frac{1}{\alpha_n}a - T_{\omega_n}\Big(\frac{1}{\alpha_n}K_2\Big)\Big|\Big\}$$

where $K_2 = B[a,s]$, $s = \big(\frac{p}{1+p}\big)^{1/N} r$.

Since $\lim\limits_{n\to\infty} d_n = \frac{1}{|\alpha_0|}s$ and $\lim\limits_{n\to\infty}\alpha_n = \alpha_0$, there exists a positive integer N_1 such that for every $n > N_1$

$$\Big|d_n - \frac{1}{|\alpha_0|}s\Big| < \frac{r-s}{2|\alpha_0|} \quad\text{and}\quad \Big|\frac{1}{\alpha_n} - \frac{1}{\alpha_0}\Big| < \frac{r-s}{2|a||\alpha_0|}.$$

According to (iii), $\lim\limits_{n\to\infty}\Big|T_{\omega_n}\Big[\frac{1}{\alpha_n}(A\cap K_2)\Big]\Big| = \Big|\frac{1}{\alpha_0}(A\cap K_2)\Big|$. So there exists a positive integer N_2 such that for $n > N_2$,

$$\bigg|\Big|T_{\omega_n}\Big[\frac{1}{\alpha_n}(A\cap K_2)\Big]\Big| - \Big|\frac{1}{\alpha_0}(A\cap K_2)\Big|\bigg| < \varepsilon\frac{|K_1|}{|\alpha_0|^N}.$$

Let $N_0 = \max(N_1, N_2)$. So for $x \in K_2$ and for $n > N_0$, we have

$$\begin{aligned}
\Big|\frac{1}{\alpha_0}a - T_{\omega_n}\Big(\frac{1}{\alpha_0}x\Big)\Big| &= \Big|\frac{1}{\alpha_0}a - \frac{1}{\alpha_n}a + \frac{1}{\alpha_n}a - T_{\omega_n}\Big(\frac{1}{\alpha_0}x\Big)\Big| \\
&\leqslant |a|\Big|\frac{1}{\alpha_0} - \frac{1}{\alpha_n}\Big| + \Big|\frac{1}{\alpha_n}a - T_{\omega_n}\Big(\frac{1}{\alpha_0}x\Big)\Big| \\
&\leqslant |a|\Big|\frac{1}{\alpha_0} - \frac{1}{\alpha_n}\Big| + d_n \leqslant |a|\frac{r-s}{2|a||\alpha_0|} + \frac{r+s}{2|\alpha_0|} = \frac{1}{|\alpha_0|}r.
\end{aligned}$$

So, $T_{\omega_n}\big(\frac{1}{\alpha_n}K_2\big) \subset \frac{1}{\alpha_0}K_1$ for $n > N_0$ and hence

$$T_{\omega_n}\Big(\frac{1}{\alpha_n}(K_2 \cap A)\Big) \subset \frac{1}{\alpha_0}K_1 \quad \text{ for } n > N_0.$$

Let $N_1, N_2, \ldots, N_p$ be p positive integers with $N_i > N_0$. Also let

$$\begin{aligned}
X = \Big[\frac{1}{\alpha_0}(A \cap K_1)\Big] \cap T_{\omega_{N_1}}\Big[\frac{1}{\alpha_{N_1}}(A \cap K_2)\Big] \cap T_{\omega_{N_2}}\Big[\frac{1}{\alpha_{N_2}}(A \cap K_2)\Big] \\
\cap \ldots \cap T_{\omega_{N_p}}\Big[\frac{1}{\alpha_{N_p}}(A \cap K_2)\Big].
\end{aligned}$$

So,

$$X = \Big(\frac{1}{\alpha_0}K_1\Big) \setminus \Big[\Big(\Big(\frac{1}{\alpha_0}K_1\Big) \setminus \frac{1}{\alpha_0}A\Big) \cup \bigcup_{i=1}^{p}\Big\{\frac{1}{\alpha_0}K_1 \setminus T_{\omega_{N_i}}\Big(\frac{1}{\alpha_{N_i}}(A \cap K_2)\Big)\Big\}\Big].$$

Hence

$$\begin{aligned}
|X| &\geqslant \Big|\frac{1}{\alpha_0}K_1\Big| - \Big[\Big|\frac{1}{\alpha_0}K_1 \setminus \frac{1}{\alpha_0}A\Big| + \sum_{i=1}^{p}\Big\{\frac{1}{\alpha_0}K_1 \setminus T_{\omega_{N_i}}\Big(\frac{1}{\alpha_{N_i}}(A \cap K_2)\Big)\Big\}\Big] \\
&= \Big|\frac{1}{\alpha_0}K_1\Big| - \Big[\Big|\frac{1}{\alpha_0}(K_1 \setminus A)\Big| + p\Big|\frac{1}{\alpha_0}K_1\Big| - \sum_{i=1}^{p}\Big|T_{\omega_{N_i}}\Big(\frac{1}{\alpha_{N_i}}(A \cap K_2)\Big)\Big|\Big] \\
&= \Big|\frac{1}{\alpha_0}K_1\Big| - \Big|\frac{1}{\alpha_0}(K_1 \setminus A)\Big| - p\Big|\frac{1}{\alpha_0}K_1\Big| + \sum_{i=1}^{p}\Big|T_{\omega_{N_i}}\Big(\frac{1}{\alpha_{N_i}}(A \cap K_2)\Big)\Big| \\
&> \Big|\frac{1}{\alpha_0}K_1\Big| - \Big|\frac{1}{\alpha_0}(K_1 \setminus A)\Big| - p\Big|\frac{1}{\alpha_0}K_1\Big| + p\Big|\frac{1}{\alpha_0}(A \cap K_2)\Big| - p\varepsilon\frac{|K_1|}{|\alpha_0|^N} \\
&= \frac{1}{|\alpha_0|^N}|K_1| - \frac{1}{|\alpha_0|^N}|K_1 \setminus A| - p\frac{1}{|\alpha_0|^N}|K_1| + p\frac{1}{|\alpha_0|^N}|A \cap K_2| - p\varepsilon\frac{|K_1|}{|\alpha_0|^N} \\
&> \frac{1}{|\alpha_0|^N}|K_1| - \frac{1}{|\alpha_0|^N}\varepsilon|K_1| - p\frac{1}{|\alpha_0|^N}|K_1| + p\frac{1}{|\alpha_0|^N}\big|K_2 \setminus (K_2 \setminus A)\big| - p\varepsilon\frac{|K_1|}{|\alpha_0|^N} \\
&= \frac{1}{|\alpha_0|^N}|K_1| - \frac{1}{|\alpha_0|^N}\varepsilon|K_1| - p\frac{1}{|\alpha_0|^N}|K_1| + p\frac{1}{|\alpha_0|^N}\big[|K_2| - |K_2 \setminus A|\big] - p\varepsilon\frac{|K_1|}{|\alpha_0|^N}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{|\alpha_0|^N}|K_1| - \frac{1}{|\alpha_0|^N}\varepsilon|K_1| - p\frac{1}{|\alpha_0|^N}\big[|K_1| - |K_2|\big] - p\frac{1}{|\alpha_0|^N}|K_2 \setminus A| - p\varepsilon\frac{|K_1|}{|\alpha_0|^N}\\
&> \frac{1}{|\alpha_0|^N}|K_1| - \frac{1}{|\alpha_0|^N}\varepsilon|K_1| - \frac{1}{|\alpha_0|^N}|K_2| - p\frac{1}{|\alpha_0|^N}|K_1 \setminus A| - p\varepsilon\frac{|K_1|}{|\alpha_0|^N}\\
&> \frac{1}{|\alpha_0|^N}\big[|K_1| - |K_2|\big] - \frac{1}{|\alpha_0|^N}\varepsilon|K_1| - p\frac{1}{|\alpha_0|^N}\varepsilon|K_1| - p\varepsilon\frac{|K_1|}{|\alpha_0|^N}\\
&= \frac{1}{|\alpha_0|^N}\frac{1}{p}|K_2| - \frac{1}{|\alpha_0|^N}[1+2p]\varepsilon|K_1|\\
&= \frac{1}{|\alpha_0|^N}\frac{1}{p}\frac{p}{1+p}|K_1| - \frac{1}{|\alpha_0|^N}(1+2p)\varepsilon|K_1|\\
&= \frac{1}{|\alpha_0|^N}\Big[\frac{1}{1+p} - (1+2p)\varepsilon\Big]|K_1|\\
&= \frac{1+2p}{|\alpha_0|^N}\Big[\frac{1}{(1+p)(1+2p)} - \varepsilon\Big]|K_1|\\
&> 0, \qquad \text{since } 0 < \varepsilon < \frac{1}{(1+p)(1+2p)}.
\end{aligned}
$$

Hence X is a set of positive measure and so by (II′), for $N_1, N_2, \ldots, N_p \geqslant N_0$ the set

$$
\frac{1}{\alpha_0}A \cap T_{\omega_{N_1}}\Big(\frac{1}{\alpha_{N_1}}A\Big) \cap T_{\omega_{N_2}}\Big(\frac{1}{\alpha_{N_2}}A\Big) \cap \ldots \cap T_{\omega_{N_p}}\Big(\frac{1}{\alpha_{N_p}}A\Big)
$$

is a set of positive measure.

This completes the proof. □

Corollary. *Let* $\alpha_n = 1$, $n = 1, 2, \ldots$ *Then Theorem* 1 *of* [2] *follows immediately.*

Now we introduce the following condition which is equivalent to (I′).

Condition (i′)**.** Let $a, b \in \mathbb{R}_N$, let there exist $\omega_0 \in \Omega$ (a metric space) and a sequence $\{\omega_n\}$ converging to ω_0 such that for every ball $K = B[b, r]$ $(r > 0)$ and for every sequence $\{\alpha_n\}$ of non-zero numbers converging to a non-zero real number α_0,

$$
\limsup_{n\to\infty}\big\{\big|\alpha_n a - T_{\omega_n}(\alpha_n K)\big|\big\} = |\alpha_0| r
$$

holds. For the following theorems we denote the condition (ii) as the condition (ii′), and the condition (iii) is replaced by the condition (iii′) which is the condition (iii) with $\omega_0 \in \Omega$ as in (i′).

Theorem 2. *Let A and B be sets of positive Lebesgue measure in $\mathbb{R}_N$ and let a and b be points of density of A and B, respectively. Let there exist an element $\omega_0 \in \Omega$ and a sequence $\{\omega_n\}$ $(\omega_n \in \Omega)$ converging to ω_0 such that for a sequence*

$\{\alpha_n\}$ of non-zero real numbers converging to $\alpha_0 (\neq 0)$ the transformations T_{ω_n} satisfy the conditions (i′), (ii′), (iii′) *with respect to (a, b, ω_0). Then there exists a natural number N_0 such that for a system of p elements $\omega_{N_1}, \omega_{N_2}, \ldots, \omega_{N_p}$ of the sequence $\{\omega_n\}$ with $N_i > N_0$,*

$$\frac{1}{\alpha_0} A \cap T_{\omega_{N_1}}\Big(\frac{1}{\alpha_{N_1}} B\Big) \cap T_{\omega_{N_2}}\Big(\frac{1}{\alpha_{N_2}} B\Big) \cap \ldots \cap T_{\omega_{N_p}}\Big(\frac{1}{\alpha_{N_p}} B\Big)$$

is a set of positive measure.

Proof. Since A and B are sets of positive measure, there exist balls $K_A = B[a, r]$ $(a \neq 0)$ and $K_B = [b, r]$ such that $|K_A \setminus A| < \varepsilon |K_A|$, $|K_B \setminus B| < \varepsilon |K_B|$ where $0 < \varepsilon < \frac{1}{2p^2+2p+1}$. Let $\sup\big\{\big|\frac{1}{\alpha_n} a - T_{\omega_n}\big(\frac{1}{\alpha_n} K_2\big)\big|\big\} = d_n$ where $K_2 = B[b, s]$,

$$s = \Big(\frac{p}{1+p}\Big)^{1/N} r.$$

Since $\lim_{n\to\infty} d_n = \frac{1}{|\alpha_0|} s$ and $\lim_{n\to\infty} \alpha_n = \alpha_0$, there exists a positive integer N_1 such that for every $n > N_1$ we have

$$\Big|d_n - \frac{1}{|\alpha_0|} s\Big| < \frac{r-s}{2|\alpha_0|} \quad \text{and} \quad \Big|\frac{1}{\alpha_n} - \frac{1}{\alpha_0}\Big| < \frac{r-s}{2|a||\alpha_0|}.$$

In virtue of (iii) $K_2 = B[a, s]$,

$$\lim_{n\to\infty} \Big|T_{\omega_n}\Big[\frac{1}{\alpha_n}(A \cap K_2')\Big]\Big| = \Big|\frac{1}{\alpha_0}(A \cap K_2')\Big|.$$

So, there exists a positive integer N_2 such that for $n > N_2$ we have

$$\bigg|\Big|T_{\omega_n}\Big[\frac{1}{\alpha_n}(A \cap K_2')\Big]\Big| - \Big|\frac{1}{\alpha_0}(A \cap K_2')\Big|\bigg| < \varepsilon_1$$

where $0 < \varepsilon_1 < \varepsilon\big|\frac{1}{\alpha_0} K_2\big|$. Let $N_0 = \max(N_1, N_2)$. Then, for $x \in K_2$ and for $n > N_0$,

$$\begin{aligned}\Big|\frac{1}{\alpha_0} a - T_{\omega_n}\Big(\frac{1}{\alpha_0} x\Big)\Big| &= \Big|\frac{1}{\alpha_0} a - \frac{1}{\alpha_n} a + \frac{1}{\alpha_n} a - T_{\omega_n}\Big(\frac{1}{\alpha_n} x\Big)\Big| \\ &\leqslant |a|\Big|\frac{1}{\alpha_0} - \frac{1}{\alpha_n}\Big| + \Big|\frac{1}{\alpha_n} a - T_{\omega_n}\Big(\frac{1}{\alpha_n} x\Big)\Big| \\ &\leqslant |a|\frac{r-s}{2|a||\alpha_0|} + d_n \leqslant |a|\frac{r-s}{2|a||\alpha_0|} + \frac{r+s}{2|\alpha_0|} = \frac{1}{|\alpha_0|} r.\end{aligned}$$

So $T_{\omega_n}\big(\frac{1}{\alpha_n} K_2\big) \subset \frac{1}{\alpha_0} K_A$ and hence

$$T_{\omega_n}\Big(\frac{1}{\alpha_n}(K_2 \cap B)\Big) \subset \frac{1}{\alpha_0} K_A \quad \text{for } n > N_0.$$

Let $N_1, N_2, \ldots, N_p$ be positive integers with $N_i > N_0$. Also let

$$X = \Big[\frac{1}{\alpha_0}(A \cap K_A)\Big] \cap T_{\omega_{N_1}}\Big[\frac{1}{\alpha_{N_1}}(B \cap K_2)\Big] \cap T_{\omega_{N_2}}\Big[\frac{1}{\alpha_{N_2}}(B \cap K_2)\Big]$$
$$\cap \ldots \cap T_{\omega_{N_p}}\Big[\frac{1}{\alpha_{N_p}}(B \cap K_2)\Big].$$

Then

$$X = \Big(\frac{1}{\alpha_0}K_A\Big) \setminus \Big[\Big(\Big(\frac{1}{\alpha_0}K_A\Big) \setminus \frac{1}{\alpha_0}(K_A \cap A)\Big) \cup \bigcup_{i=1}^{p}\Big\{\frac{1}{\alpha_0}K_A \setminus T_{\omega_{N_i}}\Big(\frac{1}{\alpha_{N_i}}(B \cap K_2)\Big)\Big\}\Big].$$

Hence

$$\begin{aligned}
|X| &\geqslant \Big|\frac{1}{\alpha_0}K_A\Big| - \Big[\Big|\frac{1}{\alpha_0}K_A \setminus \frac{1}{\alpha_0}A\Big| + \sum_{i=1}^{p}\Big\{\Big|\frac{1}{\alpha_0}K_A \setminus T_{\omega_{N_i}}\Big(\frac{1}{\alpha_{N_i}}(B \cap K_2)\Big)\Big|\Big\}\Big] \\
&= \Big|\frac{1}{\alpha_0}K_A\Big| - \Big[\Big|\frac{1}{\alpha_0}(K_A \setminus A)\Big| + p\Big|\frac{1}{\alpha_0}K_A\Big| - \sum_{i=1}^{p}\Big|T_{\omega_{N_i}}\Big(\frac{1}{\alpha_{N_i}}(B \cap K_2)\Big)\Big|\Big] \\
&= \Big|\frac{1}{\alpha_0}K_A\Big| - \Big|\frac{1}{\alpha_0}(K_A \setminus A)\Big| - p\Big|\frac{1}{\alpha_0}K_A\Big| + \sum_{i=1}^{p}\Big|T_{\omega_{N_i}}\Big(\frac{1}{\alpha_{N_i}}(B \cap K_2)\Big)\Big| \\
&> \Big|\frac{1}{\alpha_0}K_A\Big| - \Big|\frac{1}{\alpha_0}(K_A \setminus A)\Big| - p\Big|\frac{1}{\alpha_0}K_A\Big| + p\Big|\frac{1}{\alpha_0}(B \cap K_2)\Big| - p\varepsilon_1 \\
&= \frac{1}{|\alpha_0|^N}|K_A| - \frac{1}{|\alpha_0|^N}|K_A \setminus A| - p\frac{1}{|\alpha_0|^N}|K_A| + p\frac{1}{|\alpha_0|^N}|B \cap K_2| - p\varepsilon_1 \\
&> \frac{1}{|\alpha_0|^N}|K_A| - \frac{1}{|\alpha_0|^N}\varepsilon|K_A| - p\frac{1}{|\alpha_0|^N}|K_A| + p\frac{1}{|\alpha_0|^N}\big|K_2 \setminus (K_2 \setminus B)\big| - p\varepsilon_1 \\
&= \frac{1}{|\alpha_0|^N}|K_A| - \frac{1}{|\alpha_0|^N}\varepsilon|K_A| - p\frac{1}{|\alpha_0|^N}|K_A| + p\frac{1}{|\alpha_0|^N}\big[|K_2| - |K_2 \setminus B|\big] - p\varepsilon_1 \\
&= \frac{1}{|\alpha_0|^N}|K_A| - \frac{1}{|\alpha_0|^N}\varepsilon|K_A| - p\frac{1}{|\alpha_0|^N}\big[|K_A| - |K_2|\big] - p\frac{1}{|\alpha_0|^N}|K_2 \setminus B| - p\varepsilon_1 \\
&> \frac{1}{|\alpha_0|^N}|K_A| - \frac{1}{|\alpha_0|^N}\varepsilon|K_A| - \frac{1}{|\alpha_0|^N}|K_2| - p\frac{1}{|\alpha_0|^N}|K_B \setminus B| - p\varepsilon_1 \\
&> \frac{1}{|\alpha_0|^N}\big[|K_A| - |K_2|\big]\frac{1}{|\alpha_0|^N}\varepsilon|K_A| - p\frac{1}{|\alpha_0|^N}\varepsilon|K_B| - p\varepsilon\frac{|K_2|}{|\alpha_0|^N} \\
&= \frac{1}{|\alpha_0|^N}\frac{1}{p}|K_2| - \frac{1}{|\alpha_0|^N}\varepsilon\frac{1+p}{p}|K_2| - p\frac{1}{|\alpha_0|^N}\varepsilon\frac{1+p}{p}|K_2| - \frac{p}{|\alpha_0|^N}\varepsilon|K_2| \\
&= \frac{1}{|ao|^N}\Big[\frac{1}{p} - \Big(\frac{1+p}{p} + 1 + p + p\Big)\varepsilon\Big]|K_2|
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{|\alpha_0|^N}\Big[\frac{1}{p} - \Big(\frac{1+p+p+2p^2}{p}\Big)\varepsilon\Big]|K_2| \\
&= \frac{1+2p+2p^2}{|\alpha_0|^N p}\Big[\frac{1}{1+2p+2p^2} - \varepsilon\Big]|K_2| \\
&> 0, \qquad \text{since } 0 < \varepsilon < \frac{1}{2p^2+2p+1}.
\end{aligned}
$$

Hence X is a set of positive measure and so by (ii), for $N_1, N_2, \ldots, N_p \geqslant N_0$, the set

$$
\frac{1}{\alpha_0}A \cap T_{\omega_{N_1}}\Big(\frac{1}{\alpha_{N_1}}B\Big) \cap T_{\omega_{N_2}}\Big(\frac{1}{\alpha_{N_2}}B\Big) \cap \ldots \cap T_{\omega_{N_p}}\Big(\frac{1}{\alpha_{N_p}}B\Big)
$$

is a set of positive measure.

This completes the proof. □

Theorem 3. *Let A and $B_1, B_2, \ldots, B_p$ be sets of positive measure in $\mathbb{R}_N$ and let a and b_i $(i = 1, 2, \ldots, p)$ be points of density of A and B_i $(i = 1, 2, \ldots, p)$, respectively. Let there exist an element $\omega_0 \in \Omega$ and a sequence $\{\omega_n^i\}(\omega_n^i \in \Omega)$ $(i = 1, 2, \ldots, p)$ converging to ω_0 such that for a sequence $\{\alpha_n\}$ of non-zero real number converging to a non-zero real number α_0, the sequence of transformations $\{T_{\omega_n} i\}$ $(i = 1, 2, \ldots, p)$ satisfies the conditions* (i′), (ii′), (iii′) *with respect to (a, b_i, ω_0). Then there exists a natural number N_0 such that for a system of p^2 elements $\omega_{N_1} i$, $\omega_{N_2} i$, …, $\omega_{N_p} i$ of the sequence $\{T_{\omega_n} i\}$ with $N_k^i > N_0$ and for a system of p numbers α_{N_1}, α_{N_2}, …, α_{N_p} of the sequence $\{\alpha_n\}$ with $N_k > N_0$ the set*

$$
\begin{aligned}
\frac{1}{\alpha_0}A \cap T_{\omega^1_{N_1}}\Big(\frac{1}{\alpha_{N_1}}B_1\Big) \cap T_{\omega^1_{N_2^1}}\Big(\frac{1}{\alpha_{N_2}}B_1\Big) \cap \ldots \cap T_{\omega^1_{N_p^1}}\Big(\frac{1}{\alpha_{N_p}}B_1\Big) \\
\cap\, T_{\omega^2_{N_1^2}}\Big(\frac{1}{\alpha_{N_1}}B_2\Big) \cap T_{\omega^2_{N_2^2}}\Big(\frac{1}{\alpha_{N_2}}B_2\Big) \cap \ldots \cap T_{\omega^2_{N_p^2}}\Big(\frac{1}{\alpha_{N_p}}B_2\Big) \cap \ldots \\
\cap\, T_{\omega^p_{N_1^p}}\Big(\frac{1}{\alpha_{N_1}}B_p\Big) \cap T_{\omega^p_{N_2^p}}\Big(\frac{1}{\alpha_{N_2}}B_p\Big) \cap \ldots \cap T_{\omega^p_{N_p^p}}\Big(\frac{1}{\alpha_{N_p}}B_p\Big)
\end{aligned}
$$

is a set of positive measure.

P r o o f. Since A and B_i $(i = 1, 2, \ldots, p)$ are sets of positive measure, there exist balls $K_A = B[a, r]$, $a \neq 0$ and $K_{B_i} = B[b_i, r]$ $(i = 1, 2, \ldots, p)$ such that

$$
|K_A \setminus A| < \varepsilon|K_A| \quad \text{and} \quad |K_{B_i} \setminus B_i| < \varepsilon|K_{B_i}|, \quad i = 1, 2, \ldots, p,
$$

where $0 < \varepsilon < \frac{1+p}{1+p+2p^2}$. Let $\sup\big\{\big|\frac{1}{\alpha_n}a - T_{\omega_n}\big(\frac{1}{\alpha_n}K_2^i\big)\big|\big\} = d_n^i$ where $K_2^i = B[b_i, s]$,

$$
s = \Big(\frac{p}{1+p}\Big)^{1/N} r.
$$

Since $\lim\limits_{n\to\infty} d_n^i = \frac{1}{|\alpha_0|}s$ and $\lim\limits_{n\to\infty}\alpha_n = \alpha_0$, there exists a positive integer N_1 such that for every $n > N_1$,

$$\Big|d_n^i - \frac{1}{|\alpha_0|}s\Big| < \frac{r-s}{2|\alpha_0|} \quad \text{and} \quad \Big|\frac{1}{\alpha_n} - \frac{1}{\alpha_0}\Big| < \frac{r-s}{2|a||\alpha_0|}.$$

In virtue of (iii) we have

$$\lim_{n\to\infty}\Big|T_{\omega_n^i}\Big[\frac{1}{\alpha_n}(A\cap K_2^i)\Big]\Big| = \Big|\frac{1}{\alpha_0}(A\cap K_2^i)\Big|.$$

So there exists a positive integer N_2 such that for $n > N_2$,

$$\bigg|\Big|T_{\omega_n^i}\Big[\frac{1}{\alpha_n}(A\cap K_2^i)\Big]\Big| - \Big|\frac{1}{\alpha_0}(A\cap K_2^i)\Big|\bigg| < \varepsilon_1$$

where $0 < \varepsilon_1 < \varepsilon\big|\frac{1}{\alpha_0}K_2^i\big|$. Let $N_0 = \max(N_1, N_2)$. Then, for $x \in K_2^i$ and for $n > N_0$ we obtain

$$\begin{aligned}
\Big|\frac{1}{\alpha_0}a - T_{\omega_n^i}\Big(\frac{1}{\alpha_0}x\Big)\Big| &= \Big|\frac{1}{\alpha_0}a - \frac{1}{\alpha_n}a + \frac{1}{\alpha_n}a - T_{\omega_n^i}\Big(\frac{1}{\alpha_0}x\Big)\Big| \\
&\leqslant |a|\Big|\frac{1}{\alpha_0} - \frac{1}{\alpha_n}\Big| + \Big|\frac{1}{\alpha_n}a - T_{\omega_n^i}\Big(\frac{1}{\alpha_n}x\Big)\Big| \\
&\leqslant |a|\frac{r-s}{2|a||\alpha_0|} + d_n^i \leqslant |a|\frac{r-s}{2|a||\alpha_0|} + \frac{r+s}{2|\alpha_0|} = \frac{1}{|\alpha_0|}r.
\end{aligned}$$

Hence $T_{\omega_n^i}\big(\frac{1}{\alpha_n}K_2^i\big) \subset \frac{1}{\alpha_0}K_A$ and for $n > N_0$ and for $i = 1, 2, \ldots, p$. Let $N_1^i, N_2^i, \ldots, N_p^i$ and N_i $(i = 1, 2, \ldots, p)$ be positive integers with $N_k^i > N_0$ and $N_i > N_0$. Also let

$$\begin{aligned}
X = {} & \Big[\frac{1}{\alpha_0}(A\cap K_A)\Big] \cap T_{\omega^1_{N_1^1}}\Big[\frac{1}{\alpha_{N_1}}(B\cap K_2^1)\Big] \cap T_{\omega^1_{N_2^1}}\Big[\frac{1}{\alpha_{N_2}}(B_1\cap K_2^1)\Big] \cap \ldots \\
& \cap T_{\omega^1_{N_p^1}}\Big[\frac{1}{\alpha_{N_p}}(B_1\cap K_2^1)\Big] \\
& \cap T_{\omega^2_{N_1^2}}\Big[\frac{1}{\alpha_{N_1}}(B_2\cap K_2^2)\Big] \cap T_{\omega^2_{N_2^2}}\Big[\frac{1}{\alpha_{N_2}}(B_2\cap K_2^2)\Big] \cap \ldots \\
& \cap T_{\omega^2_{N_p^2}}\Big[\frac{1}{\alpha_{N_p}}(B_2\cap K_2^2)\Big] \cap \ldots \\
& \cap T_{\omega^p_{N_1^p}}\Big[\frac{1}{\alpha_{N_2}}(B_p\cap K_2^p)\Big] \cap T_{\omega^p_{N_2^p}}\Big[\frac{1}{\alpha_{N_2}}(B_p\cap K_2^p)\Big] \cap \ldots \\
& \cap T_{\omega^p_{N_p^p}}\Big[\frac{1}{\alpha_{N_p}}(B_p\cap K_2^p)\Big].
\end{aligned}$$

Then

$$X = \Big(\frac{1}{\alpha_0}K_A\Big)\setminus\Big\{\Big[\Big(\frac{1}{\alpha_0}K_A\Big)\setminus\frac{1}{\alpha_0}(K_A\cap A)\Big]\cup\bigcup_{i=1}^{p}\bigcup_{j=2}^{p}\Big[\frac{1}{\alpha_0}K_A\setminus T_{\omega^i_{N_j^i}}\Big(\frac{1}{\alpha_{N_j}}(B_i\cap K_2^i)\Big)\Big]\Big\}.$$

Hence

$$\begin{aligned}
|X| &\geqslant \Big|\frac{1}{\alpha_0}K_A\Big| - \Big[\Big|\frac{1}{\alpha_0}K_A \setminus \frac{1}{\alpha_0}A\Big| + \sum_{i=1}^{p}\sum_{j=1}^{p}\Big\{\Big|\frac{1}{\alpha_0}K_A \setminus T_{\omega^i_{N^i_j}}\Big(\frac{1}{\alpha_{N_j}}(B_i \cap K^i_2)\Big)\Big|\Big\}\Big] \\
&= \Big|\frac{1}{\alpha_0}K_A\Big| - \Big[\Big|\frac{1}{\alpha_0}(K_A \setminus A)\Big| + p^2\Big|\frac{1}{\alpha_0}K_A\Big| - \sum_{i=1}^{p}\sum_{j=1}^{p}\Big|T_{\omega^i_{N^i_j}}\Big(\frac{1}{\alpha_{N_j}}(B_i \cap K^i_2)\Big)\Big|\Big] \\
&= \Big|\frac{1}{\alpha_0}K_A\Big| - \Big|\frac{1}{\alpha_0}(K_A \setminus A)\Big| - p^2\Big|\frac{1}{\alpha_0}K_A\Big| + \sum_{i=1}^{p}\sum_{j=1}^{p}\Big|T_{\omega^i_{N^i_j}}\Big(\frac{1}{\alpha_{N_j}}(B_i \cap K^i_2)\Big)\Big| \\
&> \Big|\frac{1}{\alpha_0}K_A\Big| - \Big|\frac{1}{\alpha_0}(K_A \setminus A)\Big| - p^2\Big|\frac{1}{\alpha_0}K_A\Big| + p\frac{1}{|\alpha_0|^N}\sum_{i=1}^{p}\big|[K^i_2 \setminus \{K^i_2 \setminus B_i\}]\big| - p^2\varepsilon_1 \\
&= \frac{1}{|\alpha_0|^N}|K_A| - \frac{1}{|\alpha_0|^N}|K_A \setminus A| - p^2\frac{1}{|\alpha_0|^N}|K_A| \\
&\quad + p\frac{1}{|\alpha_0|^N}\sum_{i=1}^{p}\big[|K^i_2| - |K^i_2 \setminus B_i|\big] - p^2\varepsilon_1 \\
&> \frac{1}{|\alpha_0|^N}|K_A| - \frac{1}{|\alpha_0|^N}\varepsilon|K_A| - p^2\frac{1}{|\alpha_0|^N}|K_A| + p\frac{1}{|\alpha_0|^N}\big[p|K_2| - \sum_{i=1}^{p}\varepsilon|K_{B_i}|\big] - p^2\varepsilon_1 \\
&= \frac{1}{|\alpha_0|^N}|K_A| - \frac{1}{|\alpha_0|^N}\varepsilon|K_A| - p^2\frac{1}{|\alpha_0|^N}|K_A| + p^2\frac{1}{|\alpha_0|^N}|K_2| - \frac{p^2\varepsilon}{|\alpha_0|^N}|K_A| - p^2\varepsilon_1 \\
&= \frac{1}{|\alpha_0|^N}|K_A| - \frac{1}{|\alpha_0|^N}\varepsilon|K_A| - \frac{p^2}{|\alpha_0|^N}\big[|K_A| - |K_2|\big] - \frac{p^2}{|\alpha_0|^N}|K_A| - p^2\varepsilon_1 \\
&> \frac{1}{|\alpha_0|^N}\frac{p}{1+p}|K_2| - \frac{1}{|\alpha_0|^N}\varepsilon\frac{p}{1+p}|K_2| - \frac{p^2}{|\alpha_0|^N}\Big[\frac{p}{1+p}|K_2| - |K_2|\Big] \\
&\quad - \frac{p^2\varepsilon}{|\alpha_0|^N}\frac{p}{1+p}|K_2| - p^2\frac{1\varepsilon}{\alpha_0^N}|K_2| \\
&= \frac{1}{|\alpha_0|^N}\frac{p}{1+p}|K_2| - \frac{1}{|\alpha_0|^N}\,\varepsilon\,\frac{p}{1+p}|K_2| - p\frac{p^2}{|\alpha_0|^N}\Big[\frac{p}{1+p} - 1\Big]|K_2| \\
&\quad - \frac{p^3}{|\alpha_0|^N}\,\varepsilon\,\frac{|K_2|}{1+p} - \frac{p^2}{|\alpha_0|^N}\varepsilon|K_2| \\
&= \frac{1}{|ao|^N}\Big[\frac{p}{1+p} + \frac{p^2}{1+p}\Big]|K_2| - \frac{1}{|\alpha_0|^N}\varepsilon\Big[\frac{p}{1+p} + \frac{p^3}{1+p} + p^2\Big]|K_2| \\
&= \frac{p}{|\alpha_0|^N}|K_2| - \frac{p}{|\alpha_0|^N}\varepsilon\frac{1+p+2p^2}{1+p}|K_2| \\
&= \frac{p}{|\alpha_0|^N}\frac{1+p+2p^2}{1+p}\Big[\frac{1+p}{1+p+2p^2} - \varepsilon\Big]|K_2| \\
&> 0, \qquad \text{since } 0 < \varepsilon < \frac{1+p}{1+p+2p^2}.
\end{aligned}$$

Hence X is a set of positive measure and so by (II), for $N_1^i, N_2^i, \ldots, N_p^i$, $N_i \geqslant N_0$ $(i = 1, 2, \ldots, p)$ the set

$$\frac{1}{\alpha_0} A \cap T_{\omega^1_{N_1^1}}\Big(\frac{1}{\alpha_{N_1}} B_1\Big) \cap T_{\omega^1_{N_2^1}}\Big(\frac{1}{\alpha_{N_2}} B_1\Big) \cap \ldots \cap T_{\omega^1_{N_p^1}}\Big(\frac{1}{\alpha_{N_p}} B_1\Big)$$
$$\cap\, T_{\omega^2_{N_1^2}}\Big(\frac{1}{\alpha_{N_1}} B_2\Big) \cap T_{\omega^2_{N_2^2}}\Big(\frac{1}{\alpha_{N_2}} B_2\Big) \cap \ldots \cap T_{\omega^2_{N_p^2}}\Big(\frac{1}{\alpha_{N_p}} B_2\Big) \cap \ldots$$
$$\cap\, T_{\omega^p_{N_1^p}}\Big(\frac{1}{\alpha_{N_1}} B_p\Big) \cap T_{\omega^p_{N_2^p}}\Big(\frac{1}{\alpha_{N_2}} B_p\Big) \cap \ldots \cap T_{\omega^p_{N_p^p}}\Big(\frac{1}{\alpha_{N_p}} B_p\Big)$$

is a set of positive measure.

This completes the proof. □

References

[1] *T. Neubrunn, T. Šalát*: Distance sets, ratio sets and certain transformations of sets of real numbers. Čas. Pěst. Mat. *94* (1969), 381–393.

[2] *M. Pal*: On certain transformations of Sets in $\mathbb{R}_N$. Acta F.R.N. Univ. Comen. Mathematica XXIX, 1974, pp. 43–58.

[3] *K.C. Ray*: On two theorems of S. Kurepa. Glasnik Mat-Fiz. *19* (1964), 207–210.

[4] *N.G. Saha, K.C. Ray*: On sets under certain transformations in $\mathbb{R}_N$. Publ. Inst. Math. *22(36)* (1977), 237–244.

[5] *R.L. Wheeden, A. Zygmund*: Measure and Integration. Marcel Dekker, Inc., New York, 1977, pp. 45.

Authors' address: Department of Mathematics, University of Kalyani, Kalyani-741235, West Benga, India.