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ON A CLASS OF REAL NORMED LATTICES

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Abstract. We say that a real normed lattice is quasi-Baire if the intersection of each sequence of monotonic open dense sets is dense. An example of a Baire-convex space, due to M. Valdivia, which is not quasi-Baire is given. We obtain that E is a quasi-Baire space iff $(E, T(\mathscr{U}), T(\mathscr{U}^{-1}))$, is a pairwise Baire bitopological space, where $\mathscr{U}$, is a quasi-uniformity that determines, in L. Nachbin's sense, the topological ordered space E.

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1. Introduction

In functional analysis it is important to establish a classification of the types of barrelled spaces. In order to do so, M. Valdivia [6] p. 281–287, gave an example of a convex-Baire space which is not Baire. In section 2 of this paper we define, in a natural way, the concept of a quasi-Baire space for a real linear normed lattice which generalizes the Baire concept in such spaces and we show the mentioned example is not even quasi-Baire. All linear spaces under consideration are assumed to be defined over the field of the real numbers.

Following Fletcher and Lindgren's terminology of [3] we introduce the following definitions and notation.

By a topological ordered space we mean a triple (X, T) where X is a nonvoid set, T is a topology in X and $\leqslant$ is a partial order in X such that its graph $G(\leqslant) = \{(x, y) \colon x \leqslant y\}$ is closed in $X \times X$.

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A subset S of a topological ordered space (X,T) is said to be *increasing (decreasing)* provided $y \in S$ whenever $x \leqslant y$ $(y \leqslant x)$ and $x \in S$. S is monotonic if it is either increasing or decreasing.

As L. Nachbin observed [5] p. 51, there is a natural relation between quasi-uniform spaces and topological ordered spaces. As usual, if $\mathscr{U}$, is a quasi-uniformity in X, then $\mathscr{U}^{-1}$ and $\mathscr{U}^*$ will denote the conjugate quasi-uniformity and the uniformity of the subbase $\mathscr{U} \cap \mathscr{U}^{-1}$, respectively. The quasi-uniformity $\mathscr{U}$ is said to determine the topological ordered space $(X,T,\leqslant)$ whenever T is the topology $T(\mathscr{U}^*)$ deduced from $\mathscr{U}^*$ and

$$G(\leqslant) = \bigcap_{V \in \mathscr{U}} V$$

A non-negative real valued function q defined on a linear space E is said to be a quasi-norm [2], provided it satisfies the following conditions for $x, y \in E$ and $t \geqslant 0$:

$$\begin{gathered} \text{if } q(x) = q(-x) = 0, \\ \text{then } x = 0,\ q(tx) = tq(x),\ q(x+y) \leqslant q(x) + q(y). \end{gathered}$$

If q is a quasi-norm on E, then (E,q) is a quasi-normed space.

In a quasi-normed space (E,q) the function $d(x,y) = q(y-x)$ is a quasi-pseudometric in E that defines a quasi-uniformity q in E. We will consider in the sequel the quasi-normed space (E,q) provided with the topology $T(q)$ deduced from q. Besides, the function $q^{-1}(x) = q(-x)$ defines another quasi-norm (called the conjugate of q) in E such that its induced quasi-uniformity q^{-1} is the conjugate of q.

In [2] Corollary 3.2 we gave the following result: Every normed lattice $(E, \| \, \|, \leqslant)$ is determined by the quasi-uniformity q (or its conjugate) deduced from the quasi-norm $q(x) = \|x^+\|$ where $x^+ = \sup\{x, 0\}$ in the lattice order. In this case we will say that (E,q) is the quasi-normed space associated to the normed lattice E.

For a normed lattice $(E, \| \, \|, \leqslant)$ we will consider the topologies deduced from the norm $\| \, \|$ and from the associated quasi-norm. Then we will say G is open when it is open for the norm and q-open when it is open for the associated quasi-norm. The same consideration is valid for dense, neighborhood, ... We will denote, for $\delta > 0$, by $B_\delta(x_0)$ the open disk $\{x \in E\colon\ x - x_0 < \delta\}$ and by $V_\delta(x_0)$ the q-open set $\{x \in E\colon\ q(x - x_0) < \delta\}$. The family of sets $\{x + V_\delta(x_0)\colon\ \delta > 0\}$ or equivalently $\{V_\delta(x)\colon\ \delta > 0\}$, is a base of q-neighborhoods for $x \in E$.

A bitopological space (X,P,L) is a set X with two topologies P and L. J.C. Kelly [4] introduced these spaces initially to restore some of the symmetries of the classical metric situations to a bitopological space (X,P,L) where P and L are the topologies induced by a quasi-metric on X and its conjugate quasi-metric respectively, and in consequence to be able to obtain systematic generalizations of standard results in general topology. After Kelly, several others authors have contributed to the development of the theory and so, a large list of pairwise bitopological spaces have been studied.

In Section 3 we give a definition of a pairwise Baire bitopological space. We point out this is an interesting bitopological concept because it makes it possible to characterize the quasi-Baire normed lattices that were defined in terms of general topology.

2. Quasi-Baire normed lattices

Lemma 1. *Let be $(E, \|\ \|, \leqslant)$ a normed lattice. The set G is q-open iff G is open and decreasing. Analogously, G is q^{-1}-open iff G is open and increasing.*

Proof. Suppose G is q-open. Since E is determined by q, the topology deduced from the norm $\|\ \|$ is finer than the q-topology and G is open. Now, suppose $y \leqslant x$ with $x \in G$. Then there is a q-neighborhood of the origin $V_\delta(0)$ such that $x + V_\delta(0) \subset G$. We have $q(y-x) = \|(y-x)^+\| = 0$, then $y - x \in V_\delta(0)$ and in consequence $y \in G$.

Conversely, suppose G is open and decreasing and let $x \in G$. Then there is an open disk $B_\delta(x)$ such that $B_\delta(x) \subset G$. We see that $x + V_\delta(0) \subset G$.

Let $y \in x + V_\delta(0)$. Then for $z = x + (y-x)^+$ we have

$$\|z - x\| = \|(y-x)^+\| = q(y-x) < \delta$$

and thus $z \in G$.

Also, because $z - y = 0$ if $y \geqslant x$ and $z - y = x - y > 0$ if $y < x$, then $z \geqslant y$ and since G is a decreasing set we have $y \in G$ and, in consequence, G is q-open.

The proof for q^{-1}-open is similar. □

Proposition 1. *If $(E, \|\ \|, \leqslant)$ is a normed lattice then its associated quasi-normed space (E, q) is never Baire.*

Proof. First we will see that the non-empty q-open sets of E are q-dense. To this end, we consider two different q-open sets G and H and let $x \in G$ and $y \in H$ with $x \neq y$. Suppose $i = \inf\{x, y\}$; by the previous lemma G and H are decreasing sets and so $i \in G \cap H$.

Now, we consider the sequence $\{V_1(nx)\}_{n=1}^{\infty}$ for a fixed $x < 0$. We will show that this family has empty intersection. Suppose $y \in \bigcap\limits_{n=1}^{+\infty} V_1(nx)$, then $\|(y - nx)^+\| < 1, \forall n$

and thus

$$n\|x\| = \|nx\| = \|-nx\| = \|(-nx)^+\| = \|(y-nx-y)^+\|$$
$$\leqslant \|(y-nx)^+\| + \|(-y)^+\| < 1 + \|(-y)^+\|$$

and therefore

$$\|(-y)^+\| > n\|x\| - 1$$

which is a contradiction since the norm is always finite.

We note that if G is an increasing open dense set of a normed lattice, then $-G$ is a decreasing open set. This observation and the above lemma lead to the following definition:

Definition 1. A normed lattice $(E, \| \|, \leqslant)$ is called quasi-Baire if the intersection of each sequence of monotonic open dense sets is dense.

If a normed lattice E is Baire, it is obvious that E is quasi-Baire but the converse is false as we will see in Example 1 but before we need result.

Let E' be the dual of E (i.e., the set of all continuous linear forms on the normed lattice E) with the supremum norm.

Definition 2. A set F of positive continuous linear forms of norm one is said to determine the order of the normal lattice E provided it satisfies

$$x \leqslant y \quad \text{if and only if} \quad u(x) \leqslant u(y), \forall u \in F.$$

Proposition 2. *Let $(E, \| \|, \leqslant)$ be a normed lattice determined by $F \subset E'$. If*

$$\sup\{\inf\{u(x)\colon\ u \in F\}\colon\ x \in E\} > 0$$

then E is quasi-Baire.

Proof. We will see that E has no proper decreasing dense sets.

If $\sup\{\inf\{u(x)\colon\ u \in F\}\colon\ x \in E\} > 0$, then there is $x_0 \in E$ such that

$$\delta = \inf\{u(x_0)\colon\ u \in F\} > 0.$$

Now, suppose G is a decreasing dense set of E. Let $x \in E$ and put $y = x + x_0$. Since G is dense, there is $z \in G$ such that $\|z - y\| < \delta$. We have

$$|u(z) - u(y)| \leqslant \|u\| \|z - y\| < \delta$$

then

$$-\delta < u(z) - u(y) = u(z) - u(z) - u(x_0)$$

therefore

$$u(z) > u(x) + u(x_0) - \delta \geqslant u(x)$$

and since G is decreasing, then $x \in G$ and thus $G = E$. □

The next example proves that a quasi-Baire space can not be a Baire space and also that the quasi-Baire property in a normed lattice does not imply the barrelled property.

Example 1. Let Σ be an algebra of subsets of a non-empty set Ω .Consider the linear subspace $l_0^\infty(\Omega,\Sigma) = \langle \chi_A \colon A \in \Sigma\rangle$ generated by the characteristic functions χ_A over Σ, i.e., $\chi_A(w) = 1$ if $w \in A$ and $\chi_A(w) = 0$ if $w \notin A$.

In $l_0^\infty(\Omega,\Sigma)$ we consider the norm given by $\|x\| = \sup\{|x(w)| \colon w \in \Omega\}$. It is well known that the dual of $l_0^\infty(\Omega,\Sigma)$ is the set of all additive finite measures defined on Σ. Then the set of Dirac's measures $F = \{\delta_w \colon w \in \Sigma\}$ determines the order of since $x \geqslant 0$ iff

$$\langle \delta_{w_0}, x\rangle = \int_\Omega x(w)\,\mathrm{d}\delta_{w_0} = x(w_0) \geqslant 0, \quad \forall w_0 \in \Omega.$$

Now, for $\chi_\Omega \in l_0^\infty(\Omega,\Sigma)$ we have

$$\langle \delta_w, \chi_\Omega\rangle = \chi_\Omega(w) = 1, \quad \forall w \in \Omega$$

and therefore

$$\inf\{\langle \delta_w, \chi_\Omega\rangle \colon w \in \Omega\} = 1$$

and thus $l_0^\infty(\Omega,\Sigma)$ is quasi-Baire by Proposition 2.

Now, if we take the clopen sets of a Hausdorff zero-dimensional space Ω, then from [1] we know that $l_0^\infty(\Omega,\Sigma)$ is not barrelled and therefore it is not Baire.

The next example shows the class of normed convex-Baire spaces to be far from the class of Baire spaces.

Example 2. Take a real number p, $0 < p < 1$. Let

$$E = \Bigl\{(x_m) \in \mathbb{R}^{\mathbb{N}} \colon \sum_{m=1}^{\infty} |x_m|_p < \infty\Bigr\}.$$

M. Valdivia [6] 281–287 proved that $E \subset l_1$ and also E with the norm of l_1 is a normed convex-Baire space that is not Baire.

For $n \in \mathbb{N}$ let $G_n = \bigl\{(x_m) \in E\colon \sum\limits_{m=1}^{\infty} (x_m^+)^p > n\bigr\}$. First we will see that G_n is open:

Let (x^j) be a convergent sequence of $E - G_n$ to a point $z = (z_k) \in E$. Then

$$\sum_{k=1}^{m} (z_k^+)^p = \sum_{k=1}^{m} \Bigl(\lim_{j\to\infty} (x_k^j)^+\Bigr)^p = \lim_{j\to\infty} \sum_{k=1}^{m} ((x_k^j)^+)^p \leqslant n, \quad \forall m \in \mathbb{N}$$

and thus $\sum\limits_{k=1}^{\infty}(z_k^+)^p \leqslant n$ and therefore $z \in E - G_n$

We will see that G_n is increasing. Let $(x_m) \in G_n$ and $(y_m) \geqslant (x_m)$; since $y_k^+ \geqslant x_k^+, k \in \mathbb{N}$, we have

$$\sum_{k=1}^{\infty}(y_k^+)^p \geqslant \sum_{k=1}^{\infty}(x_k^+)^p > n$$

and then $(y_m) \in G_n$.

Now we will prove that G_n is dense. Let $(x_k) \in E$ and $\varepsilon > 0$. Take $r \in \mathbb{N}$ such that $\sum\limits_{k=r+1}^{\infty} |x_k| < \frac{\varepsilon}{4}$.

Put $\alpha = \sum\limits_{k=1}^{\infty} \frac{1}{k^{1/p}} < \infty$ and take $s \in \mathbb{N}$, $s > r$, such that $\sum\limits_{k=r+1}^{s} \frac{1}{k} > \left(\frac{2\alpha}{\varepsilon}\right)^p n$.

We define a sequence $(y_k) \in \mathbb{R}^{\mathbb{N}}$ as follows:

$$y_k = \begin{cases} x_k, & \text{if } k \leqslant r \text{ or } k \geqslant s+1, \\ |x_k| + \dfrac{\varepsilon}{2\alpha k^{1/p}}, & \text{if } r+1 \leqslant k \leqslant s. \end{cases}$$

Then we have

$$\sum_{k=1}^{\infty}|y_k|^p = \sum_{k=1}^{r}|x_k|^p + \sum_{k=r+1}^{s}\left(|x_k| + \frac{\varepsilon}{2\alpha k^{1/p}}\right)^p + \sum_{k=r+1}^{\infty}|x_k|^p < \infty$$

and then $(y_k) \in E$. We will see that $(y_k) \in G_n$. Indeed,

$$\begin{aligned} \sum_{k=1}^{\infty}(y_k^+)^p &\geqslant \sum_{k=r+1}^{s}\left(|x_k| + \frac{\varepsilon}{2\alpha k^{1/p}}\right)^p \geqslant \sum_{k=r+}^{s}\left(\frac{\varepsilon}{2\alpha k^{1/p}}\right)^p \\ &= \left(\frac{\varepsilon}{2\alpha}\right)^p \sum_{k=r+1}^{s}\frac{1}{k} > \left(\frac{\varepsilon}{2\alpha}\right)^p\left(\frac{2\alpha}{\varepsilon}\right)^p n = n \end{aligned}$$

and therefore $(y_k) \in G_n$.

Now,

$$\begin{aligned} \|(y_k) - (x_k)\| &= \sum_{k=r+1}^{s}\left||x_k| + \frac{\varepsilon}{2\alpha k^{1/p}} - x_k\right| \\ &\leqslant 2\sum_{k=r+1}^{s}|x_k| + \frac{\varepsilon}{2\alpha}\sum_{k=r+1}^{\infty}\frac{1}{k^{1/p}} < 2\frac{\varepsilon}{4} + \frac{\varepsilon}{2\alpha}\alpha = \varepsilon. \end{aligned}$$

Finally, we will see that $\bigcap\limits_{n=1}^{\infty} G_n = \emptyset$. Indeed, if $x \in \bigcap\limits_{n=1}^{\infty} G_n$ then $\sum\limits_{k=1}^{\infty}(x_k^+)^p = \infty$ and therefore $\sum\limits_{k=1}^{\infty}|x_k|^p = \infty$ which implies $x \notin E$, which is a contradiction.

3. A characterization of quasi-Baire spaces in bitopological terms

Definition 3. We say that a bitopological space (X, P, L) is pairwise Baire if the intersection of each sequence of P-open L-dense sets is L-dense and the intersection of each sequence of L-open P-dense sets is P-dense.

Lemma 2. *If G is q-dense (q^{-1}-dense) and increasing (decreasing) then G is dense.*

Proof. Let $x \in E$ and $\varepsilon > 0$. Since G is q-dense there exists $y \in V_\varepsilon(x) \cap G$. We have

$$-y = -x - (y - x) \geqslant -x - (y - x)^+ \textit{ and } y \leqslant x + (y - x)^+.$$

If we denote $z = x + (y - x)^+$ then, since G is increasing, $z \in G$.

Finally

$$\|z - x\| = \|(y - x)^+\| = q(y - x) < \varepsilon$$

and G is dense.

The proof when G is q^{-1}-dense and decreasing is similar. □

Proposition 3. *A normed lattice E is quasi-Baire if and only if (E, q, q^{-1}) is pairwise Baire.*

Proof. Suppose E is quasi-Baire and let G_n be a sequence of q-open and q^{-1}-dense sets. By Lemma 1 G_n is decreasing $(n \in \mathbb{N})$ and then $\bigcap_{n=1}^{\infty} G_n$ is dense and since the topology deduced from the norm is finer than the deduced from q^{-1}, then $\bigcap_{n=1}^{\infty} G_n$ is q^{-1}-dense.

Analogously, if G_n is a sequence of q^{-1}-open and q-dense sets then $\bigcap_{n=1}^{\infty} G_n$ is q-dense. For the converse, suppose (E, q, q^{-1}) is pairwise Baire and let G_n be a sequence of decreasing open dense sets. Then G_n is q^{-1}-dense $(n \in \mathbb{N})$ and by Lemma 1 and the hypothesis $\bigcap_{n=1}^{\infty} G_n$ is q^{-1}-dense. Now, since $\bigcap_{n=1}^{\infty} G_n$ is a decreasing set, by the previous lemma $\bigcap_{n=1}^{\infty} G_n$ is dense. □

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