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Cihan Orhan

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TRANSLATIVITY OF ABSOLUTE WEIGHTED MEAN SUMMABILITY

C. Orhan, Ankara

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Abstract. In this paper, we give necessary and sufficient conditions on (p_n) for $|R, p_n|_k$, $k \geqslant 1$, to be translative. So we extend the known results of Al-Madi [1] and Cesco [4] to the case $k > 1$.

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1. Introduction

Let $\sum\limits_{n=0}^{\infty} a_n$ be a given series with a sequence of its partial sums (s_n) and let (p_n) be a sequence of positive real numbers such that

$$P_n := \sum_{v=0}^{n} p_v \to \infty \quad \text{as } n \to \infty, \quad P_{-1} = p_{-1} = 0.$$

The sequence-to-sequence transformation given by

$$t_n := \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

defines the (R, p_n) means of the sequence (s_n). We say that the series $\sum\limits_{n=0}^{\infty} a_n$ is $|R, p_n|_k$summable, $k \geqslant 1$, if (see [3], [9])

$$\sum_{n=1}^{\infty} n^{k-1} |t_n - t_{n-1}|^k < \infty. \tag{1}$$

We now recall the discussion of an interesting phenomenon that occurs in the study of ordinary summability. Let $x = (x_k)$ be a sequence and define the sequence

$w=(w_k)$ by $w_k=x_{k+1}$, $(k=0,1,\ldots)$. The convergence of the sequence z implies the convergence of the sequence w to the same value and conversely. One might expect that, given an infinite matrix A, if x is A-summable to s then w is A-summable to s and conversely. This, however, is not necessarily the case (see, e.g., [2], [5], [10]). For any given sequence x, if x is A-summable to s implies that w is A-summable to s, then A is called right translative. If the converse holds, then A is called left translative. If A is both right and left translative then A is translative.

Following the concept of translativity in ordinary summability, Cesco [4] introduced the concept of left translativity for $|R,p_n|$ summability. Also Fridy [6] studied ℓ-translativity for a matrix A. Analogously we call $|R,p_n|_k$, $k\geqslant 1$, left translative if the $|R,p_n|_k$summability of the series $\sum\limits_{n=0}^{\infty} a_n$ implies the $|R,p_n|_k$summability of the series $\sum\limits_{n=0}^{\infty} a_{n-1}$, $(a_{-1}=0)$; it is right translative if the converse holds, and translative if it is both left and right translative.

In [4], Cesco has given sufficient conditions for $|R,p_n|$ to be left translative.

Al-Madi [1] has also studied the problem of translativity for $|R,p_n|$ and provided some examples to illustrate the differences between right and left translativity of $|R,p_n|$. Some results for translativity of $|\overline{N},p_n|_k$ summability may also be found in [7].

In the present paper, we give necessary and sufficient conditions on (p_n) for $|R,p_n|_k$, $k\geqslant 1$, to be translative. So, we extend the known results of Al-Madi [1] and Cesco [4] to the case $k>1$.

2. Some remarks on $|R,p_n|_k \Longrightarrow |R,q_n|_k$, $k\geqslant 1$

This section is devoted to the main result of [3] that we extremely need for our purposes.

Given two summability methods A and B we write $A\Rightarrow B$ meaning that every series summable A is also summable B.

Theorem 2.1. *If $|R,p_n|_k \Longrightarrow |R,q_n|_k, k\geqslant 1$, then*

$$\frac{q_nP_n}{p_nQ_n}=O(1). \tag{2}$$

If we further suppose that

$$\sum_{n=v}^{\infty} n^{k-1}\left(\frac{q_n}{Q_nQ_{n-1}}\right)^k = O\left(\frac{v^{k-1}}{Q_v^k}\right) \tag{3}$$

then (2) *is also sufficient (see* [3]*).*

Note that condition (3) is a modified version of condition (3.1) in [3]. Actually (3) is much more useful than (3.1) in the proof of the Theorem in [3]. To see this, it is enough to apply the well-known inequality

$$(a+b)^k \leqslant 2^k(a^k+b^k), \quad (a \geqslant 0,\ b \geqslant 0,\ k \geqslant 1), \tag{4}$$

instead of applying Hölder's inequality on p. 1012, line 4 in [3]. We omit the details and refer the reader to [3].

3. The main results

Let $(\overline{s}_n)$ denote the n-th partial sum of the series $\sum\limits_{n=0}^{\infty} a_{n-1}, (a_{-1} = 0)$. Hence $\overline{s}_n = s_{n-1}, s_{-1} = 0$. Let (z_n) be the (R, p_{n+1}) transform of (s_n) and $(\overline{t}_n)$ the (R, p_n) transform of $(\overline{s}_n)$. Thus we have

$$\overline{t}_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v \overline{s}_v = \frac{1}{P_n}\sum_{v=0}^{n-1} p_{v+1} s_v, \tag{5}$$

$$z_n = \frac{1}{P_{n+1}-p_0}\sum_{v=0}^{n-1} p_{v+1} s_v = \frac{P_{n+1}}{P_{n+1}-p_0}\overline{t}_{n+1}. \tag{6}$$

We shall need the following

Lemma 3.1. *Suppose that*

$$\sum_{n=v}^{n-1} n^{k-1}\left(\frac{p_{n+1}}{P_{n+1}P_n}\right)^k = O(v^{k-1}) \tag{7}$$

where $k \geqslant 1$. Then the series $\sum\limits_{n=0}^{\infty} a_{n-1}$ is summable $|R, p_n|_k$ if and only if the series $\sum\limits_{n=0}^{\infty} a_n$ is summable $|R, p_{n+1}|_k$.

P r o o f. Assume that $\sum\limits_{n=0}^{\infty} a_{n-1}$ is summable $|R, p_n|_k$, $k \geqslant 1$. Hence

$$\sum_{n=1}^{\infty} n^{k-1}|\overline{t}_n - \overline{t}_{n-1}|^k < \infty. \tag{8}$$

By (6) we get

$$z_n - z_{n-1} = \frac{P_{n+1}}{P_{n+1}-p_0}(\overline{t}_{n+1}-\overline{t}_n) - \frac{\overline{t}_n p_0 p_{n+1}}{(P_{n+1}-p_0)(P_n-p_0)}.$$

To complete the proof, it is enough to show, by Minkowski's inequality, that

$$\sum_{n=1}^{\infty} n^{k-1} \left(\frac{P_{n+1}}{P_{n+1} - p_0} |\overline{t}_{n+1} - \overline{t}_n| \right)^k < \infty \tag{9}$$

and

$$\sum_{n=1}^{\infty} n^{k-1} \left\{ \frac{p_{n+1} |\overline{t}_n|}{(P_{n+1} - p_0)(P_n - p_0)} \right\}^k < \infty. \tag{10}$$

Since $P_n \to \infty$ as $n \to \infty$, we see that $\dfrac{P_{n+1}}{P_{n+1} - p_0} \to 1$ as $n \to \infty$. Hence (9) follows from (8).

Observe that (10) holds if and only if

$$\sum_{n=1}^{\infty} n^{k-1} \left(\frac{p_{n+1}}{P_{n+1} P_n} |\overline{t}_n| \right)^k < \infty. \tag{11}$$

Now write $\overline{u}_n := \overline{t}_n - \overline{t}_{n-1}, \overline{t}_{-1} = 0$. Then $\overline{t}_n = \sum\limits_{v=0}^{n} \overline{u}_v$. Thus (8) is equivalent to

$$\sum_{n=1}^{\infty} n^{k-1} |\overline{u}_n|^k < \infty. \tag{12}$$

Using inequality (4) we show that (11) holds:

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{k-1} \left(\frac{p_{n+1}}{P_{n+1} P_n} |\overline{t}_n| \right)^k &\leqslant \sum_{n=1}^{\infty} n^{k-1} \left(\frac{p_{n+1}}{P_{n+1} P_n} \right)^k \left(\sum_{v=0}^{n} |\overline{u}_v| \right)^k \\
&\leqslant 2^k \sum_{n=1}^{\infty} n^{k-1} \left(\frac{p_{n+1}}{P_{n+1} P_n} \right)^k \sum_{v=0}^{n} |\overline{u}_v|^k \\
&= 2^k \sum_{v=0}^{\infty} |\overline{u}_v|^k \sum_{n=\max(1,v)}^{\infty} n^{k-1} \left(\frac{p_{n+1}}{P_{n+1} P_n} \right)^k \\
&= 2^k \left\{ |\overline{u}_0|^k \sum_{n=1}^{\infty} n^{k-1} \left(\frac{p_{n+1}}{P_{n+1} P_n} \right)^k + O(1) \sum_{v=1}^{\infty} v^{k-1} |\overline{u}_v|^k \right\} < \infty
\end{aligned}$$

by assumption and (12).

Conversely, suppose that the series $\sum\limits_{n=1}^{\infty} a_n$ is summable $|R, p_{n+1}|_k$. So we have

$$\sum_{n=1}^{\infty} n^{k-1} |z_n - z_{n-1}|^k < \infty. \tag{13}$$

Now (5) and (6) yield that

$$\overline{t}_{n+1} - \overline{t}_n = \left(1 - \frac{p_0}{P_{n+1}}\right)(z_n - z_{n-1}) + \frac{p_0 p_{n+1}}{P_n P_{n+1}} z_{n-1}.$$

To prove sufficiency it suffices to show, by Minkowski's inequality, that

$$\sum_{n=1}^{\infty} n^{k-1} \left\{ \left(1 - \frac{p_0}{P_{n+1}}\right) |z_n - z_{n-1}| \right\}^k < \infty \tag{14}$$

and

$$\sum_{n=1}^{\infty} n^{k-1} \left(\frac{p_{n+1}}{P_n P_{n+1}} |z_{n-1}| \right)^k < \infty. \tag{15}$$

Since $(1 - p_0/P_{n+1}) \to 1$ as $n \to \infty$, (14) follows from (13). Writing $z_n = \sum\limits_{v=0}^{n} w_v$ we see that (13) is equivalent to

$$\sum_{n=1}^{\infty} n^{k-1} |w_n|^k < \infty. \tag{16}$$

Using (4) we show that (15) holds:

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{k-1} \left(\frac{p_{n+1}}{P_n P_{n+1}} \right)^k |z_{n-1}|^k &\leqslant 2^k \sum_{n=1}^{\infty} n^{k-1} \left(\frac{p_{n+1}}{P_{n+1} P_n} \right)^k \sum_{v=0}^{n-1} |w_v|^k \\
&= 2^k \sum_{v=0}^{\infty} |w_v|^k \sum_{n=v+1}^{\infty} n^{k-1} \left(\frac{p_{n+1}}{P_n P_{n+1}} \right)^k \\
&= 2^k \left\{ |w_0|^k \sum_{n=1}^{\infty} n^{k-1} \left(\frac{p_{n+1}}{P_{n+1} P_n} \right)^k + O(1) \sum_{v=1}^{\infty} |w_v|^k v^{k-1} \right\} \\
&< \infty
\end{aligned}$$

by assumption and (16), whence the result. □

Theorem 3.2. *Suppose that* (7) *holds. Then*

(i) $|R, p_n|_k$ *is right translative if and only if* $|R, p_{n+1}|_k \Rightarrow |R, p_n|_k$.

(ii) $|R, p_n|_k$ *is left translative if and only if* $|R, p_n|_k \Rightarrow |R, p_{n+1}|_k$.

Proof. We just prove (i); the proof of (ii) follows similar lines. Suppose that $|R, p_n|_k$ is right translative and $\sum\limits_{n=0}^{\infty} a_n$ is summable $|R, p_{n+1}|_k$. By Lemma 3.1, the series $\sum_{n=0}^{\infty} a_{n-1}$ $(a_{-1}=0)$, is summable $|R,p_n|_k$. Right translativity of $|R,p_n|_k$ implies that $\sum_{n=0}^{\infty} a_n$ is summable $|R,p_n|_k$, i.e., $|R,p_{n+1}|_k \Rightarrow |R,p_n|_k$.

Conversely, suppose that $|R,p_{n+1}|_k \Rightarrow |R,p_n|_k$ and the series $\sum_{n=0}^{\infty} a_{n-1}$ $(a_{-1}=0)$, is summable $|R,p_n|_k$. By Lemma 3.1, $\sum_{n=0}^{\infty} a_n$ is summable $|R,p_{n+1}|_k$. Then, by assumption, it is summable $|R,p_n|_k$, i.e., $|R,p_n|_k$ is right translative. □

We are now ready to present the main result:

Theorem 3.3. *Suppose that*

$$\sum_{n=v}^{\infty} n^{k-1}\left(\frac{p_{n+1}}{P_{n+1}P_n}\right)^k = O\left(\frac{v^{k-1}}{P_{v+1}^k}\right) \tag{17}$$

and

$$\sum_{n=v}^{\infty} n^{k-1}\left(\frac{p_n}{P_nP_{n-1}}\right)^k = O\left(\frac{v^{k-1}}{P_v^k}\right) \tag{18}$$

hold, where $k \geqslant 1$. Then $|R,p_n|_k$ is translative if and only if

$$\text{(a)}\ \frac{P_n}{P_{n+1}} = O\left(\frac{p_n}{p_{n+1}}\right) \quad \textit{and} \quad \text{(b)}\ \frac{P_{n+1}}{P_n} = O\left(\frac{p_{n+1}}{p_n}\right). \tag{19}$$

P r o o f. We first note that each of the conditions (17) and (18) implies (7) since $P_{v+1} \geqslant P_v \geqslant p_0$ for all $v \geqslant 0$. By Theorem 3.2, we have that $|R,p_n|_k$ is translative if and only if $|R,p_n|_k$ is equivalent to $|R,p_{n+1}|_k$. Now the conclusion follows from Theorem 2.1 and the fact that $P_{n+1} \sim P_{n+1} - p_0$ as $n \to \infty$.

Note that if we take $k=1$ in Theorem 3.3 and recall the condition $P_n \to \infty$ (i.e., (R,p_n) is regular) we see that (17) and (18) respectively reduce to the equivalent conditions

$$\text{(a)}\ \frac{P_{v+1}}{P_v} = O(1) \qquad \text{(b)}\ \frac{P_v}{P_{v-1}} = O(1). \tag{20}$$

□

So we immediately get the following

Corollary 1. *If (R,p_n) is regular and* (20a) *or* (20b) *holds, then $|R,p_n|$ is translative if and only if* (19a) *and* (19b) *hold.*

Cesco [4] proved that if (R, p_n) is absolutely regular (i.e., $\sum a_n$ is $|R, p_n|$ summable whenever $\sum |a_n| < \infty$) then condition (19 a) is sufficient for $|R, p_n|$ to be left translative. Now, an application of Theorem 1 of Mears [8] shows that if (R, p_n) is regular then it is absolutely regular but not conversely. So the conditions given by Cesco [4] are weaker than ours. However, observe that we prove the theorem in the "necessary and sufficient" form for translativity, not only for left translativity. So our result is more applicable than that of Cesco.

We conclude the paper with the following observation. If $p_n = 1$ for all n, then $|R, p_n|$ reduces to the usual $|C, 1|$ summability. In this case all the conditions of Corollary 1 hold. Hence $|C, 1|$ summability is translative. Actually, Theorem 3.3 yields the following more general result:

Corollary 2. $|C, 1|_k$ $(k \geqslant 1)$, *summability is translative.*

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Author's address: Department of Mathematics, Faculty of Science, Ankara University, 06100 Tandoğan, Ankara, Turkey, e-mail: `orhan@science.ankara.edu.tr`.