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ON THE NEUMANN-POINCARÉ OPERATOR

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Abstract. Let Γ be a rectifiable Jordan curve in the finite complex plane $\mathbb{C}$ which is regular in the sense of Ahlfors and David. Denote by $L^2_C(\Gamma)$ the space of all complex-valued functions on Γ which are square integrable w.r. to the arc-length on Γ. Let $L^2(\Gamma)$ stand for the space of all real-valued functions in $L^2_C(\Gamma)$ and put

$$L^2_0(\Gamma) = \{h \in L^2(\Gamma);\ \int_\Gamma h(\zeta)\,|\,\mathrm{d}\zeta| = 0\}.$$

Since the Cauchy singular operator is bounded on $L^2_C(\Gamma)$, the Neumann-Poincaré operator C^Γ_1 sending each $h \in L^2(\Gamma)$ into

$$C^\Gamma_1 h(\zeta_0) := \operatorname{Re}(\pi \mathrm{i})^{-1}\,\mathrm{P.V.}\int_\Gamma \frac{h(\zeta)}{\zeta-\zeta_0}\,\mathrm{d}\zeta, \quad \zeta_0 \in \Gamma,$$

is bounded on $L^2(\Gamma)$. We show that the inclusion

$$C^\Gamma_1(L^2_0(\Gamma)) \subset L^2_0(\Gamma)$$

characterizes the circle in the class of all AD-regular Jordan curves Γ.

Keywords: Cauchy's singular operator, the Neumann-Poincaré operator, curves regular in the sense of Ahlfors and David

MSC 2000: 30E20

In what follows Γ will always be a simple closed oriented curve in the Euclidean plane $\mathbb{R}^2$ which is AD-regular in the sense that

$$\sup r^{-1}\mathscr{H}^1(D(z,r)\cap\Gamma) < +\infty,$$

where $\mathscr{H}^1$ denotes the usual 1-dimensional Hausdorff measure ($\equiv$ length as defined in [12], chap. II, par. 8) and the least upper bound is taken over all discs

$$D(z,r)=\{\zeta\in\mathbb{R}^2;\ |\zeta-z|<r\}.$$

Let $L^2_C(\Gamma)$ denote the complex Banach space of all square-integrable (w.r. to $\mathscr{H}^1$) complex-valued functions f on Γ with the usual norm

$$\|f\|_{L^2_C(\Gamma)}=\left(\int_\Gamma |f|^2\,\mathrm{d}\mathscr{H}^1\right)^{\frac12}.$$

As shown by G. David in [2], for any $f\in L^2_C(\Gamma)$ the Cauchy singular integral

$$C^\Gamma f(z)=\frac{1}{\pi\mathrm{i}}\,\mathrm{P.V.}\int_\Gamma\frac{f(\zeta)}{\zeta-z}\,\mathrm{d}\zeta$$

is defined for $\mathscr{H}^1$-a.e. $z\in\Gamma$ and represents an element of $L^2_C(\Gamma)$ again; besides that, the operator

$$C^\Gamma\colon\ f\mapsto C^\Gamma f$$

is bounded on $L^2_C(\Gamma)$. We shall use the symbol $L^2(\Gamma)$ for the space of all real-valued functions in $L^2_C(\Gamma)$ which is a Banach space over the reals with the norm $\|\dots\|_{L^2(\Gamma)}$ given by the same expression as in $L^2_C(\Gamma)$. The Neumann-Poincaré operator C_1^Γ sends each $h\in L^2(\Gamma)$ into

$$C_1^\Gamma h:=\operatorname{Re}C^\Gamma h;$$

C_1^Γ is a bounded operator on $L^2(\Gamma)$. Let

$$L^2_0(\Gamma)=\left\{h\in L^2(\Gamma);\ \int_\Gamma h\,\mathrm{d}\mathscr{H}^1=0\right\}.$$

Let us recall that a Jordan curve $\Gamma\subset\mathbb{R}^2$ is termed chord-arc, if it satisfies the following Lavrentjev condition (which turns out to be stronger than AD-regularity):

There is a positive constant k such that for any $z_1,z_2\in\Gamma$ the complementary subarcs Γ_1,Γ_2 of Γ with end-points z_1, z_2 satisfy

$$\min\{\mathscr{H}^1(\Gamma_1),\mathscr{H}^1(\Gamma_2)\}\leqslant k|z_1-z_2|.$$

Various properties of the Neumann-Poincaré operator corresponding to a chord-arc curve have been investigated in [7]. We shall start with an elementary example of a chord-arc curve Γ and a function $h\in L^2_0(\Gamma)$ such that $C_1^\Gamma h\in L^2_0(\Gamma)$ and

$$\|C_1^\Gamma h\|_{L^2(\Gamma)}>\|h\|_{L^2(\Gamma)}; \tag{1}$$

this answers in the negative a question posed in [8].

Remark 1. In what follows $\mathbb{R}^2$ will be identified in the usual way with $\mathbb{C}$; functions of the complex variable $z = x + \mathrm{i}y$ (where i is the imaginary unit and $x, y \in \mathbb{R}$) are considered as functions of two real variables x, y; complex-valued functions are not distinguished from two-dimensional vector-valued functions, etc.

Example 1. Fix $\alpha \in \left]0, \frac{\pi}{12}\right[$ and consider the circular arc

$$\Gamma_1 = \{\mathrm{e}^{\mathrm{i}\theta};\ \alpha < \theta < 2\pi - \alpha\}$$

and the oriented segments

$$\Gamma_2 = \{-t\cos\alpha + (1-t)\mathrm{e}^{-\mathrm{i}\alpha};\ 0 < t < 1\}$$

(whose end-points are $\mathrm{e}^{-i\alpha}$, $-\cos\alpha$) and

$$\Gamma_3 = \{t\mathrm{e}^{\mathrm{i}\alpha} - (1-t)\cos\alpha;\ 0 < t < 1\}$$

(whose end-points are $-\cos\alpha, \mathrm{e}^{\mathrm{i}\alpha}$). Joining the arc and the segments with the corresponding end-points we arrive at an oriented simple closed curve

$$\Gamma = \Gamma_1 \cup \{\mathrm{e}^{-\mathrm{i}\alpha}\} \cup \Gamma_2 \cup \{-\cos\alpha\} \cup \Gamma_3 \cup \{\mathrm{e}^{\mathrm{i}\alpha}\}$$

which is obviously chord-arc.

Define now

$$h = \begin{cases} 0 & \text{on} \quad \Gamma_1 \cup \{-\cos\alpha\}, \\ 1 & \text{on} \quad \Gamma_3, \\ -1 & \text{on} \quad \Gamma_2. \end{cases}$$

Clearly, $h \in L^2_0(\Gamma)$. If ζ_0 is fixed in the complement of the closure of Γ_j, then $\Delta\arg(\zeta - \zeta_0;\ \zeta \in \Gamma_j)$ will denote the increment of the argument of $\zeta - \zeta_0$ as ζ describes the oriented arc Γ_j ($j = 1, 2, 3$). We have

$$\begin{aligned} C_1^\Gamma h(\zeta_0) &= \frac{1}{\pi}[-\Delta\arg(\zeta - \zeta_0; \zeta \in \Gamma_2) + \Delta\arg(\zeta - \zeta_0; \zeta \in \Gamma_3)] \quad \text{for} \quad \zeta_0 \in \Gamma_1, \\ C_1^\Gamma h(\zeta_0) &= -\frac{1}{\pi}\Delta\arg(\zeta - \zeta_0; \zeta \in \Gamma_2) \quad \text{for} \quad \zeta_0 \in \Gamma_3, \\ C_1^\Gamma h(\zeta_0) &= \frac{1}{\pi}\Delta\arg(\zeta - \zeta_0; \zeta \in \Gamma_3) \quad \text{for} \quad \zeta_0 \in \Gamma_2. \end{aligned}$$

Denoting by $\overline{\zeta_0}$ the complex conjugate of $\zeta_0 \in \mathbb{C}$ we have thus

$$C_1^\Gamma h(\overline{\zeta_0}) = -C_1^\Gamma h(\zeta_0), \quad \zeta_0 \in \Gamma,$$

so that $C_1^\Gamma h \in L_0^2(\Gamma)$. We are now going to compare $\|C_1^\Gamma h\|_{L^2(\Gamma)}$ and $\|h\|_{L^2(\Gamma)}$. Consider first $\zeta_0 = \mathrm{e}^{\mathrm{i}\theta}$ with $\alpha < \theta < \pi - \alpha$. Elementary geometric consideration yields

$$\Delta \arg(\zeta - \zeta_0; \zeta \in \Gamma_3) = -\Delta \arg(\zeta - \zeta_0; \zeta \in \Gamma_2) + \alpha.$$

Since all the half-lines emanating from ζ_0 and intersecting the segment $\{(1-t)\mathrm{e}^{\mathrm{i}(\pi+\alpha)} + t\mathrm{e}^{\mathrm{i}(2\pi-\alpha)}; 0 \leqslant t \leqslant 1\}$ meet also Γ_2, we get

$$-\Delta \arg(\zeta - \zeta_0; \zeta \in \Gamma_2) \geqslant \frac{\pi}{2} - \alpha.$$

We have thus

$$\begin{aligned} C_1^\Gamma h(\zeta_0) &= \frac{1}{\pi}[\Delta \arg(\zeta - \zeta_0; \zeta \in \Gamma_3) - \Delta \arg(\zeta - \zeta_0; \zeta \in \Gamma_2)] \\ &\geqslant \frac{1}{\pi}(\pi - 2\alpha + \alpha) = 1 - \frac{\alpha}{\pi} > 1 - \frac{1}{12}. \end{aligned}$$

Symmetrically, if $\zeta_0 = \mathrm{e}^{\mathrm{i}\theta}$ with $\pi + \alpha < \theta < 2\pi - \alpha$, then

$$C_1^\Gamma h(\zeta_0) < -\frac{11}{12}.$$

Summarizing we arrive at

$$\begin{aligned} \int_\Gamma |C_1^\Gamma h(\zeta_0)|^2 \,\mathrm{d}\mathscr{H}^1(\zeta_0) &> \left(\frac{11}{12}\right)^2 \mathscr{H}^1(\{\mathrm{e}^{\mathrm{i}\theta}; \theta \in]\alpha, \pi - \alpha[\cup]\pi + \alpha, 2\pi - \alpha[\}) \\ &= \left(\frac{11}{12}\right)^2 (2\pi - 4\alpha) > \left(\frac{11}{12}\right)^2 \cdot \left(2\pi - \frac{\pi}{3}\right) > 4. \end{aligned}$$

On the other hand,

$$\int_\Gamma |h(\zeta_0)|^2 \,\mathrm{d}\mathscr{H}^1(\zeta_0) = 2|\mathrm{e}^{\mathrm{i}\alpha} + \cos\alpha| < 4,$$

which shows that Γ and h satisfy (1).

Remark 2. If $\mathscr{K}(\Gamma)$ denotes the subspace of all constant functions then $C_1^\Gamma(\mathscr{K}(\Gamma)) = \mathscr{K}(\Gamma)$. Hence it is possible to consider C_1^Γ as an operator acting on the quotient space $L^2(\Gamma)/\mathscr{K}(\Gamma)$ formed by the classes

$$\hat{f} = \{f + c;\ c \in \mathscr{K}(\Gamma)\}$$

with the quotient norm

$$\|\hat{f}\|_{L^2(\Gamma)/\mathscr{K}(\Gamma)} = \inf\{\|f + c\|_{L^2(\Gamma)};\ c \in \mathscr{K}(\Gamma)\}, \quad f \in L^2(\Gamma).$$

Example 1 shows that this operator C_1^Γ on $L^2(\Gamma)/\mathscr{K}(\Gamma)$ need not be contractive for chord-arc curves Γ. Indeed, taking Γ and h from Example 1 we observe that $h \in L_0^2(\Gamma)$ and $C_1^\Gamma h \in L_0^2(\Gamma)$; as $L_0^2(\Gamma)$ and $\mathscr{K}(\Gamma)$ are orthogonal subspaces w.r. to the usual scalar product in $L^2(\Gamma)$, we have by (1)

$$\|\hat{h}\|_{L^2(\Gamma)/\mathscr{K}(\Gamma)} = \|h\|_{L^2(\Gamma)} > \|C_1^\Gamma h\|_{L^2(\Gamma)} = \|C_1^\Gamma \hat{h}\|_{L^2(\Gamma)/\mathscr{K}(\Gamma)}.$$

We include another example showing that, for general chord-arc curve Γ, the operator C_1^Γ need not act in $L_0^2(\Gamma)$ (i.e. $C_1^\Gamma(L_0^2(\Gamma)) \not\subset L_0^2(\Gamma)$ is possible).

Example 2. Fix $\alpha \in \left]0, \frac{\pi}{2}\right[$ and consider the circular arc

$$\Gamma_1 = \{\mathrm{e}^{\mathrm{i}t};\ \alpha < t < 2\pi - \alpha\}$$

with end-points $\mathrm{e}^{\mathrm{i}\alpha}$, $\mathrm{e}^{-\mathrm{i}\alpha}$ and another circular arc (situated in the circumference centered at $2\cos\alpha$ on the real axis)

$$\Gamma_2 = \{2\cos\alpha + \mathrm{e}^{-\mathrm{i}t};\ \pi - \alpha < t < \pi + \alpha\}$$

whose end points are $\mathrm{e}^{-\mathrm{i}\alpha}$, $\mathrm{e}^{\mathrm{i}\alpha}$. Joining these arcs with the common end-points we get the curve

$$\Gamma = \Gamma_1 \cup \{\mathrm{e}^{-\mathrm{i}\alpha}\} \cup \Gamma_2 \cup \{\mathrm{e}^{\mathrm{i}\alpha}\}.$$

Define now the function f by

$$f = \begin{cases} \frac{1}{\pi-\alpha} & \text{on } \Gamma_1, \\ -\frac{1}{\alpha} & \text{on } \Gamma_2, \\ 0 & \text{on } \{\mathrm{e}^{\mathrm{i}\alpha}, \mathrm{e}^{-\mathrm{i}\alpha}\}. \end{cases}$$

Since $\mathscr{H}^1(\Gamma_1) = 2\pi - 2\alpha$ and $\mathscr{H}^1(\Gamma_2) = 2\alpha$, we have $f \in L_0^2(\Gamma)$. We are going to show that $C_1^\Gamma f \notin L_0^2(\Gamma)$. If $\zeta_0 \in \Gamma_1$, then

$$\begin{aligned} C_1^\Gamma f(\zeta_0) &= \frac{1}{\pi-\alpha} \operatorname{Re} \frac{1}{\pi\mathrm{i}} \,\mathrm{P.V.}\!\int_{\Gamma_1} \frac{\mathrm{d}\zeta}{\zeta - \zeta_0} - \frac{1}{\alpha} \operatorname{Re} \frac{1}{\pi\mathrm{i}} \int_{\Gamma_2} \frac{\mathrm{d}\zeta}{\zeta-\zeta_0} \\ &= \frac{1}{\pi-\alpha} \cdot \frac{\pi-\alpha}{\pi} - \frac{1}{\alpha} \cdot \frac{\alpha}{\pi} = 0. \end{aligned}$$

If $\zeta_0 \in \Gamma_2$, then

$$\operatorname{Re} \frac{1}{\pi\mathrm{i}} \int_{\Gamma_1} \frac{\mathrm{d}\zeta}{\zeta-\zeta_0} = \frac{2\pi - (\pi-\alpha)}{\pi} = \frac{\pi+\alpha}{\pi},$$

while

$$\operatorname{Re}\frac{1}{\pi \mathrm{i}}\,\mathrm{P.V.}\int_{\Gamma_2}\frac{\mathrm{d}\zeta}{\zeta-\zeta_0}=-\frac{\pi-(\pi-\alpha)}{\pi}=-\frac{\alpha}{\pi};$$

consequently,

$$C_1^{\Gamma}f(\zeta_0)=\frac{1}{\pi-\alpha}\cdot\frac{\pi+\alpha}{\pi}-\frac{1}{\alpha}\left(-\frac{\alpha}{\pi}\right)=\frac{2}{\pi-\alpha}.$$

We see that $C_1^{\Gamma}f$ does not belong to $L_0^2(\Gamma)$, because it vanishes on Γ_1 and remains positive on Γ_2.

This example might make the impression that the occurence of $f\in L_0^2(\Gamma)$ with $C_1^{\Gamma}f\notin L_0^2(\Gamma)$ is exceptional. Actually, the inclusion

$$C_1^{\Gamma}(L_0^2(\Gamma))\subset L_0^2(\Gamma) \tag{2}$$

characterizes the circle among all AD-regular curves Γ; this follows from Propositions 1, 2 below.

Proposition 1. *If* (2) *holds, then the logarithmic potential*

$$u(z):=\int_{\Gamma}\ln\frac{1}{|z-\zeta|}\,\mathrm{d}\mathscr{H}^1(\zeta) \tag{3}$$

remains constant on the bounded complementary domain G of Γ.

Proof. Since Γ is AD-regular, the logarithmic potential (3) is defined for all $z\in\mathbb{C}$ and represents a finite continuous function on $\mathbb{C}$ which is harmonic on the complement of Γ ([1]; concerning logarithmic potentials, cf. also section 3 in [4]). Differentiation under the integral sign yields for $z\in\mathbb{C}\setminus\Gamma$ (we denote by ∂_j the partial derivative w.r. to the j-th variable for $j=1,2$)

$$\operatorname{grad}u(z):=\partial_1u(z)+\mathrm{i}\partial_2u(z)=\int_{\Gamma}\frac{\zeta-z}{|\zeta-z|^2}d\mathscr{H}^1(\zeta),\qquad z\in\mathbb{C}\setminus\Gamma. \tag{4}$$

Consider a parametrization of Γ given by a 1–1 absolutely continuous complex-valued function of the real parameter t

$$\gamma\colon\ [a,b[\to\Gamma$$

such that

$$\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t}\neq 0 \tag{5}$$

for a.e. $t \in [a,b[$. (It is possible to choose arc-length as parameter.) Then $A \subset [a,b[$ is Lebesgue measurable iff $\gamma(A)$ is $\mathscr{H}^1$-measurable in which case

$$\mathscr{H}^1(\gamma(A)) = \int_A \left|\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t}\right| \mathrm{d}t.$$

If (5) holds, then the unit tangent vector $\tau(\zeta)$ of Γ at $\zeta = \gamma(t)$ may be introduced by

$$\tau(\zeta) = \frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} \Big/ \left|\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t}\right|, \quad \zeta = \gamma(t);$$

τ is defined a.e. w.r. to $\mathscr{H}^1$ on Γ and represents an $\mathscr{H}^1$-measurable complex-valued ($\equiv$ vector-valued) function on Γ. We have for any $f \in L^2(\Gamma)$

$$\int_\Gamma f \,\mathrm{d}\mathscr{H}^1 = \int_a^b f(\gamma(t)) \cdot |\gamma'(t)| \,\mathrm{d}t = \int_\Gamma f(\zeta)\bar{\tau}(\zeta)\,\mathrm{d}\zeta,$$

where $\bar{\tau}(\zeta)$ is the complex conjugate of $\tau(\zeta)$ and the last integral is the usual complex curvilinear integral w.r. to the complex variable $\zeta \in \Gamma$.

We shall now consider the function ψ of the complex variable $z \in G$ given by

$$(6) \quad \psi(z) := \int_\Gamma \frac{\mathrm{d}\mathscr{H}^1(\zeta)}{\zeta - z} = \int_\Gamma \frac{\bar{\tau}(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \int_\Gamma \frac{\overline{\zeta - z}}{|\zeta - z|^2}\,\mathrm{d}\mathscr{H}^1(\zeta) = \partial_1 u(z) - \mathrm{i}\partial_2 u(z).$$

According to [2], the principal value of the singular integral of the Cauchy type

$$\mathrm{P.V.} \int_\Gamma \frac{\bar{\tau}(\zeta)}{\zeta - \zeta_0}\,\mathrm{d}\zeta = \lim_{\varepsilon \downarrow 0} \int_{\Gamma \setminus D(\zeta_0,\varepsilon)} \frac{\bar{\tau}(\zeta)}{\zeta - \zeta_0}\,\mathrm{d}\zeta$$

exists for $\mathscr{H}^1$-a.e. $\zeta_0 \in \Gamma$. It follows from section 2.3 in chap. III in [11] that the angular limits of $\psi(z)$ as $z \in G$ approaches $\zeta_0 \in \Gamma$ (to be denoted by $\psi_G(\zeta_0)$) exist for $\mathscr{H}^1$-a.e. $\zeta_0 \in \Gamma$ and are given by

$$(7) \qquad \psi_G(\zeta_0) = \mathrm{P.V.} \int_\Gamma \frac{\bar{\tau}(\zeta)}{\zeta - \zeta_0}\,\mathrm{d}\zeta + \pi\mathrm{i}\bar{\tau}(\zeta_0)$$

provided Γ is positively oriented w.r. to G. Using the notation from chap. 10 in [3] and referring to section 7 in [2] and theorems 10.6, 10.4 in [3] we have

$$(8) \qquad \begin{aligned} \psi &\in E_2(G) \cap E_1(G), \\ \psi_G &\in L^2(\Gamma), \\ \psi(z) &= \frac{1}{2\pi\mathrm{i}} \int_\Gamma \frac{\psi_G(\zeta)}{\zeta - z}\,\mathrm{d}\zeta, \quad z \in G. \end{aligned}$$

Let us denote by $\langle c, d\rangle := \operatorname{Re} c\bar{d}$ the scalar product of vectors $c, d \in \mathbb{R}^2 \equiv \mathbb{C}$ and

$$\nu(\zeta) := i\tau(\zeta) \ (= \nu_1(\zeta) + i\nu_2(\zeta))$$

the unit interior normal of G at $\zeta \in \Gamma$ (which is defined for $\mathscr{H}^1$-a.e. $\zeta \in \Gamma$); the normal derivative $\frac{\partial u}{\partial \nu}(\zeta)$ will always be interpreted as the angular limit of the expression

$$\langle \nu(\zeta), \operatorname{grad} u(z)\rangle := \nu_1(\zeta)\partial_1 u(z) + \nu_2(\zeta)\partial_2 u(z)$$

as $z \in G$ tends to $\zeta \in \Gamma$. It follows from (4), (6), (7) that, for $\mathscr{H}^1$-a.e. $\zeta_0 \in \Gamma$,

$$\begin{aligned} \frac{\partial u}{\partial \nu}(\zeta_0) &= \nu_1(\zeta_0) \operatorname{Re} \mathrm{P.V.} \int_\Gamma \frac{\bar{\tau}(\zeta)}{\zeta - \zeta_0}\,\mathrm{d}\zeta - \nu_2(\zeta_0) \operatorname{Im} \mathrm{P.V.} \int_\Gamma \frac{\bar{\tau}(\zeta)}{\zeta - \zeta_0}\,\mathrm{d}\zeta \\ &\quad + \nu_1(\zeta_0) \cdot \operatorname{Re}[\pi \mathrm{i} \bar{\tau}(\zeta_0)] - \nu_2(\zeta_0) \operatorname{Im}[\pi \mathrm{i}\bar{\tau}(\zeta_0)] \\ &= -\pi + \mathrm{P.V.} \int_\Gamma \frac{\langle \nu(\zeta_0), \zeta - \zeta_0\rangle}{|\zeta - \zeta_0|^2}\,\mathrm{d}\mathscr{H}^1(\zeta). \end{aligned} \tag{9}$$

We shall now compute the last integral under the present assumption (2). Let e denote the function which is identically equal to 1 on Γ. We shall first observe that the operator $(C_1^\Gamma)^*$ adjoint to C_1^Γ w.r. to the usual scalar product in $L^2(\Gamma)$ must map e onto ke for suitable $k \in \mathbb{R}$. Indeed,

$$\int_\Gamma (C_1^\Gamma)^* e \cdot h \,\mathrm{d}\mathscr{H}^1 = \int_\Gamma e \cdot C_1^\Gamma h \,\mathrm{d}\mathscr{H}^1 = \int_\Gamma C_1^\Gamma h \,\mathrm{d}\mathscr{H}^1 = 0$$

for any $h \in L_0^2(\Gamma)$; we conclude that $(C_1^\Gamma)^* e$, being orthogonal to the subspace $L_0^2(\Gamma)$ in $L^2(\Gamma)$, must be constant a.e. on Γ:

$$(C_1^\Gamma)^* e = ke.$$

For $\varepsilon > 0$ introduce the operator $C_1^{\Gamma\varepsilon}$ on $L^2(\Gamma)$ sending each $f \in L^2(\Gamma)$ into

$$C_1^{\Gamma\varepsilon} f(\zeta_0) = \operatorname{Re} \frac{1}{\pi \mathrm{i}} \int_{\Gamma \setminus D(\zeta_0, \varepsilon)} \frac{f(\zeta)\tau(\zeta)}{\zeta - \zeta_0}\,\mathrm{d}\mathscr{H}^1(\zeta), \quad \zeta_0 \in \Gamma.$$

For any $g \in L^2(\Gamma)$ we have by Fubini's theorem

$$\int_\Gamma C_1^{\Gamma\varepsilon} f(\zeta_0) \cdot g(\zeta_0)\,\mathrm{d}\mathscr{H}^1(\zeta_0) = \int_\Gamma f(\zeta) \operatorname{Re} \frac{1}{\pi \mathrm{i}} \tau(\zeta) \int_{\Gamma \setminus D(\zeta, \varepsilon)} \frac{g(\zeta_0)}{\zeta - \zeta_0}\,\mathrm{d}\mathscr{H}^1(\zeta_0)\,\mathrm{d}\mathscr{H}^1(\zeta)$$

which shows that the adjoint operator $(C_1^{\Gamma\varepsilon})^*$ sends $g \in L^2(\Gamma)$ into

$$(C_1^{\Gamma\varepsilon})^* g(\zeta) = \operatorname{Re} \frac{\tau(\zeta)}{\pi i} \int_{\Gamma \setminus D(\zeta, \varepsilon)} \frac{g(\zeta_0)}{\zeta - \zeta_0}\,\mathrm{d}\mathscr{H}^1(\zeta_0), \quad \zeta \in \Gamma.$$

It is known from [2] that, as $\varepsilon \downarrow 0$, for any $g \in L^2_C(\Gamma)$ the functions

$$\int_{\Gamma \setminus D(\zeta,\varepsilon)} \frac{g(\zeta_0)}{\zeta - \zeta_0}\,\mathrm{d}\zeta_0$$

of the variable $\zeta \in \Gamma$ converge in $L^2_C(\Gamma)$ and $\mathscr{H}^1$-a.e. on Γ to the function

$$\mathrm{P.V.}\int_\Gamma \frac{g(\zeta_0)}{\zeta - \zeta_0}\,\mathrm{d}\zeta_0.$$

Since $|\tau(\zeta)| = 1$ for $\mathscr{H}^1$-a.e. $\zeta \in \Gamma$, for $g \in L^2(\Gamma)$ we get that, as $\varepsilon \downarrow 0$, the functions

$$(C_1^{\Gamma\varepsilon})^* g(\zeta) = \mathrm{Re}\,\frac{\tau(\zeta)}{\pi \mathrm{i}} \int_{\Gamma \setminus D(\zeta,\varepsilon)} \frac{g(\zeta_0)\bar{\tau}(\zeta_0)}{\zeta - \zeta_0}\,\mathrm{d}\zeta_0$$

converge in $L^2(\Gamma)$ and $\mathscr{H}^1$-a.e. to the function

$$\begin{aligned}\mathrm{Re}\,\frac{\tau(\zeta)}{\pi \mathrm{i}}\,\mathrm{P.V.}\int_\Gamma \frac{g(\zeta_0)\bar{\tau}(\zeta_0)}{\zeta - \zeta_0}\,\mathrm{d}\zeta_0 &= \mathrm{Re}\,\frac{\nu(\zeta)}{\pi}\,\mathrm{P.V.}\int_\Gamma \frac{g(\zeta_0)}{\zeta_0 - \zeta}\,\mathrm{d}\mathscr{H}^1(\zeta_0)\\ &= \frac{1}{\pi}\,\mathrm{P.V.}\int_\Gamma \frac{g(\zeta_0)\langle \nu(\zeta), \zeta_0 - \zeta\rangle}{|\zeta - \zeta_0|^2}\,\mathrm{d}\mathscr{H}^1(\zeta).\end{aligned}$$

In particular, $(C_1^{\Gamma\varepsilon})^* e$ converge (as $\varepsilon \downarrow 0$) to the function

$$\zeta \mapsto \frac{1}{\pi}\,\mathrm{P.V.}\int_\Gamma \frac{\langle \nu(\zeta), \zeta_0 - \zeta\rangle}{|\zeta - \zeta_0|^2}\,\mathrm{d}\mathscr{H}^1(\zeta_0)$$

which is therefore in $L^2(\Gamma)$. We have for any $f \in L^2(\Gamma)$

$$\begin{aligned}\int_\Gamma kf\,\mathrm{d}\mathscr{H}^1 = \int_\Gamma kef\,\mathrm{d}\mathscr{H}^1 &= \int_\Gamma (C_1^\Gamma)^* e \cdot f\,\mathrm{d}\mathscr{H}^1 = \int_\Gamma e \cdot C_1^\Gamma f\,\mathrm{d}\mathscr{H}^1\\ &= \lim_{\varepsilon\downarrow 0}\int_\Gamma e \cdot C_1^{\Gamma\varepsilon} f\,\mathrm{d}\mathscr{H}^1 = \lim_{\varepsilon\downarrow 0}\int_\Gamma (C_1^{\Gamma\varepsilon})^* e \cdot f\,\mathrm{d}\mathscr{H}^1\\ &= \int_\Gamma f(\zeta)\left\{\frac{1}{\pi}\,\mathrm{P.V.}\int_\Gamma \frac{\langle \nu(\zeta), \zeta_0 - \zeta\rangle}{|\zeta_0 - \zeta|^2}\,\mathrm{d}\mathscr{H}^1(\zeta_0)\right\}\mathrm{d}\mathscr{H}^1(\zeta),\end{aligned}$$

which shows that

$$\frac{1}{\pi}\,\mathrm{P.V.}\int_\Gamma \frac{\langle \nu(\zeta), \zeta_0 - \zeta\rangle}{|\zeta - \zeta_0|^2}\,\mathrm{d}\mathscr{H}^1(\zeta_0) = k \text{ for } \mathscr{H}^1 - \text{a.e. } \zeta \in \Gamma.$$

Combining this with (9) we get

$$\frac{\partial u}{\partial \nu} = \pi(k-1) \quad \mathscr{H}^1\text{-a.e. on } \Gamma. \tag{10}$$

Let Φ be a conformal map of the unit disc $D = D(0,1)$ onto G; it is well known that Φ extends to a homeomorphism of the closed disc $\operatorname{cl} D$ onto $\operatorname{cl} G = G \cup \Gamma$ (cf. [11], §4, section 4.1). Define

$$w(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} u \circ \Phi(e^{it}) \, dt \quad z \in D,$$

where $u \circ \Phi(.) = u(\Phi(.))$ denotes the composition of Φ and u which is continuous on cl D and harmonic on D. Consequently, w is holomorphic on D and

$$\operatorname{Re} w = u \circ \Phi \quad \text{on } D. \tag{11}$$

Hence $u = \operatorname{Re}(w \circ \Phi^{-1})$, and the complex derivative of $w \circ \Phi^{-1}$ must coincide with

$$(w \circ \Phi^{-1})' = \partial_1 u - i\partial_2 u = \psi \text{ on } G$$

by (6). Using (8) we get from Corollary to Theorem 10.1 in [3] that

$$w' = [(w \circ \Phi^{-1})' \circ \Phi]\Phi' = [\psi \circ \Phi] \cdot \Phi' \in H^1(D). \tag{12}$$

Consequently, also the function $z \mapsto zw'(z)$ of the variable $z \in D$ belongs to $H^1(D)$:

$$zw'(z) \in H^1(D). \tag{13}$$

It follows that $zw'(z)$ has angular limits $\zeta w'(\zeta)$ at $\mathscr{H}^1$-almost all points ζ in the boundary $\partial D = \{\zeta \in \mathbb{C};\ |\zeta| = 1\}$ of D. By Theorem 3.12 in [3] we have $\Phi' \in H^1(D)$. According to Theorem 3.11 in [3], Φ is absolutely continuous on ∂D and

$$\frac{d}{d\theta}\Phi(e^{i\theta}) = ie^{i\theta} \lim_{r\uparrow 1} \Phi'(re^{i\theta}) \tag{14}$$

for a.e. $\theta \in [0, 2\pi[$. Denoting by $\varphi\colon\ \theta \mapsto e^{i\theta}$ $(0 \leqslant \theta < 2\pi)$ the natural parametrization of ∂D we have $|\frac{d\varphi(\theta)}{d\theta}| = 1$ for all θ and, consequently, for any $A \subset [0, 2\pi[\,, \mathscr{H}^1(\varphi(A))$ coincides with the one-dimensional Lebesgue measure of A. Let us recall (cf. [10], Theorem 6.8) that for $E \subset \partial D$ the following equivalence holds

$$\mathscr{H}^1(E) = 0 \Longleftrightarrow \mathscr{H}^1(\Phi(E)) = 0$$

and, for $\mathscr{H}^1$-a.e. $\zeta \in \partial D$, the angular limit of Φ' at ζ (to be denoted by $\Phi'(\zeta)$) exists and satisfies

$$\Phi'(\zeta) = \lim_{\substack{z\to\zeta \\ z\in\operatorname{cl} D}} \frac{\Phi(z) - \Phi(\zeta)}{z - \zeta} \neq 0, \infty. \tag{15}$$

It follows from these facts that, for a.e. $\theta \in [0, 2\pi[$,

$$\left|\frac{\mathrm{d}}{\mathrm{d}\theta}\Phi(\mathrm{e}^{\mathrm{i}\theta})\right| = |\Phi'(\mathrm{e}^{\mathrm{i}\theta})|,$$

$$\frac{\frac{\mathrm{d}}{\mathrm{d}\theta}\Phi(\mathrm{e}^{\mathrm{i}\theta})}{|\Phi'(\mathrm{e}^{\mathrm{i}\theta})|} = \tau(\Phi(\mathrm{e}^{\mathrm{i}\theta})). \tag{16}$$

Fix now a $\theta \in [0, 2\pi[$ such that the angular limits of w', Φ' exist at $\zeta = \mathrm{e}^{\mathrm{i}\theta}$, the relations (14), (15), (16) hold, (10) holds at $\Phi(\zeta)$ and the angular limit of ψ exists at $\Phi(\zeta)$ (all this is true for a.e. θ). We have

$$\begin{aligned}
\mathrm{Re}[-\zeta w'(\zeta)] &= (see(12)) = \lim_{r\uparrow 1} \mathrm{Re}\{\mathrm{i}^2 \mathrm{e}^{\mathrm{i}\theta}[\psi \circ \Phi(r\mathrm{e}^{\mathrm{i}\theta})]\Phi'(r\mathrm{e}^{\mathrm{i}\theta})\} \\
&= (see(14)) = \lim_{r\uparrow 1} \mathrm{Re}\{i\left[\frac{\mathrm{d}}{\mathrm{d}\theta}\Phi(\mathrm{e}^{\mathrm{i}\theta})\right] \cdot [\psi \circ \Phi(r\mathrm{e}^{\mathrm{i}\theta})]\} \\
&= (see(16)) = \lim_{r\uparrow 1} \mathrm{Re}\{\mathrm{i}|\Phi'(\mathrm{e}^{\mathrm{i}\theta})|\tau(\Phi(\mathrm{e}^{\mathrm{i}\theta})) \cdot [\psi \circ \Phi(r\mathrm{e}^{\mathrm{i}\theta})]\} \\
&= |\Phi'(\mathrm{e}^{\mathrm{i}\theta})| \lim_{r\uparrow 1} \mathrm{Re}\{\nu(\Phi(\mathrm{e}^{\mathrm{i}\theta}))[\psi \circ \Phi(r\mathrm{e}^{\mathrm{i}\theta})]\}.
\end{aligned}$$

Note that (15) guarantees that the radial approach to $\mathrm{e}^{\mathrm{i}\theta} = \zeta$ in D is transformed by Φ into non-tangential (angular) approach to $\Phi(\zeta)$ in G. Employing (6) and using the validity of (10) at $\Phi(\zeta)$ we conclude that

$$\begin{aligned}
-\mathrm{Re}[\zeta w'(\zeta)] &= |\Phi'(\mathrm{e}^{\mathrm{i}\theta})| \lim_{r\uparrow 1} \mathrm{Re}\{\nu(\Phi(\zeta))[(\partial_1 u - \mathrm{i}\partial_2 u) \circ \Phi(r\mathrm{e}^{\mathrm{i}\theta})]\} \\
&= |\Phi'(\mathrm{e}^{\mathrm{i}\theta})| \cdot \frac{\partial u}{\partial \nu}(\Phi(\zeta)) = |\Phi'(\zeta)|\pi(k-1).
\end{aligned}$$

We have thus verified that

$$-\mathrm{Re}[\zeta w'(\zeta)] = |\Phi'(\zeta)| \cdot \pi(k-1)$$

for $\mathscr{H}^1$- a.e. $\zeta \in \partial D$.

Recalling (13) we employ the Theorem of Fichtengol'c (cf. [11], Th. 5.3 in chap. II) to get

$$-\mathrm{Re}[zw'(z)] = \frac{1}{2\pi}\int_0^{2\pi} |\Phi'(\mathrm{e}^{\mathrm{i}\theta})| \cdot \pi(k-1)\frac{1-|z|^2}{|z-\mathrm{e}^{\mathrm{i}\theta}|^2}\,\mathrm{d}\theta, \quad z \in D.$$

We see that the harmonic function $z \mapsto \mathrm{Re}[zw'(z)]$ does not change sign in D; since it vanishes at the origin it must vanish on D, so that $z \mapsto zw'(z)$ is constant on D. Taking the value at $z = 0$ into account again we see that $w' = 0$ on D, so that w is constant on D and, in view of (11), u is constant on G. □

Remark 3. It follows from continuity of u that (2) implies that u remains constant on $G \cup \Gamma = \operatorname{cl} G$.

Proposition 2. *If the logarithmic potential* (3) *is constant on* Γ, *then* Γ *is a circle.*

P r o o f. Assuming

$$\int_\Gamma \ln |z-\zeta| \,\mathrm{d}\mathscr{H}^1(\zeta) = C, \quad \forall z \in \Gamma,$$

we get using continuity of u and its harmonicity on G that

$$\int_\Gamma \ln |z-\zeta| \,\mathrm{d}\mathscr{H}^1(\zeta) = C, \quad \forall z \in \operatorname{cl} G.$$

Put $L = \mathscr{H}^1(\Gamma)$ and consider the function

$$v\colon\ z \mapsto \frac{1}{L}\int_\Gamma \ln |z-\zeta| \,\mathrm{d}\mathscr{H}^1(\zeta) - \frac{C}{L}$$

which is harmonic on $H = \mathbb{R}^2 \setminus \operatorname{cl} G$, $v(z) \to 0$ as z approaches Γ. Using the equality

$$v(z) = \frac{1}{L}\int_\Gamma \ln\Big|1-\frac{\zeta}{z}\Big| \,\mathrm{d}\mathscr{H}^1(\zeta) + \ln|z| - \frac{C}{L}$$

we observe that

$$v(z) = \ln|z| - \frac{C}{L} + \mathscr{O}(|z|^{-1}) \quad \text{as} \quad |z| \to \infty,$$

whence it follows that $v > 0$ on H. We conclude from Theorem 9.8 in [10] that v is Green's function of $H \cup \{\infty\}$ with pole at ∞ and that the transfinite diameter (=logarithmic capacity) of Γ equals

$$\operatorname{cap}\Gamma = \mathrm{e}^{\mathrm{C/L}}.$$

Consider now the function ψ_∞ on $H \cup \{\infty\}$ defined by $\psi_\infty(\infty) = \infty$,

$$\psi_\infty(z) = \mathrm{e}^{-C/L} \exp\Big\{\frac{1}{L}\int_\Gamma \ln(z-\zeta) \,\mathrm{d}\mathscr{H}^1(\zeta)\Big\}, \quad z \in H; \tag{17}$$

it is meromorphic on $H \cup \{\infty\}$ with a simple pole at ∞, which is the only point z in $H \cup \{\infty\}$ with $\psi_\infty(z) = \infty$. Observing that $|\psi_\infty(z)| \to 1$ as z approaches Γ we conclude (cf. Theorem 1.9 in [10]) that ψ_∞ is a conformal map of $H \cup \{\infty\}$ onto

$$D_\infty = \{\zeta \in \mathbb{C};\ |\zeta| > 1\} \cup \{\infty\}.$$

We have for $z \in H$

$$\psi'_\infty(z) = \psi_\infty(z) \cdot \frac{1}{L} \int_\Gamma \frac{\mathrm{d}\mathscr{H}^1(\zeta)}{z-\zeta}. \tag{18}$$

Since the logarithmic potential (3) remains constant on $\mathrm{cl}\, G$ we conclude from (6) that

$$\int_\Gamma \frac{1}{z-\zeta}\, \mathrm{d}\mathscr{H}^1(\zeta) = 0, \quad \forall z \in G.$$

Denoting, as above, by $\tau(\zeta)$ the unit tangent vector of Γ at ζ (which is defined for $\mathscr{H}^1$- a.e. $\zeta \in \Gamma$) we have

$$\int_\Gamma \frac{\bar{\tau}(\zeta)}{z-\zeta}\, \mathrm{d}\zeta = 0, \quad \forall z \in G,$$

whence we conclude by the Golubev-Privalov theorem (cf. section 2.3 in chap. III in [11]) that the angular limits of the function

$$z \mapsto \frac{1}{2\pi\mathrm{i}} \int_\Gamma \frac{\bar{\tau}(\zeta)}{z-\zeta}\, \mathrm{d}\zeta$$

as $z \in H$ approaches $\zeta \in \Gamma$ are for $\mathscr{H}^1$- a.e. $\zeta \in \Gamma$ equal to $\bar{\tau}(\zeta)$. Since the conformal map ψ_∞ extends continuously from H to $H \cup \Gamma$ we get from (18) that the angular limit of $\psi'_\infty(z)$ as $z \in H$ tends non-tangentially to $\zeta \in \Gamma$ (to be denoted by $\psi'_\infty(\zeta)$) is for $\mathscr{H}^1$- a.e. $\zeta \in \Gamma$ given by

$$\psi'_\infty(\zeta) = \psi_\infty(\zeta) \cdot \frac{2\pi i}{L} \bar{\tau}(\zeta)$$

so that, in view of $|\bar{\tau}(\zeta)| = 1 = |\psi_\infty(\zeta)|$,

$$|\psi'_\infty(\zeta)| = \frac{2\pi}{L} \quad \text{for} \quad \mathscr{H}^1\text{- a.e. } \zeta \in \Gamma. \tag{19}$$

Let now φ_∞ be the mapping which is inverse to ψ_∞. It maps D_∞ conformally onto $H \cup \{\infty\}$ $(\psi_\infty(\infty) = \infty)$ and (cf. (17))

$$\varphi_\infty(w) = \mathrm{e}^{C/L} w + \gamma_0 + \gamma_1 w^{-1} + \ldots, \quad |w| > 1; \tag{20}$$

by continuity, φ_∞ extends to a homeomorphism of $\mathrm{cl}\, D_\infty$ onto $\mathrm{cl}\, H$, and the angular limit of $\varphi'_\infty(w)$ as $w \in D_\infty$ approaches $\zeta \in \partial D$ (to be denoted by $\varphi'_\infty(\zeta)$) is available for $\mathscr{H}^1$ – a.e. $\zeta \in \partial D$. We have

$$\psi'_\infty(\varphi_\infty(w)) \cdot \varphi'_\infty(w) = 1, \quad |w| > 1. \tag{21}$$

Recall that both the extended mappings φ_∞ and ψ_∞ are absolutely continuous w.r. to $\mathscr{H}^1$ on the corresponding boundaries ∂D and Γ. Let us now pass from the unbounded domains D_∞, H to bounded domains D, G^* by means of the inversion; we shall suppose (as we may) that $0 \in G$. The mapping $z \mapsto \varphi_\infty(\frac{1}{z})(0 \mapsto \infty)$ maps D onto $H \cup \{\infty\}$, and the mapping

$$\Phi(z) = \frac{1}{\varphi_\infty(1/z)}, \quad z \in D,$$

maps D onto the bounded complementary domain G^* of the curve $\Gamma^* = \{\frac{1}{p};\ p \in \Gamma\}$ which is obtained by inversion from Γ. For $\mathscr{H}^1$- a.e. $\zeta \in \partial D$ the angular limit of Φ' at ζ (to be denoted by $\Phi'(\zeta)$) exists and satisfies (15). If such a ζ is fixed and z tends radially to ζ from D, then $\Phi(z)$ tends to $1/\varphi_\infty(\bar\zeta)$ along a path which is non-tangential to $\Gamma^*, w = \frac{1}{z}$ tends radially to $\bar\zeta$ from D_∞ and $\varphi_\infty(w) = \frac{1}{\Phi(z)}$ tends to $\varphi_\infty(\bar\zeta)$ along a path which is non-tangential to Γ at $\varphi_\infty(\bar\zeta)$. Using (21), (19) and the above mentioned absolute continuity w.r. to $\mathscr{H}^1$ we get

$$|\varphi'_\infty(\bar\zeta)| = \frac{1}{|\psi'_\infty(\varphi_\infty(\bar\zeta))|} = \frac{L}{2\pi} \quad \text{for} \quad \mathscr{H}^1\text{- a.e. } \bar\zeta \in \partial D$$

and from

$$\Phi'(z) = \frac{\varphi'_\infty(1/z)}{z^2\varphi^2_\infty(1/z)}, \quad z \in D, \tag{22}$$

we obtain

$$|\Phi'(\zeta)| = \frac{L}{2\pi} \cdot \frac{1}{|\zeta^2\varphi^2_\infty(1/\zeta)|} \quad \text{for} \quad \mathscr{H}^1\text{- a.e. } \zeta \in \partial D. \tag{23}$$

Since Γ is AD-regular, the same holds of Γ^* (cf. [2], Corollary to Proposition 2 on p. 160) and G^* is a Smirnov domain (cf. 7.3 in [10]), so that

$$\ln|\Phi'(z)| = \frac{1}{2\pi}\int_{\partial D} \ln|\Phi'(\zeta)| \frac{1-|z|^2}{|\zeta - z|^2}\,\mathrm{d}\mathscr{H}^1(\zeta), \quad z \in D. \tag{24}$$

It follows from (20) that the function

$$z \mapsto z\varphi_\infty(1/z) = \mathrm{e}^{C/L} + \gamma_0 z + \gamma_1 z^2 + \dots \tag{25}$$

does not vanish at $z = 0$. Since φ_∞ maps D_∞ conformly onto $H \cup \{\infty\}$ and $0 \in G$, the function (25) is holomorphic and $\neq 0$ on D, so that

$$z \mapsto \ln|z\varphi_\infty(1/z)|$$

is harmonic on D and continuously extendable to $\operatorname{cl} D$, whence

$$\frac{1}{2\pi}\int_{\partial D}\ln\Big|\zeta\varphi_\infty\Big(\frac{1}{\zeta}\Big)\Big|\frac{1-|z|^2}{|\zeta-z|^2}\,\mathrm{d}\mathscr{H}^1(\zeta)=\ln|z\varphi_\infty(1/z)|,\quad z\in D.$$

Combining this with (24), (23) we get

$$\ln|\Phi'(z)|=\ln\frac{L}{2\pi}-\frac{1}{\pi}\int_{\partial D}\ln\Big|\zeta\varphi_\infty\Big(\frac{1}{\zeta}\Big)\Big|\frac{1-|z|^2}{|\zeta-z|^2}\,\mathrm{d}\mathscr{H}^1(\zeta)=\ln\frac{L}{2\pi}-2\ln|z\varphi_\infty(1/z)|.$$

According to (22)

$$\ln|\Phi'(z)|=\ln|\varphi'_\infty(1/z)|-2\ln|z\varphi_\infty(1/z)|,$$

so that

$$\ln|\varphi'_\infty(1/z)|=\ln\frac{L}{2\pi},\quad z\in D,$$

$$|\varphi'_\infty(w)|=\frac{L}{2\pi},\quad |w|>1.$$

Since φ'_∞ is a holomorphic function with constant absolute value on the domain $\{w;\ |w|>1\},\varphi'_\infty$ itself must be constant there. Consequently, φ_∞ is linear and as it extends by continuity to $\operatorname{cl} D_\infty$, we have

$$\varphi_\infty(w)=cw+d,\quad |w|\geqslant 1$$

for suitable constants $c,d\in\mathbb{C}$. Hence $\Gamma=\varphi_\infty(\partial D)$ is a circle. □

Remark 4. We have used the inversion to pass to bounded domains in the above proof in order to be able to refer to some results which are formulated for bounded domains in the literature we use. Actually, the above proof could be shortened treating φ_∞ directly without passing to Φ. A short proof of Proposition 2 for smooth curves of class $C^{(2)}$ has been kindly communicated to the authors by Prof. E. Martensen who also obtained an alternate proof of Proposition 2 for curves with continuously varying curvature based on an new integral identity involving the density of the equilibrium distribution [9].

It is well known (cf. [5]) that if Γ is a circle then C_1^Γ maps $L^2(\Gamma)$ onto the subspace $\mathscr{K}(\Gamma)$ of all constant functions on Γ, so that $C_1^\Gamma(L_0^2(\Gamma))=\{0\}$ is the trivial subspace. Combining Propositions 1, 2 and Remark 3 we thus obtain the following

Theorem. *The inclusion*

$$C_1^\Gamma(L_0^2(\Gamma))\subset L_0^2(\Gamma)$$

characterizes the circle in the class of all AD-regular Jordan curves Γ.

Remark 5. Another characterization of the circle in terms of the Cauchy operator C^Γ is given in [6].

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