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SECOND ORDER DIFFERENTIABILITY AND LIPSCHITZ SMOOTH POINTS OF CONVEX FUNCTIONALS

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1. Introduction

Our article is based on the interesting paper [BN] by J. M. Borwein and D. Noll, which investigates the second order differentiability of convex functions on Banach spaces. Borwein and Noll consider a rather weak notion of second order differentiability and ask whether an infinite version of Alexandrov's theorem is possible (they show that for the strong "Fréchet" notion of second order differentiability no such version holds in ℓ_2).

The classical theorem of A.D. Alexandrov says that every convex function on $\mathbb{R}^n$ is almost everywhere second order differentiable. Borwein and Noll give examples which suggest that the only infinite dimensional space where a version of Alexandrov's theorem could be possible is the separable Hilbert space.

Using a dual description of second order differentiability and a notion of generalized second order differentiability they prove a density version of Alexandrov's theorem for a certain class of convex integral functionals on any separable Hilbert space $L^2_{\mathbb{R}^n}(\Omega, P, \mu)$, $n = 1$. They claim that their basic one-dimensional Lemma 7.7 can be generalized to $\mathbb{R}^n$ and therefore the result holds for each n. We prove this "n-dimensional" result by a direct geometrical method without any use of dual functions and generalized second order differentiability. We use an intuitively obvious characterization of Lipschitz smoothness of a convex function in a Hilbert space by existence of a ball "supporting from above" the graph of the function and a method based on the notion of the inner parallel body which was used by P. McMullen in [M]. Moreover, our method does not require the assumption of [BN] that $L^2_{\mathbb{R}^n}(\Omega, P, \mu)$ is

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separable and also yields that the set D^2_f of points of second order differentiability is even uncountable in any ball.

Borwein and Noll also show that for a very special type of convex integral functionals, namely for convex continuous functions $f\colon\ \ell_2 \to \mathbb{R}$ of the form

$$f(x) = \sum_{n=1}^{\infty} f_n(x_n), \tag{1}$$

the set D^2_f of points of second order differentiability of f is not only dense, but it is in a sense big—it is not an Aronszajn null set in any nonempty open set. Note that it is still possible* (as conjectured in [BN], p. 79) that this stronger result holds also for a general continuous convex function f on ℓ_2. Nevertheless, the most interesting open problem in this area is whether such a general f is at least densely second order differentiable (cf. also Problem 9, p. 45 of [VZ], which asks whether there exists a point at which f is second order differentiable in every direction).

Our geometrical method gives immediately the stronger result mentioned above not only for convex continuous functions $f\colon\ \ell_2 \to \mathbb{R}$ of the form (1) but also for those of a slightly more general form

$$f(x) = f_1(x_1, \ldots, x_{n_1}) + f_2(x_{n_1+1}, \ldots, x_{n_2}) + \ldots. \tag{2}$$

We do not know whether this stronger result holds for general convex integral functionals on any separable $L^2_{\mathbb{R}^n}(\Omega, P, \mu)$.

Borwein and Noll proved that for general continuous convex functions on ℓ_2, the set $X \setminus D^2_f$ is not in general Aronszajn null. They conjectured ([BN], p. 54) that it need not be Haar null. We prove this conjecture showing even an example of a function f of the form (1) for which $X \setminus D^2_f$ is not Haar null.

In connection with this example we answer negatively two questions of Christensen. Namely, we construct in c_0 an uncountable family of pairwise disjoint closed sets which are not Haar null. We also construct two closed convex sets A and B in c_0 which are not Haar null but the set $F(A, B) = \{x \in c_0\colon\ (A+x)\cap B \text{ is not Haar null}\}$ is empty.

Finally, we show that the geometrical characterization of Lipschitz smooth points and a well-known result on the existence of nearest points easily gives that the set of Lipschitz smooth points of a general continuous convex function on a Hilbert space is uncountable in any ball, which (in Hilbert spaces) slightly improves a result of M. Fabian [F] which says that it is dense.

* Added in proof: This turned out not to be the case. According to [MM] there is even an equivalent norm p on l_2 which is Fréchet differentiable only on an Aronszajn null set. Therefore (see the text after Proposition 2.3) D^2_p is Aronszajn null as well.

Note that we prove also some (tedious) measurability lemmas but postpone them to the last section. Similar measurability problems are mostly ignored in [BN].

2. Preliminaries

The letters $\mathbb{R}$ and $\mathbb{N}$ will stand for sets of real and natural numbers, respectively. We consider only real normed linear spaces. The subdifferential of a convex continuous function f is denoted by ∂f. By λ_n we denote the n-dimensional Lebesgue measure, $\operatorname{epi} f$ is the closed epigraph of a function f, and $B_X(x, r)$ is the usual notation for an open ball of radius r and center x in a normed linear space X; the subscript will be often omitted. Similarly, $\overline{B}_X(x, r)$ is the coresponding closed ball. The zero vector is denoted by 0. If H is a Hilbert space, we always consider in the product space $H \times \mathbb{R}$ the canonical inner product norm $\|(h, t)\| = (\|h\|^2 + t^2)^{1/2}$. We say that a subset A of a Banach space X is c-dense if A has cardinality of continuum in any open ball in X.

Consider the following definition ([BN], p. 45) of second order differentiability:

Definition 2.1. Let f be a convex continuous function defined on a normed linear space X. We say that f is *second order differentiable at* $x \in X$ if there exists $x^* \in \partial f(x)$ and a bounded linear operator $T\colon X \to X^*$ such that f has a representation of the form

$$f(x + th) = f(x) + t\langle x^*, h\rangle + \frac{t^2}{2}\langle Th, h\rangle + o(t^2) \qquad (t \to 0) \tag{3}$$

for every $h \in X$. We write D_f^2 for the set of points of second order differentiability of f.

The symmetrization $\nabla^2 f(x)$ of the operator T from Definition 2.1 also satisfies (3); it is called the Hessian of f at x in [BN]. By $f''(x)$ we denote the symmetric bilinear form on $X \times X$ corresponding to T. Thus $f''(x)(h, k) = \langle \nabla^2 f(x)(h), k\rangle$. The definition of second order differentiability can be reformulated also in terms of "differentiability" of the subdifferential mapping.

Theorem 2.2. (J.M. Borwein, D. Noll, [BN], p. 48) *Let f be a convex continuous function on a Banach space X, let $x \in X$. Then $x \in D_f^2$ if and only if there exists a bounded linear operator $T\colon X \to X^*$ such that*

$$\mathrm{w}^*\text{-}\lim_{t\to 0}(y_t^* - y_0^*)/t = Th$$

for any fixed $h \in X$ and all $y_t^* \in \partial f(x + th)$. Moreover, in this case, $T = \nabla^2 f(x)$.

We will need also the following result of [BN] (Remark 2, p. 46) which says that in the case of a finite-dimensional X the "weak" Definition 2.1. coincides with the "strong" one.

Proposition 2.3. *Let X be a finite-dimensional Banach space and let f be a convex continuous function on X. Then a symmetric $T\colon X \to X^*$ is the Hessian of f at a point $x \in X$ if and only if there exists $x^* \in \partial f(x)$ such that*

$$f(x+h) = f(x) + \langle x^*, h\rangle + \frac{1}{2}\langle Th, h\rangle + o(\|h\|^2) \quad (\|h\|^2 \to 0).$$

Borwein and Noll ([BN], p. 47) showed that if a convex continuous function f is second order differentiable at some point of a Banach space, then it is Fréchet differentiable at that point; even more, it is Lipschitz smooth there. The notion of Lipschitz smooth points was investigated by M. Fabian in [F].

Definition 2.4. Let X be a Banach space, and let $f\colon X \to \mathbb{R}$ be a convex continuous function. For $c > 0$ and $\delta \in (0, \infty]$, we denote by $L(f, c, \delta)$ the set of all $x \in X$ for which there exists $x^* \in X^*$ such that

$$f(x+v) - f(x) - \langle x^*, v\rangle \leqslant c\|v\|^2 \tag{4}$$

whenever $\|v\| < \delta$. We say that f is *Lipschitz smooth* at x if $x \in L_f := \bigcup\{L(f, c, \delta)\colon c > 0, \delta > 0\}$.

Of course, x^* in the above definition is necessarily the Fréchet derivative of f at x. If $x \in D_f^2 \cap L(f, c, \delta)$, then (3) and (4) easily imply that $\|\nabla^2 f(x)\| \leqslant 2c$.

Finally, let us recall Christensen's [C] concept of exceptional sets.

Definition 2.5. A Borel subset C of a separable Banach space X is said to be *Haar null* if there exists a Borel probability measure μ on X such that $\mu(C + x) = 0$ for any $x \in X$.

The definition easily implies that each Haar null set has empty interior. Christensen proved in [C] that the system of Haar null sets is closed on countable unions, and in a finite dimensional space it coincides with the system of all Borel Lebesgue null sets. Note that each Aronszajn null set [A] in a separable Banach space is Haar null.

3. Lipschitz smoothness and balls supporting the graph

In this section we give a geometrical characterization of points of Lipschitz smoothness, and prove by McMullens's method [M] the basic Proposition 3.8.

The following notion enables us to characterize Lipschitz smoothness geometrically.

Definition 3.1. Let f be a convex continuous function defined on a Hilbert space $\mathcal{H}$, let epi$f$ be the closed epigraph of f, and $r > 0$. A point $x \in \mathcal{H}$ is an *r-point* of the function f if there exists a point $z \in \mathcal{H} \times \mathbb{R}$ such that the closed ball $\overline{B}_{\mathcal{H}\times\mathbb{R}}(z, r)$ supports from above the graph of f at the point $(x, f(x))$, i.e. $\overline{B}_{\mathcal{H}\times\mathbb{R}}(z, r) \subset \operatorname{epi} f$ and $(x, f(x)) \in \overline{B}_{\mathcal{H}\times\mathbb{R}}(z, r)$.

It is easy to see that if x is an r-point for some $r > 0$ then it is an r'-point for every $0 < r' < r$.

Lemma 3.2. *Let $\mathcal{H}$ be a Hilbert space, $c > 0$ and $\psi(x) := c\|x\|^2$ for $x \in \mathcal{H}$. Then, for any $z \in \mathcal{H}$, the ball*

$$\mathcal{B} := \overline{B}\left((0, c\|z\|^2 + \frac{1}{2c}), \sqrt{\frac{1}{4c^2} + \|z\|^2}\right)$$

supports from above the graph of ψ at the point $(z, c\|z\|^2)$. In particular, every point of $\mathcal{H}$ is a $\frac{1}{2c}$-point of the function $c\|x\|^2$.

Proof. The inequality

$$c\|x\|^2 \leqslant c\|z\|^2 + \frac{1}{2c} - \sqrt{\frac{1}{4c^2} + \|z\|^2 - \|x\|^2}$$

is valid for all $x \in \mathcal{H}$ with $\|x\| \leqslant \sqrt{1/(4c^2) + \|z\|^2}$, as an easy computation after substituting $t := \|z\|^2 - \|x\|^2$ reveals. Since on the right side of the inequality we have the "lower sphere function" of the ball $\mathcal{B}$, and for $x = z$ we get equality, $\mathcal{B}$ supports from above the graph of ψ at $(z, c\|z\|^2)$. $\square$

Lemma 3.3. *Let $\mathcal{H}$ be a Hilbert space, let $c > 0$, $\delta > 0$, and $x \in \mathcal{H}$ be given. Let f be a convex continuous function on $\mathcal{H}$ such that $\nabla^2 f(y)$ exists and $\|\nabla^2 f(y)\| \leqslant c$ for every $y \in B(x, \delta)$. Then $x \in L(f, c, \delta)$.*

Proof. The assumptions easily imply that f' is c-Lipschitz on $B(x, \delta)$. This fact and the mean value theorem imply that, for every $h \in \mathcal{H}$, $\|h\| < \delta$, we have

$$f(x + h) - f(x) - \langle f'(x), h\rangle = \langle f'(z), h\rangle - \langle f'(x), h\rangle \leqslant c\|h\|^2$$

for a suitable choice of $t \in (0, 1)$ and $z = x + th$. $\square$

Lemma 3.4. *Let $\mathcal{H}$ be a Hilbert space, $r > 0$ and $\psi(x) = -\sqrt{r^2 - \|x\|^2}$ for $x \in B_{\mathcal{H}}(0,r)$. Then the second Fréchet derivative T of ψ is bounded on $B(0, r-\varepsilon)$ for any $\varepsilon > 0$. Moreover,*

(i) $\|\psi'(x)\| > 1$ *for* $x \in B_{\mathcal{H}}(0,r)$, $\|x\| > \frac{\sqrt{2}}{2}r$ *and*

(ii) $\|T(x)\| < 8/r$ *for* $x \in B_{\mathcal{H}}(0, \frac{\sqrt{3}}{2}r)$.

Proof. The statement of the lemma follows easily from the fact that

$$\psi'(x) = x/\sqrt{r^2 - \|x\|^2}$$

and

$$T(x)(v) = v/\sqrt{r^2 - \|x\|^2} + x\langle x, v\rangle \cdot 1/(\sqrt{r^2 - \|x\|^2})^3$$

for $x \in B_{\mathcal{H}}(0,r)$ and $v \in \mathcal{H}$. $\square$

Lemma 3.5. *Let $r > 0$ be given. Then there exist $c > 0$ and $\delta > 0$ such that if f is a convex continuous function on a Hilbert space $\mathcal{H}$, x is an r-point of f, and $\|f'(x)\| < 1$, then $x \in L(f, c, \delta)$.*

Proof. If x is an r-point of f, there exist $y \in \mathcal{H}$ and $s \in \mathbb{R}$ such that f is "supported at x" from above by the "lower sphere function" $\psi(v) := s - \sqrt{r^2 - \|v - y\|^2}$, i.e.

$$\begin{aligned} &f(v) \leqslant \psi(v) \quad \text{if } \|v - y\| \leqslant r, \quad \text{and} \\ &f(x) = \psi(x). \end{aligned}$$

Since clearly $\psi'(x) = f'(x)$ and $\|f'(x)\| < 1$, Lemma 3.4 implies that $\|x - y\| \leqslant \sqrt{2}r/2$. Therefore Lemma 3.3 and Lemma 3.4 give $x \in L(\psi, 8/r, (\sqrt{3} - \sqrt{2})r/2)$, and consequently $x \in L(f, 8/r, (\sqrt{3} - \sqrt{2})r/2)$. $\square$

Lemma 3.6. *Let f be a continuous convex function on a Hilbert space $\mathcal{H}$ and let $x \in \mathcal{H}$ be an r-point of f. Then $x \in L_f$.*

Proof. Define y, s and ψ as in the proof of Lemma 3.5. Then $\|x - y\| < r$ and the second derivative of ψ is bounded on a neighborhood of x by Lemma 3.4. Hence Lemma 3.3 implies that $x \in L_\psi$ and consequently $x \in L_f$. $\square$

Lemma 3.7. *Let $c > 0$, $\delta > 0$ be given. Then there exists $r > 0$ such that if f is a convex continuous function on a Hilbert space $\mathcal{H}$ and $x \in L(f, c, \delta)$, then x is an r-point of f.*

P r o o f. We can put $r := \min\{\delta/2, 1/(2c)\}$. To prove this, suppose that $x \in L(f,c,\delta)$, where f is as above. Without any loss of generality we can suppose $x = 0$. Then

$$\begin{aligned}
&f(v) \leqslant \psi(v) := f(0) + \langle f'(0), v\rangle + c\|v\|^2 \quad \text{if} \;\; \|v\| \leqslant \delta, \;\; \text{and}\\
&f(0) = \psi(0).
\end{aligned}$$

A simple calculation gives that $\psi(v) = c\|v-y\|^2 + s$ for $y = \frac{-f'(0)}{2c}$ and $s = f(0) - \frac{\|f'(0)\|^2}{4c^2}$. Consequently, Lemma 3.2 implies that $x = 0$ is an $\frac{1}{2c}$-point of ψ. Now it is easy to show that $x = 0$ is an r-point of f. □

The following proposition (as well as its proof) is only a quantitative version of the McMullen's result of [M].

Proposition 3.8. *Let $c > 0$, $\delta > 0$ ($c > 0$ and $\delta = \infty$) be given. Then there exist $c' > 0$ and $\delta' > 0$ (respectively, $c' > 0$ and $\delta' = \infty$) such that if $n \in \mathbb{N}$, $f\colon \mathbb{R}^n \to \mathbb{R}$ is convex, $z \in L(f,c,\delta)$ and $\eta > 0$, then $\lambda_n(B(z,\eta) \cap L(f,c',\delta')) > 0$.*

P r o o f. First consider the case $0 < \delta < \infty$. Denote by r the constant assigned to c and δ by Lemma 3.7, and define c' and δ' as the constants which Lemma 3.5 assigns to $r/2$. Now, let some convex $f\colon \mathbb{R}^n \to \mathbb{R}$, $z \in L(f,c,\delta)$ and $\eta > 0$ be given. Without any loss of generality we can assume that $z = 0$, $f(z) = 0$, $f'(z) = 0$, and $\|x^*\| < 1$ for every $x^* \in \partial f(x)$, $x \in B(0,\eta)$. Consider the inner parallel set K of $\operatorname{epi} f$ defined by the ball $B := \overline{B}_{\mathbb{R}^{n+1}}(0, r/2)$:

$$K := \{\alpha \in \mathbb{R}^{n+1}\colon\; B + \alpha \subset \operatorname{epi} f\} = \{\alpha \in \operatorname{epi} f\colon\; \operatorname{dist}(\alpha, \operatorname{graph} f) \geqslant r/2\}.$$

Since an inner parallel set of any convex set is convex (see e.g. [Lt]), we know that K is convex. Using this fact, it is easy to prove that K is a closed epigraph of a convex function $g > f$ and $\operatorname{bdr} K = \operatorname{graph} g = \{\alpha \in \operatorname{epi} f\colon\; \operatorname{dist}(\alpha, \operatorname{graph} f) = r/2\}$. Denote

$$\begin{aligned}
G &:= \{(x, f(x)) \in \mathbb{R}^{n+1}\colon\; \operatorname{dist}((x,f(x)), K) = r/2\},\\
K' &:= \{(x, f(x)) \in G\colon\; \|x\| < \eta\},\\
K'' &:= \{x \in \mathbb{R}^n\colon\; (x, f(x)) \in K'\}.
\end{aligned}$$

It is easy to see that G is closed, and therefore K' and K'' are F_σ sets. Every $x \in K''$ is clearly an $(r/2)$-point of f. To see this, observe that the ball $\overline{B}(z, r/2)$, where $z \in K$ and $\|z - (x, f(x))\| = r/2$ “supports the graph of f from above” at $(x, f(x))$; the existence of such a z follows from the definition of G and the fact that K is closed. Since $\|x\| < \eta$, we have also that $\|f'(x)\| < 1$. Therefore, $x \in L(f, c', \delta')$

by the choice of c' and δ'. Let us prove that $\lambda_n K'' > 0$. To this end, consider the metric projection $P = P_K$ on the set K restricted to G. Since K is convex, P is a 1-Lipschitz mapping by the well-known result. Clearly, $P^{-1}(t)$ is nonempty for any $t \in \operatorname{bdr} K$. Since 0 is an r-point of f, $f(0) = 0$ and $f'(0) = 0$, it is easy to see that $\operatorname{dist}((0,r), \operatorname{graph} f) = r$ and consequently $\alpha := (0, r/2) \in \operatorname{bdr} K$. Now, we shall show that there exists $\varepsilon > 0$ such that

$$P^{-1}(B(\alpha, \varepsilon)) \subset K'. \tag{5}$$

Supposing the contrary, we obtain that there exists a sequence $\{z_n\}$ in $G \setminus K'$ such that $P(z_n) \to \alpha$. Consequently, there exists $z \in G \setminus K'$ such that $P(z) = \alpha$; this fact and the definition of G imply that $z \in \overline{B}(\alpha, r/2)$. Since $z \in G$, we have that $z \notin B(2\alpha, r)$ and consequently $z \in \overline{B}(\alpha, r/2) \setminus B(2\alpha, r) = \{0\} \subset K'$, which is a contradiction. Now (5) yields that

$$P(K') \supset \operatorname{bdr} K \cap B(\alpha, \varepsilon).$$

The n-dimensional Hausdorff measure (for the definition and properties see e.g. [Rog]) of the latter set is nonzero. Since P is Lipschitz, the n-dimensional Hausdorff measure of K' is also nonzero, and since the mapping $x \mapsto (x, f(x))$ is Lipschitz on K'', by the same reasoning $\lambda_n K'' > 0$.

Now let $c > 0$ and $\delta = \infty$. Find $c' > 0$ and $\delta' > 0$ which correspond to c and $\delta = 1$ by the just proved part of our proposition. Clearly, we can suppose that $c' > 3c$. Now let some convex $f\colon \mathbb{R}^n \to \mathbb{R}$, $z \in L(f, c, \delta)$, and $\eta > 0$ be given. Without any loss of generality we can assume that $z = 0$, $f(z) = 0$, and $f'(z) = 0$. Choose $0 < \zeta < \delta'/5$ such that $\|y^*\| \leqslant c\delta'$ for any $y^* \in \partial f(y)$, $y \in B(0, \zeta)$. To prove that

$$\lambda_n \left(B(z, \zeta) \cap L(f, c', \infty)\right) > 0,$$

it is enough to show that if $y \in L(f, c', \delta')$ and $\|y\| < \zeta$, then $y \in L(f, c', \infty)$. To this end, fix such a y and let $v \in \mathbb{R}^n$ be such that $\|v - y\| \geqslant \delta'$. Then

$$\|v\| \geqslant \frac{4\delta'}{5} \quad \text{and} \quad \|v - y\| \geqslant \frac{3\|v\|}{4}. \tag{6}$$

Because $0 \in L(f, c, \infty)$ and $f'(0) = 0$ we have $f(v) \leqslant c\|v\|^2$ and $f(y) \geqslant 0$. Therefore

$$\begin{aligned} &f(v) - f(y) - \langle f'(y), v - y\rangle \leqslant c\|v\|^2 + \|f'(y)\|\|v - y\| \leqslant \\ &16c\|v - y\|^2/9 + c\delta'\|v - y\| \leqslant 3c\|v - y\|^2 \leqslant c'\|v - y\|^2, \end{aligned}$$

where (6) was used to obtain the second inequality. $\square$

4. Cardinality of the set L_f

The set L_f of points where a convex continuous function f on a Hilbert space $\mathcal{H}$ is Lipschitz smooth is dense, but unlike the Fréchet differentiability, L_f does not have to contain a dense G_δ set, cf. [F]. The geometrical characterization of Lipschitz smoothness from Section 3 enables us to give a different proof of density of L_f which yields moreover that L_f is uncountable in any ball. Let X be a Banach space and let C be a closed nonempty subset of X. If $z \in C$, $x \in X$ and $\|x - z\| = \operatorname{dist}(x, C)$, we say that z is a *nearest point in C to x*. Let f be as above, let $C \subset \mathcal{H} \times \mathbb{R}$ be the graph of f, and let $z = (x, f(x))$ be a nearest point in C to a point in the open epigraph of f. We will use the obvious fact that then x is an r-point for some $r > 0$ and therefore $x \in L_f$ by Lemma 3.6.

Theorem 4.1. *Let f be a convex continuous function on a Hilbert space $\mathcal{H}$. Then the set L_f of points of Lipschitz smoothness is uncountable in any ball. If H is separable, then L_f is c-dense (i.e., it has cardinality of continuum in any ball).*

Proof. Let $z \in \mathcal{H}$ and $\varepsilon > 0$ be given. Let $D \subset \mathcal{H} \times \mathbb{R}$ be the intersection of the open epigraph of f and the ball $B((z, f(z)), \varepsilon/2)$. The well-known result on the existence of nearest points (cf. e.g. [BF]) implies that the set G of points in D which possess a nearest point in the graph of f is a residual subset of D. Define G' as the set of all $x \in \mathcal{H}$ for which $(x, f(x))$ is a nearest point in the graph of f to some point in G. Clearly $G' \subset B(z, \varepsilon)$. Let $x \in G'$; denote $\alpha = (x, f(x))$. Since f is Lipschitz smooth at x, there exists a unique hyperplane supporting the graph of f at the point α; denote it by P. If $y \in \operatorname{epi} f$ is any point for which α is a nearest point in C, then α is also a nearest point of y in P. Therefore y is contained in the line p_x perpendicular to P and containing the point α. Consequently, $G \subset \bigcup\limits_{x \in G'} p_x$. Therefore the set G' is not countable, and consequently $L_f \cap B(z, \varepsilon)$ is also uncountable. Since the set L_f is Borel by Lemma 8.2, Aleksandrov-Hausdorff theorem (cf. [K]) implies that L_f is c-dense if H is separable. $\square$

5. Convex functionals of the form (2)

Lemma 5.1. *Let H_i, $i \in \mathbb{N}$, be Hilbert spaces and let the Hilbert space $H := \left(\sum H_i\right)_{\ell_2} \subset \prod\limits_{i=1}^{\infty} H_i$ be the ℓ_2-sum of these spaces. Let $f_i \colon H_i \to \mathbb{R}$ be (necessarily continuous convex) functions such that $f(x_1, x_2, \ldots) := \sum\limits_{i=1}^{\infty} f_i(x_i)$ is a continuous convex function on H. Let $x^0 = (x_1^0, x_2^0, \ldots) \in H$, $c > 0$ and $\delta > 0$ be given. Then the following conditions are equivalent:*

(i) $x^0 \in L(f, c, \delta)$,

(ii) $x_i^0 \in L(f_i, c, \delta)$ *for each* $i \in \mathbb{N}$.

Proof. That (i) implies (ii) is obvious. Suppose that (ii) holds. Choose $y \in \partial f(x^0)$. Then there clearly exist $y_i \in \partial f_i(x_i^0)$ such that $\langle y, v\rangle = \sum_{i=1}^{\infty} \langle y_i, v_i\rangle$, for each $v = (v_1, v_2, \ldots) \in H$. Now choose an arbitrary $v \in H$ such that $\|v\| < \delta$. Then, for each i, $\|v_i\| < \delta$ and consequently (ii) yields

$$f_i(x_i^0 + v_i) - f_i(x_i^0) \leqslant \langle y_i, v_i\rangle + c\|v_i\|^2.$$

Hence

$$f(x^0+v)-f(x^0) = \sum_{i=1}^{\infty}(f_i(x_i^0+v_i)-f_i(x_i^0)) \leqslant \sum_{i=1}^{\infty}\langle y_i, v_i\rangle+c\sum_{i=1}^{\infty} \|v_i\|^2 = \langle y, v\rangle+c\|v\|^2,$$

which implies (i). □

Lemma 5.2. *Let* H_i *and* H *be as in Lemma* 5.1. *Suppose that all* H_i *are finite-dimensional. Let* $M_i \subset H_i$ *be bounded Borel sets of positive Lebesgue measure and such that* $M = \prod_{i=1}^{\infty} M_i \subset H$. *Then* M *is not a null set in the Aronszajn sense.*

Proof. Put

$$\nu_i(A) = \frac{\lambda_i(A \cap M_i)}{\lambda_i(M_i)},$$

where $A \subset H_i$ is Borel and λ_i is the Lebesgue measure on H_i. Put $\nu = \prod_{i=1}^{\infty} \nu_i$. Then ν is a probability Borel measure on $\prod_{i=1}^{\infty} H_i$ which is concentrated on M, such that $\mu A = 0$ for each Aronszajn null subset of H (cf. [A], Proposition 3, p. 189). Consequently, M is not Aronszajn null. □

Theorem 5.3. *Let* H_i, H, f_i, f *be as in Lemma* 5.1 *and let all* H_i *be finite-dimensional. Let* $U \subset H$ *be a non-empty open set. Then* $U \cap D_f^2$ *is not null in the Aronszajn sense.*

Proof. By Theorem 4.1 we can find $c > 0$, $\delta > 0$ and $x^0 = (x_1^0, x_2^0, \ldots) \in U \cap L(f, c, \delta)$. Lemma 5.1 gives that $x_i^0 \in L(f_i, c, \delta)$ for each i. Consequently, Proposition 3.8 easily yields that there exist $c' > 0$, $\delta' > 0$ and $M_i \subset H_i$ such that $M_i \subset L(f_i, c', \delta')$, M_i is bounded and of positive Lebesgue measure in H_i and $M = \prod_{i=1}^{\infty} M_i \subset U$. By Lemma 5.1 we obtain that $M \subset L_f$ and, since by Corollary 4.2 of [BN] we know that $L_f \backslash D_f^2$ is Aronszajn null, we obtain our assertion by Lemma 5.2. □

The following theorem is only a reformulation of the previous one.

Theorem 5.4. *Let $f\colon \ell_2 \to \mathbb{R}$ be a continuous convex function of the form*

$$f(x) = f_1(x_1, \dots, x_{n_1}) + f_2(x_{n_1+1}, \dots, x_{n_2}) + \dots .$$

Then $U \cap D_f^2$ is not null in the Aronszajn sense, whenever $U \subset \ell_2$ is an open nonempty set.

6. A convex function on ℓ_2 which is not Haar almost everywhere second order differentiable

The following example shows that even though the functionals of the form (1) are densely second order differentiable and D_f^2 is not Aronszajn null, the complement of points of Lipschitz smoothness and consequently also the complement of D_f^2 does not have to be Haar null. Borwein and Noll observe in [BN] that there exists a closed convex subset C of c_0 (namely the positive cone) with empty interior which contains a translate of any compact subset of c_0. Such a set C is clearly not Haar null, and the convex continuous function $f(x) := \operatorname{dist}(x, C)$ is Fréchet differentiable at no point of C. Consequently, C is contained in the complement of D_f^2. However, no reflexive Banach space contains a closed convex set with empty interior which contains a translate of every compact set (cf. [MS]). Therefore to construct a counterexample in ℓ_2 we have to use a different method. We will need the following simple characterization of compact sets in c_0: a closed subset K of c_0 is compact if and only if there exists some $z = (z_n) \in c_0$ such that $|x_n| < z_n$ whenever $x = (x_n) \in K$ (cf. [DS], p. 339).

Example 6.1. There exists a convex continuous function of the form $F(x) = \sum\limits_{n=1}^{\infty} F_n(x_n)$ on ℓ_2 such that the set of points where F is not Lipschitz smooth is not a Haar null set.

For an arbitrary natural number n put

$$f_n(t) = \begin{cases} 0, & t \in (-\infty, 1/(n+1)], \\ (n+1)(t - \frac{1}{n+1})^2, & t \in (1/(n+1), 1/n), \\ \frac{2t}{n} + \frac{1-2(n+1)}{(n+1)n^2}, & t \in [1/n, \infty). \end{cases}$$

Clearly, each function f_n is smooth and convex. It is easy to verify that if $t \in I_n := (1/(n+1), 1/n)$, $0 < c < n+1$, and $\delta > 0$ is arbitrary, then $t \notin L(f_n, c, \delta)$. Moreover, an easy computation reveals that

$$f_n(t) \leqslant t^2 \quad \text{for } t \in \mathbb{R}. \tag{7}$$

Let $p(n)$ be a sequence of natural numbers such that $\{n\colon p(n) = k\}$ is infinite for each natural number k, and let $F_n := f_{p(n)}$. The functions F_n are convex, and due to (7)

$$F(x) = F(x_1, x_2, \ldots) := \sum_{n=1}^{\infty} F_n(x_n) \leqslant \sum_{n=1}^{\infty} x_n^2 = \|x\|^2.$$

Hence the function F is convex and locally bounded, therefore it is continuous. Now, let us prove the following statement:

Let $\{n_k\}$ be an increasing sequence of natural numbers such that the sequence $\{p(n_k)\}$ is not bounded. Then F is Lipschitz smooth at no point of the set $A := \{x \in \ell_2\colon x_{n_k} \in I_{p(n_k)}, k \in \mathbb{N}\}$.

In fact, assume F is Lipschitz smooth at a point $x \in A$, namely let $x \in L(F, c, \delta)$ for some $c > 0$, $\delta > 0$. Then Lemma 5.1 implies that $x_n \in L(F_n, c, \delta)$ for every $n \in \mathbb{N}$. Choose some n_k such that $p(n_k) > c$. Since $x_{n_k} \in I_{p(n_k)}$, we have $x_{n_k} \notin L(F_{n_k}, c, \delta)$, which is a contradiction.

Finally, let us prove that the set C of points where F is not Lipschitz smooth is not a Haar null set. Let μ be a Borel probability measure on ℓ_2. Then there exists a compact set $K \subset \ell_2$ such that $\mu K > 0$. If we consider K as a subset of c_0, then K is also compact. Hence there exists a sequence $(a_n) \in c_0$ such that for any $x \in K$ and $n \in \mathbb{N}$ we have $|x_n| < a_n$. Now choose an increasing sequence $\{n_k\}$ such that

$$a_{n_k} \leqslant \frac{1}{2(k+1)^2}$$

and

$$p(n_k) = k \quad \text{for } k = 1, 2, \ldots.$$

Define

$$z := \sum_{k=1}^{\infty} \Big(\frac{1}{k+1} + \frac{1}{2(k+1)^2} \Big) e_{n_k},$$

where $\{e_n\}$ is the usual orthonormal basis of ℓ_2. The set $K + z$ is a subset of the set $A = \{x \in \ell_2\colon x_{n_k} \in I_{p(n_k)}, k = 1, 2, \ldots\}$, therefore $\mu(A - z) > 0$. However, no point of A is a point of Lipschitz smoothness of F, and therefore C is not a Haar null set.

Now, let us show that the above mentioned characterization of compact sets in c_0 yields easily a negative answer to two questions of Christensen (cf. [C], p. 123).

Example 6.2. There exist uncountably many pairwise disjoint closed subsets of c_0 which are not Haar null.*

Denote

$$M_\alpha = \Big\{(x_1, x_2, \dots) \in c_0 \colon\; x_n \alpha_n \geqslant \frac{1}{n}\Big\}$$

for $\alpha = (\alpha_1, \alpha_2, \dots) \in \{1, -1\}^{\mathbb{N}}$. Then the sets M_α are closed and pairwise disjoint. Also, if we use the already mentioned property of compact subsets of c_0, it is easy to show that for every compact $K \subset c_0$ and $\alpha \in \{1, -1\}^{\mathbb{N}}$ there exists some $x \in c_0$ for which $x + K \subset M_\alpha$.

Example 6.3. There exist closed subsets A and B of c_0 which are not Haar null and the set $F(A, B) = \{x \in c_0 \colon\; (x + A) \cap B$ not Haar null$\}$ is empty.

Let $\alpha = \{1, 1, 1, \dots\}$ and $\beta = \{-1, -1, -1, \dots\}$. Using the notation from the previous example define $A = M_\alpha$ and $B = M_\beta$. Then A and B are not Haar null. If we again use the characterization of compact subsets of c_0, it is easy to see that the set $(x + A) \cap B$ is compact for every $x \in c_0$ and therefore it is Haar null (cf. [C], p. 119).

7. Integral functionals

In this section we consider a class of special convex functionals on Hilbert spaces $L^2_{\mathbb{R}^n}(\Omega, P, \mu)$ of (classes of) functions $x \colon\; \Omega \to \mathbb{R}^n$.

Setting: Let (Ω, P, μ) be a measure space with a complete σ-finite measure. For a given $n \in \mathbb{N}$ let H be the Hilbert space $L^2_{\mathbb{R}^n}(\Omega, P, \mu)$ and let $\mathcal{B}$ be the σ-algebra of Borel subsets of $\mathbb{R}^n$. Let $\varphi \colon\; \mathbb{R}^n \times \Omega \to \mathbb{R}$ be a function such that

(i) φ is $\mathcal{B} \otimes P$-measurable and

(ii) $\varphi_\tau := \varphi(\cdot, \tau)$ is convex for all $\tau \in \Omega$.

In other words, φ is a finite convex integrand in the sense of R.T. Rockafellar (cf. e.g. [R2]). Consider the functional f defined in H by the formula

$$f(x) := \int_\Omega \varphi(x(\tau), \tau)\,\mathrm{d}\mu(\tau). \tag{8}$$

We will suppose that there exist some $a \geqslant 0$ and $b \in L^1_{\mathbb{R}}(\mu)$ such that φ satisfies for almost all $\tau \in \Omega$ the growth condition

$$|\varphi(x, \tau)| \leqslant a\|x\|^2 + b(\tau). \tag{9}$$

* Added in proof: After the paper was submitted, we were informed that Dougherty (cf. [D]) contructed such a family of sets in every separable Banach space. Solecki (cf. [S]) generalized this result to any Polish, abelian, non-locally compact group (see also [BL]).

(Note that it is not difficult to prove that the condition (9) is equivalent to the condition (d) of Theorem 3L. of [R2] for $p = 2$.)

Let us mention a few properties of such a functional f (cf. [R1] and [R2]). The integrand of (8) is measurable for every measurable function x on Ω. The growth condition (9) implies that f is convex, finite and continuous on H; if μ has no atoms then (9) is also a necessary condition for f to be finite everywhere. For the subdifferential of f we have that $x^* \in \partial f(x)$ if and only if $x^*(\tau) \in \partial\varphi_\tau(x(\tau))$ almost everywhere.

Often in this section we will have to know that certain sets or mappings are measurable. The proofs of these facts are contained in the appendix.

In the following, if we write $x \in H$, then we actually mean that x is an arbitrary but fixed representative defined on all of Ω, of the given class of functions.

Lemma 7.1. *Let (Ω, P, μ), φ and f be as in the setting and, moreover, let μ be nonatomic. Let $x \in H$ and c, $\delta > 0$ be given. Then $x \in L(f, c, \delta)$ if and only if $x(\tau) \in L(\varphi_\tau, c, \infty)$ for almost all τ.*

P r o o f. First let $x \in L(f, c, \delta)$. Denote $x^* := f'(x)$. It is enough to show that the set A of all $\tau \in \Omega$ for which there exist some $s \in \mathbb{R}^n$ and $k \in \mathbb{N}$ such that

$$\varphi_\tau(x(\tau) + s) - \varphi_\tau(x(\tau)) - \langle x^*(\tau), s\rangle > c\|s\|^2 + 1/k \tag{10}$$

has measure zero. Because the left side of (10) is continuous with respect to s,

$$A = \bigcup_{s\in S} \bigcup_{k\in\mathbb{N}} A_{s,k},$$

where $A_{s,k}$ is the set of all τ satisfying (10) for the particular choice of s and k, and S is a countable dense subset of $\mathbb{R}^n$. Since the left side of (10) is measurable with respect to τ, the sets $A_{s,k}$ are measurable and therefore A is measurable as well. Suppose that $\mu A > 0$. Then there exists a pair s, k such that $\mu A_{s,k} > 0$. Since μ is nonatomic, we can choose $B \subset A_{s,k}$ such that $0 < \mu B < \delta^2/\|s\|^2$ and define $v := s\chi_B$. Then $v \in H$ and $\|v\| < \delta$. We have

$$\begin{aligned} f(x+v) - f(x) - \langle x^*, v\rangle &= \int_B \varphi_\tau(x(\tau)+s) - \varphi_\tau(x(\tau)) - \langle x^*(\tau), s\rangle \,\mathrm{d}\mu(\tau) \\ &> c\mu B\|s\|^2 + \mu B/k = c\|v\|^2 + \mu B/k, \end{aligned}$$

which is a contradiction with $x \in L(f, c, \delta)$.

To prove the other implication suppose that $x^* \in \partial f(x)$ and $v \in H$ are given. Then

$$f(x+v) - f(x) - \langle x^*, v\rangle = \int_\Omega [\varphi_\tau(x(\tau)+v(\tau)) - \varphi_\tau(x(\tau)) - \langle x^*(\tau), v(\tau)\rangle]\,\mathrm{d}\mu(\tau)$$
$$\leqslant \int_\Omega c\|v(\tau)\|^2\,\mathrm{d}\mu(\tau) = c\|v\|^2.$$

Hence $x \in L(f, c, \infty)$ and therefore also $x \in L(f, c, \delta)$. $\square$

In the case of a separable space H, the following proposition follows from Proposition 6.1 and Proposition 6.4 of [BN]. Our proof does not require separability of H and is more direct.

Proposition 7.2. *Let (Ω, P, μ), φ and f be as in the setting and, moreover, let μ be nonatomic. Let $x \in L_f$ and $x(\tau) \in D^2_{\varphi_\tau}$ almost everywhere. Then $x \in D^2_f$.*

P r o o f. Choose $c, \delta > 0$ such that $x \in L(f, c, \delta)$. Denote $x^* := f'(x)$. Then $\varphi'_\tau(x(\tau)) = x^*(\tau)$ almost everywhere. By the assumptions $h(\tau) := \nabla^2\varphi_\tau(x(\tau))$ exists and $\|h(\tau)\| \leqslant 2c$ almost everywhere by Lemma 7.1 and the remark after Definition 2.4. Due to Lemma 8.10 the mapping h is measurable. Define a mapping T on H such that

$$T(y)(\tau) := h(\tau)\,(y(\tau)) \qquad \text{for } y \in H.$$

Because h is measurable and $\|h(\tau)\| \leqslant 2c$ a.e., $T(y) \in H$. Clearly T is linear and $\|T\| \leqslant 2c$. We will show that T satisfies (3) in Definition 2.1. For any $z \in H$ we have

$$\begin{aligned}&\frac{1}{t^2}(f(x+tz) - f(x) - t\langle x^*, z\rangle - \frac{t^2}{2}\langle Tz, z\rangle)\\ &= \int_\Omega \Big(\frac{1}{t^2}[\varphi_\tau(x(\tau)+tz(\tau)) - \varphi_\tau(x(\tau)) - t\langle\varphi'_\tau(x(\tau)), z(\tau)\rangle]\\ &\quad - \frac{1}{2}\langle h(\tau)\,(z(\tau))\,, z(\tau)\rangle\Big)\,\mathrm{d}\mu(\tau).\end{aligned}$$

The integrand converges on Ω almost everywhere to zero when t converges to zero. Lemma 7.1 gives that, for almost all τ and all $t \neq 0$, $|\frac{1}{t^2}[\varphi_\tau(x(\tau)+tz(\tau)) - \varphi_\tau(x(\tau)) - t\langle\varphi'_\tau(x(\tau)), z(\tau)\rangle]| \leqslant c\|z(\tau)\|^2$, and consequently the absolute value of the integrand is bounded by $2c\|z(\tau)\|^2$ for almost all τ and all $t \neq 0$. Since $\int_\Omega \|z(\tau)\|^2\,\mathrm{d}\mu(\tau) < \infty$, the Lebesgue dominated convergence theorem gives (3). $\square$

Theorem 7.3. *Let (Ω, P, μ), φ and H be as in the setting. Then for the convex functional*

$$f(x) := \int_\Omega \varphi(x(\tau), \tau)\,\mathrm{d}\mu(\tau), \quad x \in H,$$

the set D^2_f of points of second order differentiability of f is uncountable in any ball. If H is separable, then D^2_f is c-dense.

P r o o f. First let μ be a nonatomic measure. By Theorem 4.1 the set L_f is dense in H. Therefore, it is sufficient to show that for any $x \in L_f$ and $\varepsilon > 0$, the set $D^2_f \cap B(x,\varepsilon)$ is not countable.

Let $x \in L(f,c,\delta)$ for some $c,\delta > 0$ and let $\varepsilon > 0$ be given. Since μ is σ-finite, there exists a strictly positive $g \in H$ such that $\|g\| < \varepsilon$. Since x and g are measurable, we have

$$C := \{(z,\tau) \in \mathbb{R}^n \times \Omega\colon\ \|x(\tau) - z\| \leqslant \|g(\tau)\|\} \in \mathcal{B} \otimes \Omega$$

and $C_\tau := \{z\colon\ (z,\tau) \in C\}$ is clearly a neighborhood of $x(\tau)$. Lemma 7.1 implies that $x(\tau) \in L(\varphi_\tau, c, \infty)$ almost everywhere. Alexandrov's theorem applied to φ_τ and Proposition 3.8 imply that there exists $c' > 0$ such that

$$C_\tau \cap L(\varphi_\tau, c', \infty) \cap D^2_{\varphi_\tau}$$

is uncountable for almost all τ. Assume now that $D^2_f \cap B(x,\varepsilon)$ is countable, say $D^2_f \cap B(x,\varepsilon) = \{y_k\colon\ k \in \mathbb{N}\}$. Denote

$$\begin{aligned} L &:= \{(z,\tau) \in \mathbb{R}^n \times \Omega\colon\ z \in L(\varphi_\tau, c', \infty)\},\\ D &:= \{(z,\tau) \in \mathbb{R}^n \times \Omega\colon\ z \in D^2_{\varphi_\tau}\},\ \text{and}\\ G &:= \bigcup_{k=1}^{\infty} \operatorname{graph} y_k. \end{aligned}$$

By Lemma 8.9 and Lemma 8.10 the set $(C \cap L \cap D) \setminus G$ is contained in $\mathcal{B} \otimes \Omega$. The measure μ is complete and σ-finite, hence for every set there exists a measurable cover. Therefore by the Szpilrajn-Marczewski's theorem ([K], p. 95), the σ-algebra P is closed under the Suslin operation. Since the projection of the set $(C \cap L \cap D) \setminus G$ on Ω is Ω up to a set of measure zero, Theorem 8.1 implies that there exists a measurable function $\tilde{x}\colon\ \Omega \to \mathbb{R}^n$ such that

$$\tilde{x}(\tau) \in C_\tau \cap L(\varphi_\tau, c', \infty) \cap D^2_{\varphi_\tau} \quad \text{and} \quad \tilde{x}(\tau) \neq y_k(\tau) \quad \text{for all } k \in \mathbb{N}$$

almost everywhere. Clearly $\|x - \tilde{x}\| < \varepsilon$ and $\tilde{x} \neq y_k$ for all $k \in \mathbb{N}$. By Lemma 7.1 we have that $\tilde{x} \in L_f$, consequently Proposition 7.2 implies $\tilde{x} \in D^2_f$, which is a contradiction. Hence $B(x,\varepsilon) \cap D^2_f$ is not countable.

Now let μ be a σ-finite purely atomic measure. Choose a maximal set of pairwise disjoint atoms $\{\alpha_k\}_{k=1}^K \subset P$. We will suppose that $K = \infty$; the case when K is finite is similar. We have $\mu(\Omega \setminus \bigcup \alpha_k) = 0$; denote $c_k := \mu\alpha_k$. The mapping

$$S(x) := \sum_{k=1}^{\infty} \frac{x_k}{\sqrt{c_k}} \chi_{\alpha_k}, \quad x = (x_k) \in \ell^2_{\mathbb{R}^n} \ \ \text{(i.e. } x_k \in \mathbb{R}^n\text{)}$$

is an isometry of $\ell^2_{\mathbb{R}^n}$ onto H. Define functions ψ_k on $\mathbb{R}^n$ by the formula

$$\psi_k(z) = \int_{\alpha_k} \varphi\Big(\frac{z}{\sqrt{c_k}}, \tau\Big)\, \mathrm{d}\mu(\tau).$$

Then ψ_k are clearly convex. Moreover

$$g(x) := \sum_{k=1}^{\infty} \psi_k(x_k) = f(S(x)) \quad \text{ for } x = (x_k) \in \ell^2_{\mathbb{R}^n},$$

hence g is continuous. Therefore by Theorem 5.4 the set D^2_g is uncountable in any ball in $\ell^2_{\mathbb{R}^n}$. Since S is a linear isometry, $S(D^2_g) = D^2_f$ and therefore D^2_f is uncountable in any ball in H.

Finally, let us consider the general case. Choose a maximal set of pairwise disjoint atoms $\{\alpha_k\}$ in Ω. Define $\Omega_1 := \bigcup \alpha_k$, $\Omega_0 := \Omega \setminus \Omega_1$, and $P_i := \{A \subset \Omega_i\colon\ A \in P\}$ for $i = 0, 1$. Then (Ω_0, P_0, μ) is nonatomic, (Ω_1, P_1, μ) is purely atomic, and both are as in the setting. For $i = 0, 1$ denote

$$\begin{aligned} H_i &:= L^2_{\mathbb{R}^n}(\Omega_i, P_i, \mu), \\ f_i(x) &:= \int_{\Omega_i} \varphi(x(\tau), \tau)\, \mathrm{d}\mu(\tau) \ \text{ for } x \in H_i, \\ g_i(x_0, x_1) &:= f_i(x_i) \quad \text{for } (x_0, x_1) \in H_0 \times H_1. \end{aligned}$$

Then, clearly, $D^2_{g_0} = D^2_{f_0} \times H_1$ and $D^2_{g_1} = H_0 \times D^2_{f_1}$. Also, $D^2_{g_0+g_1} \supset D^2_{g_0} \cap D^2_{g_1} = D^2_{f_0} \times D^2_{f_1}$. Since we know that $D^2_{f_i}$ is uncountable in any ball in H_i for $i = 0, 1$, the set $D^2_{g_0+g_1}$ is uncountable in every ball of $H_0 \times H_1$. The mapping $Z\colon\ H_0 \times H_1 \to H$ such that

$$Z(x_0, x_1)(\tau) = \begin{cases} x_0(\tau), & \text{for } \ \tau \in \Omega_0, \\ x_1(\tau), & \text{for } \ \tau \in \Omega_1, \end{cases}$$

is a linear isometry (when $H_0 \times H_1$ is equipped with the ℓ_2-norm). We have

$$(g_0 + g_1)(x_0, x_1) = f(Z(x_0, x_1)).$$

Therefore $Z(D^2_{g_0+g_1}) = D^2_f$, and thus D^2_f is uncountable in any ball in H. $\square$

If H is a separable space, then D^2_f is a Borel set by Proposition 8.6. Consequently, the Aleksandrov-Hausdorff theorem (cf. [K]) implies that D^2_f is c-dense.

Let Ω be an arbitrary measurable set equipped with a σ-algebra P. We say that a closed-valued multifunction $T\colon\ \Omega \to 2^{\mathbb{R}^n}$ is measurable, if $\{\tau \in \Omega\colon\ T(\tau) \cap F \neq \emptyset\} \in P$ for every closed set $F \subset \mathbb{R}^n$ (or, equivalently, for every closed ball F in $\mathbb{R}^n$). We consider the set $\mathbb{R}^n \times \Omega$ always to be equipped with the σ-algebra $\mathcal{B} \otimes P$, where $\mathcal{B}$ is the σ-algebra of Borel subsets of $\mathbb{R}^n$. As usual, Q is the set of all rational numbers.

The following selection theorem is an immediate consequence of Corollary on p. 33 of [L].

Theorem 8.1. [L] *Let Ω be a space equipped with a σ-algebra P which is closed under the Suslin operation, and let $\mathcal{B}$ be the σ-algebra of Borel subsets of $\mathbb{R}^n$ for some $n \in \mathbb{N}$. Let A be an element of the product σ-algebra $\mathcal{B} \otimes P$. Denote by $\Pi_\Omega A$ the projection of A into Ω. Then $\Pi_\Omega A \in P$ and there is a measurable mapping $T\colon\ \Pi_\Omega A \to \mathbb{R}^n$ such that $(T(z), z) \in A$ for every $z \in \Pi_\Omega A$.*

Lemma 8.2. *Let f be a continuous convex function on a Banach space X. Then the set L_f of points of Lipschitz smoothness is an F_σ set.*

P r o o f. For natural numbers k and n denote $A_{k,n} = L(f, k, 1/n)$. Let $\{x_m\}$ be a sequence in $A_{k,n}$ which converges to some $x \in X$. Since f is Lipschitz on a neigbourhood of x, we can find a weak* cluster point x^* of the sequence $f'(x_m)$. Then it is easy to see that

$$f(x+v) - f(x) - \langle x^*, v\rangle \leqslant k\|v\|^2$$

holds whenever $v \in X$, $\|v\| < 1/n$. Consequently, each $A_{k,n}$ is closed, and therefore $L_f = \bigcup\limits_{k,n=1}^{\infty} A_{k,n}$ is an F_σ set. $\square$

The statement of Lemma 8.4 is implicitly contained in the proof of Theorem 4.1 of [BN]. Its proof is based on the following simple fact which is actually proved in the above reference:

Lemma 8.3. *Let $\{f_n\}$ be a sequence of K-Lipschitz functions on a metric space which converges pointwise on a dense subset. Then it converges everywhere to a K-Lipschitz function.*

Lemma 8.4. *Let X be a Banach space and let Z be a dense subset of X. Let f be a convex continuous function defined on X, $x \in L_f$ and let $T\colon\ X \to X^*$ be a bounded linear operator such that*

$$\lim_{t\to 0} \frac{1}{t^2}(f(x+tz) - f(x) - t\langle f'(x), z\rangle) = \langle Tz, z\rangle/2 \tag{11}$$

for every $z \in Z$. Then (11) *holds for every $z \in X$, and consequently $x \in D_f^2$.*

Proof. Define

$$\Delta_t(z) := \frac{1}{t^2}(f(x+tz) - f(x) - t\langle f'(x), z\rangle).$$

Because $x \in L_f$, there exist $r > 0$ and $c > 0$ such that

$$|\Delta_t(z)| \leqslant c$$

whenever $z \in B(0,r)$ and $t \in (0,1]$. The functions Δ_t, $t \in (0,1]$ are convex, therefore they all are Lipschitz with a common constant $K \in \mathbb{R}$ on $B(0, r/2)$ (cf. [P], the proof of Proposition 1.6). The function $F(z) := \langle Tz, z\rangle/2$ is continuous on X and the functions Δ_t converge pointwise to F on Z when t goes to zero. Using the previous lemma we easily get (e.g. by Heine's definition of limit) that Δ_t converge to F on $B(0, r/2)$ and therefore everywhere. □

Let X be a Banach space, let $Z \subset X$ be a finite dimensional subspace. Let f be a convex continuous function on X and $k > 0$. For $x \in X$ define on Z the function $g_{x,Z}(z) = f(x+z)$. Let $D^2_{f,Z,k}$ be the set of all $x \in X$ for which $g_{x,Z}$ is second order differentiable at 0, and $\|g''_{x,Z}(0)\| \leqslant k$.

Lemma 8.5. *Let X, Z, f, and k be as above. Then $D^2_{f,Z,k}$ is an $F_{\sigma\delta}$ set.*

Proof. Let D_f be the set of all points of Fréchet differentiability of f. Denote by $\mathcal{M}$ the set of all symmetric bilinear forms on $Z \times Z$ with the usual norm. For $\varepsilon > 0$ and $\delta > 0$ denote by $A(\varepsilon, \delta)$ the set of all points $x \in D_f$ for which there exists $M \in \mathcal{M}$ such that $\|M\| \leqslant k$ and

$$|f(x+z) - f(x) - \langle f'(x), z\rangle - \frac{1}{2}M(z,z)| \leqslant \varepsilon\|z\|^2 \quad \text{if} \;\; z \in Z, \|z\| < \delta. \tag{12}$$

First observe that each set $A(\varepsilon, \delta)$ is closed in D_f. In fact, if (x_n) is a sequence in $A(\varepsilon, \delta)$ which converges to some $x_0 \in D_f$, we can find a subsequence x_{n_i} and bilinear forms $M_i \in \mathcal{M}$, $\|M_i\| \leqslant k$ such that (12) holds with $x = x_{n_i}$, $M = M_i$ and (M_i) converges to some $M_0 \in \mathcal{M}$ with $\|M_0\| \leqslant k$. Using the well-known fact that $f'(x_n) \to f'(x_0)$ (cf. [P], Lemma 2.6), we easily obtain that (12) holds with $x = x_0$ and $M = M_0$. Thus, since D_f is a G_δ set (cf. [P]), it is sufficient to show that

$$D^2_{f,Z,k} = \bigcap_{q=1}^{\infty} \bigcup_{p=1}^{\infty} A\Big(\frac{1}{q}, \frac{1}{p}\Big).$$

The inclusion "$\subset$" is an immediate consequence of Proposition 2.3.

If $x \in \bigcap_{q=1}^{\infty} \bigcup_{p=1}^{\infty} A(\frac{1}{q}, \frac{1}{p})$, we can find for each $q \in \mathbb{N}$ some $p_q \in \mathbb{N}$ and $M_q \in \mathcal{M}$, $\|M_q\| \leqslant k$ such that (12) holds with $M = M_q$, $\varepsilon = \frac{1}{q}$ and $\delta = \frac{1}{p_q}$. Let M_0 be a cluster point of the sequence (M_q). Then it is easy to check that $M_0 = g''_{x,Z}(0)$ and $\|M_0\| \leqslant k$. $\square$

Proposition 8.6. *Let X be a separable Banach space and let f be a convex continuous function on X. Then D^2_f is an $F_{\sigma\delta\sigma}$ set.*

Proof. Let $Z_1 \subset Z_2 \subset \ldots$ be finite dimensional subspaces of X such that $\overline{\bigcup Z_p} = X$. Then, in virtue of Lemma 8.2 and Lemma 8.5, we know that it is sufficient to show that

$$D^2_f = L_f \cap \bigcup_{k=1}^{\infty} \bigcap_{p=1}^{\infty} D^2_{f,Z_p,k}.$$

The inclusion "$\subset$" can be easily obtained by Proposition 2.3. To show the other inclusion suppose that $x \in L_f$ and that $k \in \mathbb{N}$ is chosen so that $x \in D^2_{f,Z_p,k}$ for $p = 1, 2, \ldots$. Put $M_p = g''_{x,Z_p}(0)$. Clearly $\|M_p\| \leqslant k$ and $M_{p+1} = M_p$ on $Z_p \times Z_p$. It is easy to show that there exists a symmetric bilinear form M on $X \times X$ such that $M = M_p$ on $Z_p \times Z_p$ and $\|M\| \leqslant k$. From Lemma 8.4 (which we apply with $Z := \bigcup Z_p$, and with $T\colon X \to X^*$, $\langle T(z), x\rangle = M(z, x)$) it now follows that $M = f''(x)$. $\square$

The second statement of the next lemma can be found in [R2]. The first we were not able to find.

Lemma 8.7. *Let (Ω, P, μ) and φ be as in the setting from Section 7; let $x\colon \Omega \to \mathbb{R}^n$ be a measurable function. Then the mapping $\partial\varphi\colon \mathbb{R}^n \times \Omega \to 2^{\mathbb{R}^n}$, where $\partial\varphi(z,\tau) := \partial\varphi_\tau(z)$, and the mapping $\psi\colon \tau \in \Omega \mapsto \partial\varphi_\tau(x(\tau))$ are measurable.*

Proof. Let a closed ball F in $\mathbb{R}^n$ be given. To prove measurability of $\partial\varphi$, it is sufficient to show that the set

$$A := \{(z,\tau) \in \mathbb{R}^n \times \Omega\colon\ \partial\varphi_\tau(z) \cap F \neq \emptyset\}$$

is measurable. Denote $S := Q^n$. We will show that

$$\begin{aligned}&(\mathbb{R}^n \times \Omega) \setminus A \\ &= \bigcup_{h\in S} \bigcup_{m\in\mathbb{N}} \{(z,\tau) \in \mathbb{R}^n \times \Omega\colon\ m(\varphi_\tau(z + h/m) - \varphi_\tau(z)) < \inf\{\langle h, y\rangle\colon\ y \in F\}.\end{aligned}$$

Denote the latter set by B and fix $(z,\tau) \notin A$. Since $\partial\varphi_\tau(z)$ is closed, convex and bounded, it is easy to show that there exists $h \in S$ for which

$$\sup\{\langle h, y\rangle\colon\ y \in \partial\varphi_\tau(z)\} < \inf\{\langle h, y\rangle\colon\ y \in F\}.$$

Since it is well-known that

$$\sup\{\langle h, y\rangle\colon\ y \in \partial\varphi_\tau(z)\} = \lim_{t\to 0+} \frac{1}{t}(\varphi_\tau(z+th) - \varphi_\tau(z)),$$

we obtain that $(z,\tau) \in B$. Consequently $(\mathbb{R}^n \times \Omega) \setminus A \subset B$. The opposite inclusion easily follows from the fact that $\frac{1}{t}(\varphi_\tau(z+th) - \varphi_\tau(z))$ is nondecreasing with respect to t. Since φ is a measurable function and S is countable, the sets B and A are measurable. Since the transformation $\tau \mapsto (x(\tau),\tau)$ is measurable (cf. [R2], Corollary on p. 174) and $\partial\varphi$ is measurable with respect to the σ-algebra $\mathcal{B} \otimes P$, the second assertion of the lemma also holds. $\square$

Lemma 8.8. *Let φ and x be as in the previous lemma. Then the sets*

$$F := \{(z,\tau) \in \mathbb{R}^n \times \Omega\colon\ \varphi_\tau \text{ is differentiable at } z\} \quad \text{and}$$
$$F_x := \{\tau \in \Omega\colon\ \varphi_\tau \text{ is differentiable at } x(\tau)\}$$

are measurable.

P r o o f. Let $\partial\varphi$ and ψ be the mappings from Lemma 8.7. Using a well-known fact, we see that F, F_x are the sets of all points at which $\partial\varphi$, ψ, respectively, are singlevalued. Therefore the measurability of $\partial\varphi$ and ψ together with the separability of $\mathbb{R}^n$ easily imply that F and F_x are measurable as well. Indeed, e.g. the complement of F is the set of points x for which there exist two disjoint elements B_1 and B_2 from a fixed countable basis of open sets of $\mathbb{R}^n$, such that $\partial\varphi$ intersects both B_1 and B_2. $\square$

Remark. Define a mapping $\varphi'\colon F \to \mathbb{R}^n$ by $\varphi'(z,\tau) := \varphi'_\tau(z)$. Then φ' is a restriction of the multivalued measurable mapping $\partial\varphi$ to the measurable set F and consequently φ' is measurable.

Lemma 8.9. *Let φ be as in Lemma 8.7, and let $c > 0$. Then the set*

$$L := \{(z,\tau) \in \mathbb{R}^n \times \Omega\colon\ z \in L(\varphi_\tau, c, \infty)\}$$

is measurable.

P r o o f. The set F from the previous lemma is measurable and we know that $L \subset F$. Denote $S := Q^n$. It is easy to see that

$$F \setminus L = \bigcup_{y\in S} \{(z,\tau) \in F\colon\ |\varphi_\tau(y) - \varphi_\tau(z) - \langle \varphi'_\tau(z), y - z\rangle| > c\|y - z\|^2\}.$$

Due to the measurability of φ and φ' on F the set $F \setminus L$, and consequently also L, is measurable. $\square$

In the following we will identify in the standard way $\mathbb{R}^{n^2}$, the space of all real $n \times n$ matrices, and the space $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ of linear operators. We will consider them with the Euclidean norm $\|.\|_e$ of $\mathbb{R}^{n^2}$.

Lemma 8.10. *Let φ and x be as in Lemma 8.7. Then the sets*

$$D := \{(z,\tau) \in \mathbb{R}^n \times \Omega\colon\ z \in D^2_{\varphi_\tau}\} \ \text{ and } \ D_x := \{\tau \in \Omega\colon\ x(\tau) \in D^2_{\varphi_\tau}\}$$

are measurable. Moreover, the mapping $d\colon\ \tau \to \nabla^2\varphi_\tau(x(\tau))$ is measurable on D_x.

Proof. For $r > 0$, let $\mathcal{M}_r = Q^{n^2} \cap B(0,r)$ and $S(r) = Q^n \cap B(0,r)$. Let g be a convex function on $\mathbb{R}^n$, and let $x \in \mathbb{R}^n$. By Proposition 2.3 we know that $x \in D^2_g$ if and only if there exist $x^* \in \partial g(x)$ and a linear operator $T\colon\ \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\lim_{h\to 0} \|h\|^{-2}(g(x+h) - g(x) - \langle x^*, h\rangle - \frac{1}{2}\langle Th, h\rangle) = 0. \tag{13}$$

Of course, the symmetrization of T necessarily equals to $\nabla^2 g(x)$ and x^* is the (Fréchet) derivative $g'(x)$. Using this fact we shall prove that $D^2_g = V$, where

$$V = \bigcup_{m\in\mathbb{N}} \bigcap_{k\in\mathbb{N}} \bigcup_{T\in\mathcal{M}_m} \bigcup_{p\in\mathbb{N}} \bigcap_{0\neq h\in S(1/p)} \{x \in D_g\colon\ \|h\|^{-2}\ |g(x+h) - g(x) - \langle g'(x), h\rangle - \frac{1}{2}\langle Th, h\rangle| \leqslant 1/k\},$$

and D_g is the set of points where g is differentiable. If $x \in D^2_g$, choose a natural number $m > \|\nabla^2 g(x)\|_e$. Now, for each $k \in \mathbb{N}$, choose a $p \in \mathbb{N}$ such that

$$\|h\|^{-2}|g(x+h) - g(x) - \langle g'(x), h\rangle - \frac{1}{2}\langle \nabla^2 g(x)(h), h\rangle| \leqslant \frac{1}{2k} \ \text{ for } h \in B(0,1/p) \setminus \{0\}$$

and find $T \in \mathcal{M}_m$ such that $\|h\|^{-2}|\langle Th, h\rangle - \langle \nabla^2 g(x)(h), h\rangle| \leqslant \frac{1}{k}$ for each $h \neq 0$. Then clearly

$$\|h\|^{-2}|g(x+h) - g(x) - \langle g'(x), h\rangle - \frac{1}{2}\langle Th, h\rangle| \leqslant \frac{1}{k} \ \text{ for } h \in B(0,1/p) \setminus \{0\}$$

and thus $x \in V$.

If $x \in V$, we choose $m \in \mathbb{N}$ and sequences $T_k \in \mathcal{M}_m$, $p_k \in \mathbb{N}$, k=1,2, ..., such that

$$\|h\|^{-2}|g(x+h) - g(x) - \langle g'(x), h\rangle - \frac{1}{2}\langle T_k h, h\rangle| \leqslant \frac{1}{k}$$

for $h \in S(1/p_k) \setminus \{0\}$, and consequently for each $h \in B(0,1/p_k) \setminus \{0\}$. Since $\|T_k\|_e \leqslant m$, there exists a cluster point T of $\{T_k\}$, and it is easy to check that T satisfies (13).

Consequently

$$D = \bigcup_{m\in\mathbb{N}} \bigcap_{k\in\mathbb{N}} \bigcup_{T\in\mathcal{M}_m} \bigcup_{p\in\mathbb{N}} \bigcap_{0\neq h\in S(1/p)}$$
$$\{(z,\tau)\in F\colon \ \|h\|^{-2}|\varphi_\tau(z+h)-\varphi_\tau(z)-\langle\varphi_\tau'(z),h\rangle-\frac{1}{2}\langle Th,h\rangle|\leqslant\frac{1}{k}\},$$

where F is the set defined in Lemma 8.8. Using Remark after Lemma 8.8 we easily obtain that D is measurable.

We have $D_x = G^{-1}(D)$,where $G(\tau)=(x(\tau),\tau)$. Since the mapping G is measurable, we conclude that D_x is measurable as well.

It remains to prove that d is measurable.

Let $\{e_i\}_{i=1}^n$ be the usual orthonormal basis of $\mathbb{R}^n$ and let $i,j\in\{1,\dots,n\}$ be fixed. By Lemma 8.7 the mapping $\tau\mapsto\partial\varphi_\tau(x(\tau))$ is measurable and closed valued, therefore it has a measurable selection s (cf. [R2], p. 163). Similarly, let s_k be a measurable selection for the mapping $\tau\mapsto\partial\varphi_\tau(x(\tau)+e_i/k)$. By Theorem 2.2 we have for $\tau\in D_x$ that

$$d_{i,j}(\tau) := \langle\nabla^2\varphi_\tau(x(\tau))e_i,e_j\rangle = \lim_{k\to\infty} k\langle s_k(\tau)-s(\tau),e_j\rangle.$$

Therefore the function $d_{i,j}$ is measurable, being a pointwise limit of a sequence of measurable functions, and consequently also d is measurable on D_x. □

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