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PSEUDO-SYMMETRIC IDEALS OF SEMIGROUP AND THEIR RADICALS

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1. Introduction

Pseudo-symmetric ideals and pseudo-symmetric semigroups were studied for the first time by A. Anjaneyulu in 1980 ([2]). The aim of this paper is to establish some further properties of such ideals and semigroups. Also, the radicals of a pseudo-symmetric ideal of an arbitrary semigroup will be characterized. In fact, we will see that the concept of "pseudo-symmetricity" plays an important role in the study of radicals of semigroups.

Let S be a semigroup and I an ideal of S. Then we have the following well known definitions:

(i) I is prime $\Longleftrightarrow$ for all $x, y \in S$, $xSy \subseteq I$ implies $x \in I$ or $y \in I$;
(ii) I is completely prime $\Longleftrightarrow$ for all $x, y \in S$, $xy \in I$ implies $x \in I$ or $y \in I$;
(iii) I is semiprime $\Longleftrightarrow$ for all $x \in S$, $xSx \subseteq I$ implies $x \in I$;
(iv) I is completely semiprime $\Longleftrightarrow$ for all $x \in S$, $x^2 \in I$ implies $x \in I$.

In addition, if S has a zero element 0, then

(v) I is nilpotent $\Longleftrightarrow$ $I^n = 0$ for some integer $n > 0$;
(vi) I is nil $\Longleftrightarrow$ x is nilpotent for all $x \in I$;
(vii) I is locally nilpotent $\Longleftrightarrow$ the subsemigroup generated by any finite number of elements of I is nilpotent.

The following radicals occured in J. Bosák [3]. Let S be a semigroup and A an ideal of S. Then we define:

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(i) $R_A(S)$: the union of all ideals of S nilpotent with respect to A, i.e., $R_A(S) = \bigcup_{i\in\wedge} I_i$, where I_i is an ideal of S for each i and $I_i^{n_i} \subseteq A$ for some $n_i \geqslant 1$;

(ii) $M_A(S)$: the intersection of all prime ideals of S containing A;

(iii) $L_A(S)$: the union of all ideals of S locally nilpotent with respect to A;

(iv) $R_A^*(S)$: the union of all ideals of S nil with respect to A;

(v) $N_A(S)$: the set of all elements of S nilpotent with respect to A;

(vi) $C_A(S)$: the intersection of all completely prime ideals of S containing A.

The following beautiful result was also given in J. Bosák [3].

Lemma 1.1. ([3] Theorem 2) *Let A be an ideal of a semigroup S. Then*

(i) $A \subseteq R_A(S) \subseteq M_A(S) \subseteq L_A(S) \subseteq R_A^*(S) \subseteq N_A(S) \subseteq C_A(S) \subseteq S$;

(ii) *there exists a periodic semigroup U with a zero element 0 such that*

$$0 \subset R_0(U) \subset M_0(S) \subset L_0(S) \subset R_0^*(S) \subset N_0(S) \subset C_0(S) \subset S,$$

where "$\subset$" means the proper inclusion.

Remark 1.2. Let A be an ideal. Then, by Lemma 1.1, we can see that all the above radicals, except $N_A(S)$ are ideals of S.

Remark 1.3. It is clear that

$$R_A(S) = \{x \in S \mid \langle x\rangle^n \subseteq A \text{ for some integer } n > 0\},$$

where $\langle x\rangle$ means the principal ideal generated by x.

Definition 1.4. Let S be a semigroup and T a non-empty subset of S. We call T a pseudo-symmetric subset of S if for all $x, y \in S$, $xy \in T$ implies $xSy \subseteq T$. An ideal A of S is called a pseudo-symmetric ideal if A is also a pseudo-symmetric subset. A semigroup S is said to be pseudo-symmetric if every ideal of S is pseudo-symmetric.

Remark 1.5. All normal, quasi-commutative, left zero, right zero semigroups and bands are pseudo-symmetric semigroups (see [2] and [7]).

The following example shows that a pseudo-symmetric subset of S need not be a subsemigroup of S.

Example 1.6. Let $S = \{0, a, b, c\}$ with the following Cayley table:

	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	a	a
c	0	0	0	a

Then $T = \{0, c\}$ is clearly a pseudo-symmetric subset but not a subsemigroup of S.

In this paper, we will prove that $C_A(S) = R_A(S)$ if A is a pseudo-symmetric ideal of the semigroup S. We will also give a characterization for the radical $N_A(S)$ to be an ideal of S. Some results in [1] are generalized.

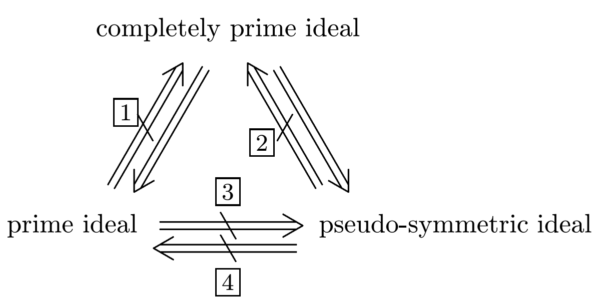
2. Pseudo-symmetric ideals

We first discuss the relationships among prime, completely prime and pseudo-symmetric ideals.

Proposition 2.1. *Let S be a semigroup. Then the following statements hold:*
(i) *Every completely prime ideal is both prime and pseudo-symmetric.*
(ii) *Let A be a pseudo-symmetric ideal of S. Then A is prime $\Longleftrightarrow$ A is completely prime.*
(iii) *Let A be a prime ideal of S. Then A is pseudo-symmetric $\Longleftrightarrow$ A is completely prime.*

P r o o f. (i) This statement is easy to observe. We hence omit the proof.
(ii) This result follows from Lemma 1 in [2].
(iii) ($\Longrightarrow$) Let A be a pseudo-symmetric ideal of S. If $xy \in A$ for some $x, y \in S$, then $xSy \subseteq A$. Since A is prime, we have $x \in A$ or $y \in A$. This shows that A is completely prime.
$\Longleftarrow$) This part follows immediately from (i). $\square$

In general, we have the following diagram:



Example 2.2. Let $S = \{0, a, \ldots, a^{n-1}\}$ be a semigroup with $a^n = 0$. Then S clearly is a commutative semigroup and $\{0\}$ is a pseudo-symmetric ideal which is neither prime nor completely prime. This shows that $\boxed{2}$ and $\boxed{4}$ are valid in the above diagram. The following example shows that $\boxed{1}$ and $\boxed{3}$ hold in the above diagram.

Example 2.3. Let $S = \{0, e, f, a, b\}$ be a set with the following Cayley table:

	0	e	f	a	b
0	0	0	0	0	0
e	0	e	0	0	b
f	0	0	f	a	0
a	0	a	0	0	f
b	0	0	b	e	0

Then S is a 0-simple semigroup. Clearly, $\{0\}$ is a prime ideal but not a pseudo-symmetric ideal of S because $ef = 0$ but $ebf = b \neq 0$. It is obvious that $\{0\}$ is not completely prime.

A semigroup S is called a left (right) duo semigroup if every left (right) ideal of S is a two sided ideal. We call S a duo semigroup if S is both a left and a right duo semigroup.

We now describe the relationship between the one-side duo semigroup and the pseudo-symmetric semigroup.

Proposition 2.4. *A left (right) duo semigroup is pseudo-symmetric.*

Proof. Let S be a left duo semigroup. Clearly, for any $x \in S$, $x \cup Sx$ is a left ideal of S containing x. Then we have $xS \subseteq x \cup Sx$ for all $x \in S$ since S is a left duo semigroup. Now, let A be an arbitrary ideal of S with $xy \in A$ for some $x, y \in S$. Then we have $xSy \subseteq (x \cup Sx)y \subseteq xy \cup Sxy \subseteq A$. This shows that A is a pseudo-symmetric ideal and hence S is a pseudo-symmetric semigroup.

Similarly, we can prove that if S is a right duo semigroup then S is a pseudo-symmetric semigroup. □

The following example illustrates that a pseudo-symmetric semigroup need not be a left (right) duo semigroup.

Example 2.5. Let $S = \{0, a, b, c\}$ be a set with the following Cayley table:

	0	a	b	c
0	0	0	0	0
a	0	0	a	0
b	0	0	b	0
c	0	a	a	c

Then it is clear that

(i) S contains five ideals, namely, $I_1 = \{0\}$, $I_2 = \{0, a\}$, $I_3 = \{0, a, c\}$, $I_4 = \{0, a, b\}$ and $I_5 = S$. These ideals are all pseudo-symmetric;

(ii) S is neither a left nor a right duo semigroup since $bS = \{0, b\}$ and $Sc = \{o, c\}$ are not ideals of S.

Lemma 3.1. *Let S be a semigroup and A an ideal of S. Then*

(i) *for any $x \in S$, $x \in N_A(S) \Longleftrightarrow x^2 \in N_A(S)$;*

(ii) *for any $x, y \in S$, $xy \in N_A(S) \Longleftrightarrow yx \in N_A(S)$.*

Proof. (i) If $x \in N_A(S)$ then there exists $n > 0$ such that $x^n \in A$. Since A is an ideal, we have $x^{2n} = (x^2)^n \in A$. This leads to $x^2 \in N_A(S)$.

Conversely, if $x^2 \in N_A(S)$, then there exists $n > 0$ such that $(x^2)^n = x^{2n} \in A$. Therefore, $x \in N_A(S)$.

(ii) Let $xy \in N_A(S)$. Then we have $(xy)^n \in A$ for some $n > 0$. We can assume that $n \geqslant 2$. Since $x(yx)^{n-1}y = (xy)^n \in A$, $(yx)^{n+1} \in A$. Therefore, $yx \in N_A(S)$. The converse can be proved similarly. $\square$

The following theorem is the main theorem of this paper. It shows that the concept "pseudo-symmetricity" is essential for the radical $N_A(S)$ to become an ideal of S.

Theorem 3.2. *Let A be an ideal of a semigroup S. Then the following statements are equivalent:*

(i) *$N_A(S)$ is an ideal of S.*

(ii) *$N_A(S)$ is a pseudo-symmetric ideal of S.*

(iii) *$N_A(S)$ is a pseudo-symmetric subset of S.*

(iv) *$N_A(S)$ is a one-side ideal of S.*

(v) *$R^*_A(S) = N_A(S) = C_A(S)$.*

Proof. (ii) $\Longrightarrow$ (iii) and (v) $\Longrightarrow$ (i) are clear.

(i) $\Longrightarrow$ (ii). Assume that $N_A(S)$ is an ideal of S. Then, by Lemma 3.1 (ii), for any $x, y \in S$ we know that $xy \in N_A(S)$ implies $yx \in N_A(S)$. Suppose that $yx \in N_A(S)$ for some $x, y \in S$. Since $N_A(S)$ is an ideal of S we have $(yx)s \in N_A(S)$ for any $s \in S$. Thus, by using Lemma 3.1(ii) again, we have $xsy \in N_A(S)$. This shows that $xSy \subseteq N_A(S)$. Consequently, $N_A(S)$ is a pseudo-symmetric ideal of S.

(iii) $\Longrightarrow$ (iv). Let $N_A(S)$ be a pseudo-symmetric subset of S. Then, by Lemma 3.1, for all $x \in N_A(S)$ and $s \in S$, we have

$$x \in N_A(S) \Longrightarrow x^2 \in N_A(S) \Longrightarrow xsx \in N_A(S)$$
$$\Longrightarrow x(xs) \in N_A(S) \Longrightarrow xs(xs) = (xs)^2 \in N_A(S) \Longrightarrow xs \in N_A(S).$$

This shows that $N_A(S)$ is a right ideal of S. The fact that $N_A(S)$ is a left ideal can be proved similarly.

(iv) $\Longrightarrow$ (v). Without loss of generality, we may assume that $N_A(S)$ is a left ideal of S. Then for all $x \in N_A(S)$ and all $s \in S$ we have $sx \in N_A(S)$. It follows from

Lemma 3.1(ii) that $xs \in N_A(S)$. This shows that $N_A(S)$ is an ideal of S. Therefore, $R^*_A(S) = N_A(S)$. Also, it is obvious that $N_A(S)$ is a completely semiprime ideal of S. Thus, it follows from Theorem II. 3.7 in [6] that $N_A(S)$ is the intersection of some completely prime ideals of S and so $N_A(S) \supseteq C_A(S)$. By using Lemma 1.1, we obtain $N_A(S) = C_A(S)$. □

Theorem 3.3. *Let A be an ideal of a semigroup S. Then the following statements are equivalent:*

(i) $N_A(S) = A$.

(ii) *A is completely semiprime.*

(iii) $A = R_A(S) = M_A(S) = L_A(S) = R^*_A(S) = N_A(S) = C_A(S)$.

Proof. (i) $\Longrightarrow$ (ii). Assume that $N_A(S) = A$ and $x^2 \in A$. Thus $x^2 \in N_A(S)$ and so $x \in N_A(S) = A$ by Lemma 3.1. This shows that A is completely semiprime.

(ii) $\Longrightarrow$ (i). Let A be completely semiprime. We only need to prove that $N_A(S) \subseteq A$. For any $x \in N_A(S)$ there exists $n > 0$ such that $x^n \in A$. This leads to $\big(x^{\frac{n}{2}}\big)^2$ or $\big(x^{\frac{n+1}{2}}\big)^2 \in A$. As a consequence, we have $x^{\frac{n}{2}}$ or $x^{\frac{n+1}{2}} \in A$. Notice that $\frac{n}{2}, \frac{n+1}{2} < n$, so by induction on n, we can eventually obtain that $x \in A$. This shows that $N_A(S) = A$.

(iii) $\Longrightarrow$ (i). Clear.

(i) $\Longrightarrow$ (iii). By Theorem 3.2 and Lemma 1.1 we can easily obtain the required result. □

The following theorem gives a condition for the radicals described by J. Bosák in [3] to be equal.

Theorem 3.4. *Let S be an arbitrary semigroup and A a pseudo-symmetric ideal of S. Then the following equalities hold:*

$$R_A(S) = M_A(S) = L_A(S) = R^*_A(S) = N_A(S) = C_A(S).$$

Proof. In view of Lemma 1.1 and Theorem 3.2, we only need to prove that $N_A(S) \subseteq R_A(S)$. For this purpose, we let $x \in N_A(S)$ and $s \in S$. Then we have $x^n \in A$ for some $n \geqslant 1$. If $n = 1$ then clearly $x \in A \subseteq R_A(S)$. If $n > 1$, then we let S^1 be the semigroup adjoint with an identity 1. Since the ideal A is pseudo-symmetric, we have $\langle x\rangle x^{n-1} = (S^1xS^1)x^{n-1} \subseteq S^1A \subseteq A$. By using induction on n, we can easily obtain that $\langle x\rangle^{n-1} x \subseteq A$. Thus,

$$\langle x\rangle^n = \langle x\rangle^{n-1}(S^1xS^1) = \big(\langle x\rangle^{n-1} S^1x\big)S^1 \subseteq \big(\langle x\rangle^{n-1} x\big)S^1 \subseteq A.$$

Therefore, $x \in R_A(S)$. The proof is completed. □

We would like to point out here that the converse of Theorem 3.4 is not true. This can be illustrated by the following example:

Example 3.5. Let $S = \{0, a, b, c\}$ be a set with the following Cayley table:

	0	a	b	c
0	0	0	0	0
a	0	0	a	0
b	0	0	b	0
c	0	a	0	c

Then S is a semigroup. Moreover, we have the following:
(i) S contains five ideals, namely, $I_1 = \{0\}$, $I_2 = \{0, a\}$, $I_3 = \{0, a, b\}$, $I_4 = \{0, a, c\}$ and $I_5 = S$.
(ii) All the above ideals except $\{0\}$ are pseudo-symmetric ideals.
(iii) I_1 and I_2 are not prime ideals, but I_3, I_4 and I_5 are all completely prime ideals.
(iv) $R_0(S) = I_2 = C_0(S)$.

By Proposition 2.4, we can immediately deduce the following result obtained by A. Anjaneyulu in [1].

Corollary 3.6. ([1] Proposition 1.3) *For any ideal A in a left (right) duo semigroup S, we have $R_A(S) = M_A(S) = N_A(S)$.*

Finally, we discuss the one-sided primary ideal given in [1].

Definition 3.7. ([1]) Call an ideal A of a semigroup S left (right) primary provided that the following conditions hold:
(i) If X, Y are ideals of S such that $XY \subseteq A$ and $Y \nsubseteq A$ ($X \nsubseteq A$) then $X \subseteq M_A(S)$ ($Y \subseteq M_A(S)$);
(ii) $M_A(S)$ is a prime ideal of S.

The following theorem gives a characterization for a pseudo-symmetric ideal to be a one-sided primary ideal.

Theorem 3.8. *Let S be a semigroup and A a pseudo-symmetric ideal of S. Then A is left primary if and only if the following condition holds:*

$(*)$ *for all $x, y \in S$, $xy \in A$ and $y \notin A$ imply $x^n \in A$ for some $n > 0$.*

P r o o f. $\Longrightarrow$) Suppose that A is a left primary ideal of S and $xy \in A$ with $y \notin A$. Then, since A is pseudo-symmetric and $xy \in A$, we have $\langle x\rangle \langle y\rangle = S^1(xS^1S^1y)S^1 \subseteq$

$S^1AS^1 \subseteq A$. Thus, by Definition 3.7, we have $\langle x\rangle \subseteq M_A(S)$. By Theorem 3.4 we have $M_A(S) = N_A(S)$. This implies that $x \in N_A(S)$ and so $x^n \in A$ for some $n \geqslant 1$. Hence, $(*)$ holds.

$\Longleftarrow$) Suppose that $(*)$ holds. Then we have the following situations:

(i) X and Y are ideals of S with $XY \subseteq A$ but $Y \nsubseteq A$. Then there exists an element $y \in Y$ but $y \notin A$ such that for all $x \in X$, $xy \in XY \subseteq A$. By $(*)$ and Theorem 3.4, we immediately obtain that $x \in M_A(S)$ for all $x \in X$. This implies that $X \subseteq M_A(S)$.

(ii) Assume that $xy \in M_A(S)$. Then we have $xy \in N_A(S)$ and hence we can find a smallest positive integer n such that $(xy)^n \in A$. If $n = 1$ then $xy \in A$. By $(*)$ we have $x^k \in A$ for some integer $k > 0$ or $y \in A$. This means that $x \in M_A(S)$ or $y \in M_A(S)$. Now, we assume that $n > 1$. We have the following cases:

Case (i). If $y(xy)^{n-1} \notin A$, then by $(*)$ and $x\bigl(y(xy)^{n-1}\bigr) = (xy)^n \in A$ we have $x^n \in A$ for some $n > 0$. This implies that $x \in N_A(S) = M_A(S)$.

Case (ii). If $y(xy)^{n-1} \in A$ then since $(xy)^{n-1} \notin A$, we have $y \in N_A(S) = M_A(S)$.

Hence, in all cases we must have $x \in M_A(S)$ or $y \in M_A(S)$. This shows that $M_A(S)$ is completely prime, and so $M_A(S)$ is prime. □

By Proposition 2.4, we re-deduce the following result in [1]:

Corollary 3.9. ([1] Theorem 2.4) *If S is a one-side duo-semigroup then an ideal A of S is left primary if and only if $(*)$ holds.*

Remark 3.10. It is well known that every ideal of a one-side duo semigroup has a primary decomposition. Unfortunately, we can not extend this result to pseudo-symmetric semigroups. (See [1] Example 2.2)

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