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ALMOST HYPER-HERMITIAN STRUCTURES IN BUNDLE SPACES OVER MANIFOLDS WITH ALMOST CONTACT 3-STRUCTURE

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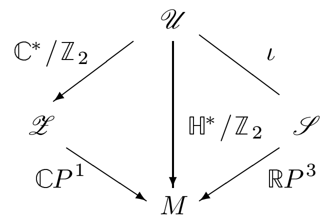
Abstract. We consider almost hyper-Hermitian structures on principal fibre bundles with one-dimensional fiber over manifolds with almost contact 3-structure and study relations between the respective structures on the total space and the base. This construction suggests the definition of a new class of almost contact 3-structure, which we called trans-Sasakian, closely connected with locally conformal quaternionic Kähler manifolds. Finally we give a family of examples of hypercomplex manifolds which are not quaternionic Kähler.

1. Introduction

An almost hyper-Hermitian (quaternion-Hermitian) manifold is a Riemannian $4n$-manifold which admits a reduction of its frame bundle to the subgroup $\mathrm{Sp}(n)$ ($\mathrm{Sp}(n)\,\mathrm{Sp}(1)$) of $\mathrm{SO}(4n)$. These two types of manifolds are of special interest because $\mathrm{Sp}(n)$ and $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$ are included in the list of Berger ([1]) of the possible holonomy groups of locally irreducible Riemannian manifolds that are not locally symmetric. An almost hyper-Hermitian (quaternion-Hermitian) manifold is said to be hyper-Kähler (quaternionic Kähler), if its reduced holonomy group is a subgroup of $\mathrm{Sp}(n)$, $n \geqslant 1$ ($\mathrm{Sp}(n)\,\mathrm{Sp}(1)$, $n > 1$). The terms "quaternionic Kähler" and "hyper-Kähler" were introduced by Calabi and Ishihara in 1973. A few years before, Kuo ([13]) defined a new type of geometric structure closely related to both quaternion-Hermitian and almost hyper-Hermitian structures, the almost contact 3-structure. A particular and interesting class of almost contact 3-structure is the Sasakian 3-structure. Riemannian manifolds with Sasakian 3-structure are called 3-Sasakian manifolds. They are Einstein and $(4n+3)$-dimensional and have many links with quaternionic Kähler and hyper-Kähler manifolds. In fact, if the distribution formed by the three Killing vector fields of a Sasakian 3-structure is regular then the space of leaves is quaternionic Kähler, which was shown by Ishihara in 1973 ([8]). Later in 1975, Konishi ([11]) proved the existence of a Sasakian 3-structure on a certain principal SO(3) bundle over any quaternionic Kähler manifold of positive scalar curvature. Recently, Boyer, Galicki and Mann ([3]) have shown that for any quaternionic Kähler manifold M of positive scalar curvature there exists a commutative diagram



where $\mathscr{U}$ is hyper-Kähler (the Swann bundle associated to M [19]), $\mathscr{Z}$ is Kähler-Einstein (the twistor space associated to M [18]) and $\mathscr{S}$ is 3-Sasakian (the Konishi bundle associated to M [11]). The map $\iota\colon \mathscr{S} \to \mathscr{U}$ is the inclusion of a level set of a natural real valued function while the other maps are fibrations where each map is denoted by its associated fiber.

In this paper we consider principal fibre bundles with one-dimensional structure group over manifolds with almost contact metric 3-structures. On the total bundle space we construct an almost hyper-Hermitian structure defined from an arbitrary connection form and the almost contact metric 3-structure of the base. In this context, we find relations among classes of the almost hyper-Hermitian structure, classes of the almost contact metric 3-structure and the curvature of the connection form. These relations lead us to consider a new class of almost contact 3-structure, called trans-Sasakian, which is closely connected with locally conformal quaternionic Kähler structures. Finally, the mentioned relations have suggested us a construction of a family of hypercomplex manifolds which are not quaternionic semi-Kähler.

2. Quaternion-Hermitian structures

Quaternion-Hermitian manifolds have been broadly treated by diverse authors (see [2], [8], [18], and [19]). In this section we review some basic definitions, known facts and prove some new results.

A $4n$-dimensional manifold M ($n > 1$) is said to be *quaternion-Hermitian* if M is equipped with a Riemanniann metric $\langle\,,\rangle$ and a rank-three subbundle $\mathscr{J}$ of the endomorfism bundle $\operatorname{End} TM$ such that locally $\mathscr{J}$ has an *adapted basis* J_1, J_2, J_3 with $J_i^2 = -1$, $J_1 J_2 = J_3 = -J_2 J_1$ and $\langle J_i X, J_i Y\rangle = \langle X, Y\rangle$, for $i = 1, 2, 3$. This is equivalent to saying that M has a reduction of its structure group to $\operatorname{Sp}(n)\operatorname{Sp}(1)$.

At each point of a $4n$-dimensional quaternion-Hermitian manifold there is a local orthonormal frame field, called *adapted frame*, given in the following way:

$$\{E_1, \dots, E_n, J_1E_1, \dots, J_1E_n, J_2E_1, \dots, J_2E_n, J_3E_1, \dots, J_3E_n\}.$$

From the three local two-forms $F^i(X,Y) = \langle X, J_iY\rangle$, one may define a global four-form Ω by the local formula

$$\Omega = F^1 \wedge F^1 + F^2 \wedge F^2 + F^3 \wedge F^3. \tag{2.1}$$

The following lemma will be useful later.

Lemma 2.1. *Let M be a quaternion-Hermitian $4n$-manifold ($n > 1$) and α a skew-symmetric p-form on M ($p \leqslant 2$). Then $\alpha \wedge \Omega = 0$ if and only if $\alpha = 0$.*

Proof. Throughout the proof (i, j, k) is always a cyclic permutation of $(1, 2, 3)$, $r, s = 1, \dots, n$ with $r \neq s$ and we consider an adapted local frame of M ordered as in (2.1). First, from

$$\begin{aligned}
\alpha \wedge \Omega(E_s, J_iE_s, E_r, J_1E_r, J_2E_r, J_3E_r) &= 0,\\
\alpha \wedge \Omega(J_jE_s, J_kE_s, E_r, J_1E_r, J_2E_r, J_3E_r) &= 0,\\
\alpha \wedge \Omega(E_r, J_iE_r, E_s, J_1E_s, J_2E_s, J_3E_s) &= 0,\\
\alpha \wedge \Omega(J_jE_r, J_kE_r, E_s, J_1E_s, J_2E_s, J_3E_s) &= 0,
\end{aligned}$$

we have

$$\begin{aligned}
3\alpha(E_s, J_iE_s) + \alpha(E_r, J_iE_r) + \alpha(J_jE_r, J_kE_r) &= 0,\\
3\alpha(J_jE_s, J_kE_s) + \alpha(E_r, J_iE_r) + \alpha(J_jE_r, J_iE_r) &= 0,\\
\alpha(E_s, J_iE_s) + \alpha(J_jE_s, J_kE_s) + 3\alpha(E_r, J_iE_r) &= 0,\\
\alpha(E_s, J_iE_s) + \alpha(J_jE_s, J_kE_s) + 3\alpha(J_jE_r, J_kE_r) &= 0.
\end{aligned}$$

From these equations $\alpha(E_r, J_iE_r) = \alpha(J_jE_r, J_kE_r) = \alpha(E_s, J_iE_s) = \alpha(J_jE_s, J_kE_s) = 0$. Secondly, we consider

$$\begin{aligned}
\alpha \wedge \Omega(E_s, J_1E_s, J_2E_s, E_r, J_1E_r, J_2E_r) &= 0,\\
\alpha \wedge \Omega(E_s, J_3E_s, J_1E_s, E_r, J_3E_r, J_1E_r) &= 0,\\
\alpha \wedge \Omega(E_s, J_2E_s, J_3E_s, E_r, J_2E_r, J_3E_r) &= 0,\\
\alpha \wedge \Omega(J_1E_s, J_2E_s, J_3E_s, J_1E_r, J_2E_r, J_3E_r) &= 0,
\end{aligned}$$

then we have

$$
\begin{aligned}
\alpha(E_s, E_r) + \alpha(J_1E_s, J_1E_r) + \alpha(J_2E_s, J_2E_r) &= 0,\\
\alpha(E_s, E_r) + \alpha(J_1E_s, J_1E_r) + \alpha(J_3E_1, J_3E_r) &= 0,\\
\alpha(E_s, E_r) + \alpha(J_2E_s, J_2E_r) + \alpha(J_3E_1, J_3E_r) &= 0,\\
\alpha(J_1E_s, J_1E_r) + \alpha(J_2E_s, J_2E_r) + \alpha(J_3E_1, J_3E_r) &= 0.
\end{aligned}
$$

Hence, $\alpha(E_s, E_r) = \alpha(J_iE_1, J_iE_r) = 0$. At this point, we can conclude $\alpha = 0$. □

Remark 2.2. In [12] it is shown that $\alpha\wedge\Omega = 0$ implies $\alpha = 0$, when α is a p-form such that $p + 4 \leqslant n + 1$.

If Ω is parallel with respect to the Levi-Civita connection ∇ of $\langle\,,\rangle$, then the holonomy group of M is a subgroup of $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$ $(n > 1)$ and M is said to be *quaternionic Kähler*. The quaternionic Kähler condition is equivalent to the existence of three local one-forms α^1, α^2, α^3 such that

$$\nabla J_i = \alpha^i \otimes J_j - \alpha^k \otimes J_k \tag{2.2}$$

for all cyclic permutations (i, j, k) of $(1, 2, 3)$ ([8]). If the exterior derivative $\mathrm{d}\Omega$ vanishes, M is said to be *quaternionic almost-Kähler*. In [19] it is shown that every quaternionic almost-Kähler manifold of dimension $\geqslant 12$ is quaternionic Kähler. The dimension eight is included in the following result.

Proposition 2.3. *Let M be a quaternion-Hermitian $4n$-manifold $(n > 1)$. Then the following statements are equivalent:*

i) *M is quaternionic Kähler.*

ii) *$\mathrm{d}\Omega = 0$ and $\mathrm{d}F^i = a^i \wedge F^i + b^i \wedge F^j + c^i \wedge F^k$.*

iii) *There exist three local one-forms α^1, α^2, α^3 such that $\mathrm{d}F^i = \alpha^i \wedge F^j - \alpha^k \wedge F^k$ where (i, j, k) is a cyclic permutation of $(1, 2, 3)$.*

P r o o f. The equivalence of the first two statements was established in [19]. Taking (2.2) into account, it is easy to see that the third statement follows from the first. Finally, it is a straightforward computation that the second statement follows from the third. □

The quaternionic nearly-Kähler condition, i.e., $\mathrm{d}\Omega = 5\nabla\Omega$, is equivalent to the quaternionic Kähler condition ([20]). If the coderivative $\delta\Omega$ vanishes, M is said to be *quaternionic semi-Kähler*. In [2] it is shown that $\delta\Omega = -*\,\mathrm{d}k\Omega^{n-1}$, where k is constant and $*$ denotes Hodge's star operator. Then every quaternionic almost-Kähler manifold is quaternionic semi-Kähler. The converse is also true for dimension eight.

A quaternion-Hermitian $4n$-manifold M $(n > 1)$ is said to be *locally conformal quaternionic Kähler*, if $\mathrm{d}F^i = \alpha \wedge F^i + \alpha^i \wedge F^j - \alpha^k \wedge F^k$ for all cyclic permutations (i, j, k) of $(1, 2, 3)$ and some one-forms α, α^1, α^2, α^3. In this case $\mathrm{d}\Omega = 2\alpha \wedge \Omega$ and using Lemma 2.1 we have $\mathrm{d}\alpha = 0$, then locally $\alpha = \mathrm{d}f$. If we consider the metric $e^{-f}\langle\, , \rangle$, the structure considered on a neighborghood of a point is also quaternion-Hermitian and satisfies the third statement of Proposition 2.3. Moreover, if we have $\mathrm{d}\Omega = 2\alpha_U \wedge \Omega = 2\alpha_V \wedge \Omega$ for all points of $U \cap V$, U, V open sets of M, by Lemma 2.1 $\alpha_U = \alpha_V$ on $U \cap V$, then the one-form α is global.

An *almost hyper-Hermitian structure* on M is a quaternion-Hermitian structure such that the subbundle $\mathscr{J}$ has an adapted basis J_1, J_2, J_3 of global tensor fields. In this case, M has a reduction of its structure group to $\mathrm{Sp}(n)$. If M has an almost hyper-Hermitian structure such that F^1, F^2, F^3 are closed, M is said to be *hyper-Kähler*. Hitchin [6] showed that this implies that J_1, J_2, J_3 are integrable and hence the holonomy group is contained in $\mathrm{Sp}(n)$. An alternative condition to impose on an almost hyper-Hermitian structure is that J_1, J_2, J_3 all be integrable. In this case the manifold M is said to be *hypercomplex* (*hyper-Hermitian*). A manifold M is said to be *locally conformal hyper-Kähler*, if M has an almost hyper-Hermitian structure such that $\mathrm{d}F^i = \alpha \wedge F^i$ for some one-form α. In this case α is closed and we can do a local conformal change of the metric such that the almost hyper-Hermitian structure considered on a neigborhood of the point is hyper-Kähler for the new metric.

3. Almost contact 3-structures

In this section we show, together with some definitions and known facts (see [13], [14], some new results about almost contact 3-structures which will be used later. An *almost contact structure* (φ, ξ, η) on a differentiable manifold is an aggregate consisting of a tensor field φ of type $(1, 1)$, a vector field ξ and a one-form η which satisfy $\eta(\xi) = 1$, $\varphi^2 = -I + \xi \otimes \eta$, where $\otimes$ means the tensor product and I is the identity tensor.

A $(4n + 3)$-manifold M $(n \geqslant 1)$ possesses an *almost contact metric* 3-*structure*, if M has a Riemannian metric $\langle\, , \rangle$ and three almost contact structures, $(\varphi_i, \xi_i, \eta^i)$, $i = 1, 2, 3$, satisfying

$$\eta^i(\xi_j) = \delta^i_j, \quad \varphi_i(\xi_j) = -\varphi_j(\xi_i) = \xi_k, \quad \eta^i \circ \varphi_j = -\eta^j \circ \varphi_i = \eta_k,$$
$$\varphi^i \circ \varphi_j - \eta^j \otimes \xi_i = -\varphi_j \circ \varphi_i + \eta^i \otimes \xi_j = \varphi_k, \quad \langle \varphi_i X, \varphi_i Y \rangle = \langle X, Y \rangle,$$

for any cyclic permutation (i, j, k) of $(1, 2, 3)$ and any X, Y vector fields on M. In this case, the structure group of M admits a reduction to $\mathrm{Sp}(n) \times I_3$. At each point of a $(4n+3)$-manifold with an almost contact metric 3-structure there is a local orthonormal frame field, called *adapted frame*, given in the following way:

$$\{E_1, \dots, E_n, \varphi_1 E_1, \dots, \varphi_1 E_n, \varphi_2 E_1, \dots, \varphi_2 E_n, \varphi_3 E_1, \dots, \varphi_3 E_n, \xi_1, \xi_2, \xi_3\}. \tag{3.1}$$

Let F^i be the two-forms given by $F^i(X,Y) = \langle X, \varphi_i Y\rangle$. Associated to an almost contact metric 3-structure there is a four-form given by

$$\Omega = F^1 \wedge F^1 + F^2 \wedge F^2 + F^3 \wedge F^3. \tag{3.2}$$

For almost contact 3-structures it is also needed to consider the three-form

$$\Psi = \eta^1 \wedge F^1 + \eta^2 \wedge F^2 + \eta^3 \wedge F^3. \tag{3.3}$$

We will make use of the following lemma in the sequel.

Lemma 3.1. *Let M be a $(4n+3)$-manifold ($n \geqslant 1$) with an almost contact 3-structure and α a skew-symmetric two-form on M. Then*

i) $\alpha \wedge \Psi = 0$ *if and only if* $\alpha = 0$.

ii) $\mathrm{C}_{12} \circ \mathrm{C}_{13}(\alpha \otimes \Omega) = 0$ *if and only if* $\alpha = 0$, *where* C *denotes the metric contraction.*

Proof. Throughout the proof (i,j,k) is always a cyclic permutation of $(1,2,3)$, $r,s = 1,\dots,n$ with $r \neq s$ and we consider an adapted local frame of M ordered as in (3.1). i) First, we develop $\alpha \wedge \Psi(E_r, E_s, \xi_1, \xi_2, \xi_3) = 0$, then we get $3\alpha(E_r, E_s) = 0$. Secondly, we consider $\alpha \wedge \Psi(\xi_i, E_r, \varphi_1 E_r, \varphi_2 E_r, \varphi_3 E_r) = 0$, $\alpha \wedge \Psi(E_r, \varphi_i E_r, \xi_1, \xi_2, \xi_3) = 0$ and $\alpha \wedge \Psi(\varphi_j E_r, \varphi_k E_r, \xi_1, \xi_2, \xi_3) = 0$, then we have

$$\begin{aligned} \alpha(E_r, \varphi_i E_r) + \alpha(\varphi_j E_r, \varphi_k E_r) &= 0,\\ \alpha(\xi_j, \xi_k) + 3\alpha(E_r, \varphi_i E_r) &= 0,\\ \alpha(\xi_j, \xi_k) + 3\alpha(\varphi_j E_r, \varphi_k E_r) &= 0. \end{aligned}$$

From these equations, $\alpha(\xi_j, \xi_k) = 0$, $\alpha(E_r, \varphi_i E_r) = 0$ and $\alpha(\varphi_j E_r, \varphi_k E_r) = 0$. Finally, we consider $\alpha \wedge \Psi(\xi_i, \xi_j, E_r, \varphi_i E_r, \varphi_j E_r) = 0$ and get $\alpha(\xi_i, \varphi_i E_r) + \alpha(\xi_j, \varphi_j E_r) = 0$. Hence, $\alpha(\xi_1, \varphi_1 E_r) = -\alpha(\xi_2, \varphi_2 E_r) = \alpha(\xi_3, \varphi_3 E_r) = -\alpha(\xi_1, \varphi_1 E_r) = 0$. In a similar way, we can get $\alpha(\xi_i, E_r) = 0$, $\alpha(\xi_i, \varphi_j E_r) = 0$ and $\alpha(\xi_i, \varphi_k E_r) = 0$. So we conclude $\alpha = 0$.

ii) From $\mathrm{C}_{12} \circ \mathrm{C}_{13}(\alpha \otimes \Omega)(\xi_i, \xi_j) = 0$, we get

$$\sum_{l=1}^{n} \alpha(E_l, \varphi_k E_l) + \sum_{l=1}^{n} \alpha(\varphi_i E_l, \varphi_k E_l) = 0. \tag{3.4}$$

Now, from $\mathrm{C}_{12} \circ \mathrm{C}_{13}(\alpha \otimes \Omega)(E_r, \varphi_i E_r) = 0$, $\mathrm{C}_{12} \circ \mathrm{C}_{13}(\alpha \otimes \Omega)(E_r, \varphi_j E_r) = 0$ and $\mathrm{C}_{12} \circ \mathrm{C}_{13}(\alpha \otimes \Omega)(E_r, \varphi_k E_r) = 0$, taking (3.4) into account, we get

$$\begin{aligned} \alpha(\xi_i, \xi_j) - \alpha(E_r, \varphi_k E_r) + 2\alpha(\varphi_i E_r, \varphi_j E_r) = 0, \\ \alpha(\xi_i, \xi_j) - \alpha(\varphi_i E_r, \varphi_j E_r) + 2\alpha(E_r, \varphi_k E_r) = 0. \end{aligned}$$

From these equations we have

$$-\alpha(\xi_i, \xi_j) = \alpha(\varphi_i E_r, \varphi_j E_r) = \alpha(E_r, \varphi_k E_r). \tag{3.5}$$

Now using (3.5) in (3.4) we obtain $-2n\alpha(\xi_i, \xi_j) = 0$. Hence

$$0 = \alpha(\xi_i, \xi_j) = \alpha(\varphi_i E_r, \varphi_j E_r) = \alpha(E_r, \varphi_k E_r). \tag{3.6}$$

Let us compute successively $\mathrm{C}_{12} \circ \mathrm{C}_{13}(\alpha\otimes\Omega)(\xi_1, \varphi_1 E_r)$, $\mathrm{C}_{12} \circ \mathrm{C}_{13}(\alpha\otimes\Omega)(\xi_2, \varphi_2 E_r)$ and $\mathrm{C}_{12} \circ \mathrm{C}_{13}(\alpha \otimes \Omega)(\xi_3, \varphi_3 E_r)$, obtaining

$$\begin{aligned} \alpha(\xi_2, \varphi_2 E_r) + \alpha(\xi_3, \varphi_3 E_r) = 0, \\ \alpha(\xi_1, \varphi_1 E_r) + \alpha(\xi_3, \varphi_3 E_r) = 0, \\ \alpha(\xi_1, \varphi_1 E_r) + \alpha(\xi_2, \varphi_2 E_3) = 0. \end{aligned}$$

From these equations we get

$$\alpha(\xi_1, \varphi_1 E_r) = \alpha(\xi_2, \varphi_2 E_r) = \alpha(\xi_3, \varphi_3 E_r) = 0. \tag{3.7}$$

In a similar way we can obtain

$$\alpha(\xi_i, E_r) = \alpha(\xi_j, \varphi_k E_r) = \alpha(\xi_k, \varphi_j E_r) = 0. \tag{3.8}$$

If the dimension of M is seven, the proof is already concluded. Let us complete the proof for dimension higher than seven. From $\mathrm{C}_{12} \circ \mathrm{C}_{13}(\alpha \otimes \Omega)(E_r, E_s) = 0$ and $\mathrm{C}_{12} \circ \mathrm{C}_{13}(\alpha \otimes \Omega)(\varphi_i E_r, \varphi_i E_s) = 0$ we have

$$\begin{aligned} \alpha(\varphi_1 E_r, \varphi_1 E_s) + \alpha(\varphi_2 E_r, \varphi_2 E_s) + \alpha(\varphi_3 E_r, \varphi_3 E_s) = 0, \\ \alpha(E_r, E_s) + \alpha(\varphi_j E_r, \varphi_j E_s) + \alpha(\varphi_k E_r, \varphi_k E_s) = 0. \end{aligned}$$

These equations yield

$$\alpha(E_r, E_s) = \alpha(\varphi_1 E_r, \varphi_1 E_s) = \alpha(\varphi_2 E_r, \varphi_2 E_s) = \alpha(\varphi_3 E_r, \varphi_3 E_s) = 0. \tag{3.9}$$

In a similar way we can get

$$\alpha(E_r, \varphi_i E_s) = \alpha(\varphi_i E_r, E_s) = \alpha(\varphi_j E_r, \varphi_k E_s) = \alpha(\varphi_k E_r, \varphi_j E_s) = 0. \tag{3.10}$$

From (3.6), (3.7), (3.8), (3.9) and (3.10) we conclude $\alpha = 0$. □

If Ω and Ψ are parallel with respect to the Levi-Civita connection, the almost contact 3-structure is said to be *cosymplectic*. If Ω and Ψ are closed, we say that M has an *almost-cosymplectic* 3*-structure*. If the forms Ω and Ψ are coclosed, i.e., $\delta\Omega = \delta\Psi = 0$, we say that M has a *semi-cosymplectic* 3*-structure*. An almost contact metric 3-structure is said to be *hypernormal*, if the three almost contact structures are normal, i.e., $N_{\varphi_i} + 2\,\mathrm{d}\eta^i \otimes \xi_i = 0$, where N_{φ_i} is the Ninjenhuis tensor of φ_i, i.e.,

$$N_{\varphi_i} = \varphi_i^2[X, Y] + [\varphi_i X, \varphi_i Y] - \varphi_i[\varphi_i X, Y] - \varphi_i[X, \varphi_i Y].$$

If we suppose that the two-forms F^1, F^2, F^3 and the one-forms η^1, η^2, η^3 are closed, we say that M has a *hypercosymplectic* 3*-structure*. One can use Hitchin's argument to deduce that in this case the three almost contact structures are normal. Therefore, the three almost contact structures are cosymplectic, i.e., $\nabla F^i = 0$ and $\nabla\eta^i = 0$, $i = 1, 2, 3$ ([5]).

Definition 3.2. An almost contact metric 3-structure is said to be *a-Sasakian* ($a \in \mathbb{R}$, $a \neq 0$), if it is hypernormal and $\mathrm{d}\eta^i = aF^i$. When $a = 1$, the almost contact 3-structure is said to be *Sasakian*. A hypercosymplectic structure can be considered a 0-Sasakian structure.

Definition 3.3. An almost contact metric 3-structure is said to be *trans-Sasakian*, if

$$\begin{aligned}
\mathrm{d}F^i &= \alpha \wedge F^i + \alpha^i \wedge F^j - \alpha^k \wedge F^k,\\
\mathrm{d}\eta^i &= aF^i + r_i F^j - r_k F^k + \alpha \wedge \eta^i + \alpha^i \wedge \eta^j - \alpha^k \wedge \eta^k
\end{aligned}$$

for some a, r_1, r_2, r_3 differentiable local functions, α, α^1, α^2, α^3 local one-forms on M and for all (i, j, k) cyclic permutations of $(1, 2, 3)$. In this case we have $\mathrm{d}\Omega = 2\alpha \wedge \Omega$ and $\mathrm{d}\Psi = 2\alpha \wedge \Psi + a\Omega$.

Lemma 3.4. *Let M be a $(4n + 3)$-manifold ($n \geqslant 1$) with a trans-Sasakian structure. Then the local functions a, r_1, r_2, r_3 and the local forms α, α^1, α^2, α^3 given in Definition* 3.3 *are global.*

Proof. Let us suppose a_U, r_{1U}, r_{2U}, r_{3U}, α_U, α_U^1, α_U^2, α_U^3 defined on U and a_V, r_{1V}, r_{2V}, r_{3V}, α_V, α_V^1, α_V^2, α_V^3 defined on V, where U, V are no disjoint open sets of M. On $U \cap V$ we have $a_U = a_V = \mathrm{d}\eta^i(E, \varphi_i E)$, $r_{iU} = r_{iV} = \mathrm{d}\eta^i(E, \varphi_j E)$, where E is a unitary vector orthogonal to ξ_1, ξ_2, ξ_3. Therefore a, r_1, r_2, r_3 are global differentiable functions on M. Now, from $\mathrm{d}\Psi = 2\alpha\wedge\Psi + a\Omega$ we have $(\alpha_U - \alpha_V)\wedge\Psi = 0$. By Lemma 3.1, $\alpha_U = \alpha_V$. Therefore α is a global one-form. Finally, from

Definition 3.3 we have

$$0 = (\alpha^i_U - \alpha^i_V) \wedge F^j - (\alpha^k_U - \alpha^k_V) \wedge F^k,$$
$$0 = (\alpha^i_U - \alpha^i_V) \wedge \eta^j - (\alpha^k_U - \alpha^k_V)\alpha^k \wedge \eta^k.$$

If E is a unitary vector orthogonal to ξ_1, ξ_2 and ξ_3 we have

$$0 = ((\alpha^i_U - \alpha^i_V) \wedge F^j - (\alpha^k_U - \alpha^k_V) \wedge F^k)(\xi_r, E, \varphi_j E) = -\alpha^i_U(\xi_r) + \alpha^i_V(\xi_r),$$

where $r = 1, 2, 3$. Moreover,

$$0 = ((\alpha^i_U - \alpha^i_V) \wedge \eta^j - (\alpha^k_U - \alpha^k_V)\alpha^k \wedge \eta^k)(E, \xi_j) = \alpha^i_U(E) - \alpha^i_V(E).$$

Hence, $\alpha^i_U = \alpha^i_V$. Then the forms α^1, α^2 and α^3 are global. □

4. Almost hyper-Hermitian structures in principal fibre bundles over manifolds with almost contact 3-structure

From now on, M will be a $(4n+3)$-manifold ($n \geqslant 1$) with an almost contact metric 3-structure $(\varphi_i, \xi_i, \eta^i, \langle\,,\rangle)$, $i = 1, 2, 3$ and $\mathfrak{X}(M)$ will denote the Lie algebra of C^∞ vector fields on M. Let ω be an arbitrary connection form on $\overline{M}$, where $\overline{M} = \overline{M}(M, G, \pi)$ denotes a principal fibre bundle with a one-dimensional connected structure group G and projection π. We use X^H and A^* to denote the horizontal lift of $X \in \mathfrak{X}(M)$ and the fundamental vector field with respect to $A \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. Then we have ([10])

$$(4.1) \qquad \begin{aligned} &[A^*, X^H] = 0, \quad [A^*, B^*] = 0,\\ &\omega([X^H, Y^H]) = -2\hat{\Omega}(X^H, Y^H), \quad \mathrm{h}[X^H, Y^H]_p = [X, Y]^H_p \end{aligned}$$

for $A, B \in \mathfrak{g}$, $X, Y \in \mathfrak{X}(M)$, where $\hat{\Omega}$ denotes the curvature form of ω and h is the horizontal component of a vector in $T_p\overline{M}$.

Making use of the connection form ω and the almost contact 3-structure on M, $(\varphi_i, \xi_i, \eta^i, \langle,\rangle)$, one can define three almost complex structures J_1, J_2, J_3 on $\overline{M}$ in the following way ([21]):

$$(4.2) \qquad (J_i)_p = -\omega_p(X_p)(\xi^H_i)_p + (\varphi_{i\pi(p)}\pi_{*p}X_p)^H + \eta^i_{\pi(p)}(\pi_{*p}X_p)\hat{\xi}^*_p, \quad i = 1, 2, 3,$$

where p is a point of $\overline{M}$, $\hat{\xi} \in \mathfrak{g}$ with $\hat{\xi} \neq 0$ and $\hat{\eta}$ is the dual form of $\hat{\xi}$.

Let $\langle\,\rangle_0$ be the tensor metric field on $\overline{M}$ given by $\langle\,\rangle_0 = \pi^*\langle\,\rangle + \hat{\eta}\omega \otimes \hat{\eta}\omega$. In [15] it is shown that each J_i is almost Hermitian with respect to $\langle\,\rangle_0$. By a straightforward computation one can check that $J_1 J_2 = J_3 = -J_2 J_1$. Then we have:

Proposition 4.1. $(J_1, J_2, J_3, \langle\,\rangle_0)$ *is an almost hyper-Hermitian structure on* $\overline{M}$.

Let us denote by $\overline{F}^i$ and F^i the respective two-forms defined from the i-th almost complex structure on $\overline{M}$ and the i-th almost contact structure on M. Analogously, $\overline{\Omega}$ and Ω represent the respective four-forms on $\overline{M}$ and M. The three-form on M is denoted by Ψ as in Section 2. The relations among all these forms is given in the following lemma.

Lemma 4.2. *We have*
i) $\overline{F}^i = \pi^* F^i + \hat{\eta}\omega \wedge \pi^*\eta^i, \quad i = 1, 2, 3;$
ii) $\overline{\Omega} = \pi^*\Omega + 2\hat{\eta}\omega \wedge \pi^*\Psi.$

P r o o f. Using (4.2) we get

$$\overline{F}^i(X^H, Y^H) = F^i(X, Y) \circ \pi, \qquad \overline{F}^i(X^H, \hat{\xi}^*) = -\eta^i(X) \circ \pi, \qquad \overline{F}^i(A^*, Y^*) = 0$$

for $X, Y \in \mathfrak{X}(M)$ and $A, B \in \mathfrak{g}$. Now it is immediate that $\pi^* F^i + \hat{\eta}\omega \wedge \pi^*\eta^i$ coincides with $\overline{F}^i$. We can deduce ii) using i), (2.1), (3.2) and (3.3). □

From now on $\{E_1, E_2, \ldots, E_{4n+3}\}$ will be an adapted frame of M ordered as in (3.1) and we will write ω, $\hat{\Omega}$ instead of $\hat{\eta}\omega$, $\hat{\eta}\hat{\Omega}$. The curvature form $\hat{\Omega}$ is tensorial of $(\mathrm{Ad}, \mathfrak{g})$ type, where Ad is the adjoint representation. But here, the Lie group G is abelian, hence we have $\hat{\Omega}_{pg}(X^H_{pg}, Y^H_{pg}) = \hat{\Omega}_p(X^H_p, Y^H_p)$ for all $p \in \overline{M}$, $g \in G$ and $X, Y \in \mathfrak{X}(M)$. Thus we can define a two-form on M, denoted also by $\hat{\Omega}$, given by $\hat{\Omega}(X, Y) = \hat{\Omega}(X^H, Y^H)$.

Now we consider $\overline{\nabla}$ and ∇, the respective Levi-Civita connections of $\langle\,\rangle_0$ and $\langle\,\rangle$. From the Koszul formula ([10]) using (4.1) we obtain the following lemma.

Lemma 4.3. *For* $A, B \in \mathfrak{g}$ *and* $X, Y \in \mathfrak{X}(M)$, *we have*
i) $\overline{\nabla}_{A^*} B^* = 0,$
ii) $\overline{\nabla}_{X^H} A^* = \overline{\nabla}_{A^*} X^H = \dfrac{1}{2}\hat{\eta}(A) \sum\limits_{i=1}^{4n+3} \hat{\Omega}(X^H, E^H_i) E^H_i,$
iii) $\overline{\nabla}_{X^H} Y^H = -\dfrac{1}{2}(\hat{\Omega}(X^H, Y^H))^* + (\nabla_X Y)^H.$

The covariant derivative of $\overline{F}^i$ in terms of F^i, η^i and ω is given in the next lemma.

Lemma 4.4. *For* $X, Y, Z \in \mathfrak{X}(M)$ *we have*

i) $\overline{\nabla}_{\hat{\xi}^*}(\overline{F}^i)(\hat{\xi}^*, X^H) = \dfrac{1}{2}\hat{\Omega}(\xi_i^H, X^H)$,

ii) $\overline{\nabla}_{\hat{\xi}^*}(\overline{F}^i)(X^H, Y^H) = -\dfrac{1}{2}\{\hat{\Omega}(X^H, (\varphi_i Y)^H) + \hat{\Omega}((\varphi_i X)^H, Y^H)\}$,

iii) $\overline{\nabla}_{X^H}(\overline{F}^i)(\hat{\xi}^*, Y^H) = \nabla_X(\eta^i)(Y) \circ \pi - \dfrac{1}{2}\hat{\Omega}(X^H, (\varphi_i Y)^H)$,

iv) $\overline{\nabla}_{X^H}(\overline{F}^i)(Y^H, Z^H) = \nabla_X(F^i)(Y, Z) \circ \pi + \dfrac{1}{2}(X \diamond \hat{\Omega} \wedge \eta^i)(Y, Z) \circ \pi$,

where $\diamond$ denotes the interior product.

Proof. We have

$$\overline{\nabla}_{\hat{\xi}^*}(\overline{F}^i)(\hat{\xi}^*, X^H) = \hat{\xi}^* \overline{F}^i(\hat{\xi}^*, X^H) - \overline{F}^i(\overline{\nabla}_{\hat{\xi}^*}\hat{\xi}^*, X^H) - \overline{F}^i(\hat{\xi}^*, \overline{\nabla}_{\hat{\xi}^*} X^H).$$

By Lemma 4.2 i) and Lemma 4.3 i) the first and second summands vanish. Now using again Lemma 4.3 ii) and Lemma 4.2 i) we have

$$\overline{\nabla}_{\hat{\xi}^*}(\overline{F}^i)(\hat{\xi}^*, X^H) = -\frac{1}{2}\sum_{j=1}^{4n+3} \hat{\Omega}(X^H, E_j^H)\eta^i(E_j) \circ \pi.$$

But $\eta^i(E_j) = 0$ if $E_j \neq \xi_i$ and $\eta^i(\xi_i) = 1$, hence we have i). Part ii) is deduced in a similar way using Lemma 4.3 ii) and taking $\hat{\xi}^* \overline{F}^i(X^H, Y^H) = \hat{\xi}^*(F^i(X, Y) \circ \pi) = 0$ into account.

To show iii) we use Lemma 4.2 and Lemma 4.3 ii) and iii) to reach

$$\overline{\nabla}_{X^H}(\overline{F}^i)(\hat{\xi}^*, Y^H) = X\eta^i(Y) \circ \pi - \frac{1}{2}\sum_{j=1}^{4n+3} \hat{\Omega}(X^H, E_j^H)F^i(E_j, Y) \circ \pi - \eta^i(\nabla_X Y) \circ \pi.$$

From this equality iii) is immediate. Part iv) can be proved in a similar way, taking Lemma 4.2 and Lemma 4.3 into account. □

In the following results we relate the almost hyper-Hermitian structure of $\overline{M}$ with the almost contact 3-structure on M. First, from Lemma 4.4 one easily gets the following result.

Theorem 4.5. *Two of the following conditions imply the remaining one:*
(a) *The almost hyper-Hermitian structure on $\overline{M}$ is hyper-Kähler.*
(b) *The almost contact metric 3-structure of M is hypercosymplectic.*
(c) *The curvature form $\hat{\Omega}$ of the connection one-form ω vanishes.*

As a direct consequence of the fact proved in [21, Proposition 3.1, p. 178] the following theorem can be obtained.

Theorem 4.6. *Two of the following conditions imply the remaining one:*

(a) *The almost hyper-Hermitian structure of* $\overline{M}$ *is hypercomplex.*
(b) *The almost contact metric* 3*-structure of* M *is hypernormal.*
(c) *The curvature form* $\hat{\Omega}$ *of the connection one-form* ω *is the pullback by* π *of a two-form on* M *belonging to* $\mathfrak{sp}(n)$*, the Lie algebra of* $\mathrm{Sp}(n) \times I_3$*, i.e.,*

$$\hat{\Omega}(\varphi_i X, \varphi_i Y) = \hat{\Omega}(X, Y)$$

for all $X, Y \in \mathfrak{X}(M)$ *and* $i = 1, 2, 3$*.*

Theorem 4.7. *Two of the following conditions imply the remaining one:*
(a) *The almost hyper-Hermitian structure of* $\overline{M}$ *is locally conformal hyper-Kähler.*
(b) *The almost contact metric* 3*-structure of* M *is a-Sasakian.*
(c) *The curvature form* $\hat{\Omega}$ *of the connection one-form* ω *vanishes.*

P r o o f. First, let us suppose that M has an a-Sasakian 3-structure and the curvature form $\hat{\Omega} = 0$. From Lemma 4.2 we have $\mathrm{d}\overline{F}^i = -a\omega \wedge \overline{F}^i$. Hence we have that $\overline{M}$ is locally conformal hyper-Kähler.

Now, we suppose a) and c), i.e., $\hat{\Omega} = 0$ and $\mathrm{d}\overline{F}^i = \overline{\alpha} \wedge \overline{F}^i$, $i = 1, 2, 3$. From the argument given in [6], it is immediate that $\overline{M}$ is hypercomplex. By Theorem 4.6 it follows that M has a hypernormal 3-structure. On the other hand, for all $p \in \overline{M}$ there is a local section through p, $\sigma\colon U \to \pi^{-1}(U)$ and $T_p\overline{M} = T_p\sigma(U) \oplus T_p\pi^{-1}(x)$, where $x = \pi(p)$. On $\sigma(U)$, the one-form $\overline{\alpha}$ can be expressed as $\overline{\alpha} = \alpha - fw$, where α is $\overline{\alpha}$ restricted to $T_q\sigma(U)$ and $f(q) = -\overline{\alpha}_q(\hat{\xi}^*_q)$, for all $q \in \sigma(U)$. From Lemma 4.2, we have

$$\alpha \wedge \pi^* F^i - \omega \wedge (\alpha \wedge \pi^* \eta^i + f\pi^* F^i) = \pi^* \,\mathrm{d}F^i - \omega \wedge \pi^* \,\mathrm{d}\eta^i.$$

Therefore, $\mathrm{d}F^i = \sigma^*\alpha \wedge F^i$, $\mathrm{d}\eta^i = \sigma^*\alpha \wedge \eta^i + \sigma^* f\, F^i$. In [16, Theorem 2.4, p. 192] it is proved that for a normal almost contact structure satisfying these equations we have $\sigma^*\alpha = \delta\eta^i\eta^i$, $i = 1, 2, 3$. Hence $\sigma^*\alpha = 0$ and $\sigma^* f$ is locally constant. In conclusion, M has an a-Sasakian 3-structure.

Finally, from a) and b), taking Lemma 4.2 into account, we have

$$\alpha \wedge \pi^* F^i - \omega \wedge (\alpha \wedge \pi^* \eta^i + f\pi^* F^i) = \hat{\Omega} \wedge \pi^* \eta^i - a\,\omega \wedge \pi^* F^i,$$

where $\overline{\alpha} = \alpha - f\omega$ on $\sigma(U)$ as before. Then

$$\sigma^*\alpha \wedge \pi^*\eta^i + (a - \sigma^* f)\pi^* F^i = 0, \qquad \sigma^*\alpha \wedge F^i - \hat{\Omega} \wedge \pi^*\eta^i = 0.$$

Hence, $\sigma^*\alpha = 0$, $a = \sigma^* f$ and $\hat{\Omega} \wedge \eta^i = 0$. If X_p and Y_p are vectors orthogonal to ξ_{ip}, then $0 = \hat{\Omega}_p \wedge \eta^i_p(X_p, Y_p, \xi_{ip}) = \hat{\Omega}_p(X_p, Y_p)$. On the other hand, from Theorem 4.6 we have $\hat{\Omega}(\varphi_i X, \varphi_i Y) = \hat{\Omega}(X, Y)$. Then $\hat{\Omega}(\xi_i, X) = 0$. Therefore $\hat{\Omega}(X, Y) = \hat{\Omega}(X - \eta^i(X)\xi_i, \varphi_i Y - \eta^i(Y)\xi_i) = 0$, noting that $X - \eta^i(X)\xi_i$ is orthogonal to ξ_i. Hence $\hat{\Omega}$ vanishes. □

Now the covariant derivative of $\overline{\Omega}$ is expressed in terms of Ω, Ψ and ω.

Lemma 4.8. *For $X, Y, Z, U, V \in \mathfrak{X}(M)$ we have*

$$\overline{\nabla}_{\hat{\xi}^*}(\overline{\Omega})(\hat{\xi}^*, X^H, Y^H, Z^H) = \mathscr{A}(\mathrm{C}_{13}\,\hat{\Omega} \otimes \Psi)(X, Y, Z) \circ \pi, \tag{4.3}$$

$$\overline{\nabla}_{\hat{\xi}^*}(\overline{\Omega})(X^H, Y^H, Z^H, U^H) = \frac{1}{2}\mathscr{A}(\mathrm{C}_{13}\,\hat{\Omega} \otimes \Omega)(X, Y, Z, U) \circ \pi, \tag{4.4}$$

$$\begin{aligned}\overline{\nabla}_{X^H}(\overline{\Omega})(\hat{\xi}^*, Y^H, Z^H, U^H) &= 2\nabla_X(\Psi)(Y, Z, U) \circ \pi \\ &\quad + \frac{1}{2}\,\mathrm{C}_{13}\,\hat{\Omega} \otimes \Omega(X, Y, Z, U) \circ \pi,\end{aligned} \tag{4.5}$$

$$\begin{aligned}\overline{\nabla}_{X^H}(\overline{\Omega})(Y^H, Z^H, U^H, V^H) &= \nabla_X(\Omega)(Y, Z, U, V) \circ \pi \\ &\quad + (X \diamond \hat{\Omega}) \wedge \Psi(Y, Z, U) \circ \pi,\end{aligned} \tag{4.6}$$

where $\mathscr{A}$ denotes the skew-symmetrization of a tensor, C the metric contraction, $\hat{\Omega}$ the curvature form of ω and $\diamond$ the interior product.

Proof. It follows by a straightforward computation, taking Lemma 4.2 and Lemma 4.3 into account. □

For the exterior derivative and the coderivative of $\overline{\Omega}$ we have the following expressions.

Lemma 4.9. *For all $X, Y, Z \in \mathfrak{X}(M)$ we have*

i) $\mathrm{d}\overline{\Omega} = \pi^*\,\mathrm{d}\Omega + 2\hat{\Omega} \wedge \pi^*\Psi - 2\omega \wedge \pi^*\,\mathrm{d}\Psi$;

ii) $\delta\overline{\Omega}(X^H, Y^H, Z^H) = \delta\Omega(X, Y, Z) \circ \pi$;

iii) $\delta\overline{\Omega}(\hat{\xi}^*, X^H, Y^H) = -2\delta\Psi(X, Y) \circ \pi + \frac{1}{2}\,\mathrm{C}_{12} \circ \mathrm{C}_{13}(\hat{\Omega} \otimes \Omega)(X, Y) \circ \pi$, *where* C *denotes the metric contraction.*

Proof. The expression i) is a direct consequence of Lemma 4.2. Let us prove ii): if $\{E_1, E_2, \ldots, E_{4n+3}\}$ is an adapted frame of M, then $\{E_1^H, E_2^H, \ldots, E_{4n+3}^H, \hat{\xi}^*\}$ is an adapted frame of $\overline{M}$, hence we have

$$\delta\overline{\Omega}(X^H, Y^H, Z^H) = -\overline{\nabla}_{\hat{\xi}^*}\overline{\Omega}(\hat{\xi}^*, X^H, Y^H, Z^H) - \sum_{j=1}^{4n+3} \overline{\nabla}_{E_j^H}\overline{\Omega}(E_j^H, X^H, Y^H, Z^H).$$

From (4.3) and (4.6) we get ii). To show iii) we have

$$\delta\overline{\Omega}(\hat{\xi}^*, X^H, Y^H) = -\overline{\nabla}_{\hat{\xi}^*}\overline{\Omega}(\hat{\xi}^*, \hat{\xi}^*, X^H, Y^H) - \sum_{j=1}^{4n+3} \overline{\nabla}_{E_j^H}\overline{\Omega}(E_j^H, \hat{\xi}^*, X^H, Y^H),$$

and using (4.6) we get

$$\delta\overline{\Omega}(\hat{\xi}^*, X^H, Y^H) = 2\sum_{j=1}^{4n+3} \nabla_{E_j}(\Psi)(E_j, X, Y) + \frac{1}{2}\sum_{j,k=1}^{4n+3} \hat{\Omega}(E_j^H, E_k^H)\Omega(E_j, E_k, X, Y)\circ\pi,$$

then iii) follows. □

Next, we give a relation between a quaternionic semi-Kähler structure and a semi-cosymplectic 3-structure.

Theorem 4.10. *Two of the following conditions imply the remaining one:*

(a) *The almost hyper-Hermitian structure on $\overline{M}$ is quaternionic semi-Kähler.*

(b) *The almost contact metric 3-structure on M is semi-cosymplectic.*

(c) *The curvature form $\hat{\Omega}$ of ω vanishes.*

P r o o f. It follows directly from Lemma 4.9 and Lemma 3.1. □

Theorem 4.11. *Two of the following conditions imply the remaining one:*

(a) *The almost hyper-Hermitian structure on $\overline{M}$ is quaternionic Kähler.*

(b) *The almost contact metric 3-structure on M is cosymplectic.*

(c) *The curvature form $\hat{\Omega}$ of ω vanishes.*

P r o o f. It follows from (4.3), (4.4), (4.5), (4.6) and Theorem 4.10. □

To study the quaternionic almost Kähler case, we need before to prove the following statement.

Proposition 4.12. *Let M be a $(4n+3)$-manifold $(n \geqslant 1)$. Then every almost cosymplectic 3-structure on M is semi-cosymplectic.*

P r o o f. We consider the product manifold $M \times \mathbb{R}$ where $\mathbb{R}$ is the set of real numbers. We take the projection map ω on the second factor as the connection form. We consider the almost hyper-Hermitian structure on $M \times \mathbb{R}$ defined, as in (4.2), from ω and the almost contact 3-structure on M. By Lemma 4.9, the almost hyper-Hermitian structure on $M \times \mathbb{R}$ is quaternionic almost-Kähler. By the argument given in [2], $M \times \mathbb{R}$ is quaternionic semi-Kähler. Using now Theorem 4.10 we deduce that M has a semi-cosymplectic 3-structure. □

Theorem 4.13. *Two of the following conditions imply the remaining one:*

(a) *The almost hyper-Hermitian structure on $\overline{M}$ is quaternionic almost Kähler.*

(b) *The almost contact metric 3-structure on M is almost cosymplectic.*

(c) *The curvature form $\hat{\Omega}$ of ω vanishes.*

Proof. It follows from Lemma 4.9, Theorem 4.10 and Proposition 4.12. □

Theorem 4.14. *Two of the following conditions imply the remaining one:*

(a) *The almost hyper-Hermitian structure of $\overline{M}$ is locally conformal quaternionic Kähler.*

(b) *The almost contact metric 3-structure of M is trans-Sasakian.*

(c) *The curvature form $\hat{\Omega}$ of ω vanishes.*

Proof. First, let us suppose that M has a trans-Sasakian 3-structure and the curvature form $\hat{\Omega} = 0$. From Lemma 4.2 we have

$$\mathrm{d}\overline{F}^i = (\pi^*\alpha - \pi^* a\omega) \wedge \overline{F}^i + (\pi^*\alpha^i - \pi^* r_i\omega) \wedge \overline{F}^j - (\pi^*\alpha^k - \pi^* r_k\omega) \wedge \overline{F}^k.$$

Hence we have that $\overline{M}$ is locally conformal quaternionic Kähler.

Now, we suppose a) and c), i.e., $\hat{\Omega} = 0$ and $\mathrm{d}\overline{F}^i = \overline{\alpha} \wedge \overline{F}^i + \overline{\alpha}^i \wedge \overline{F}^j - \overline{\alpha}^k \wedge \overline{F}^k$. For all $p \in \overline{M}$ there is a local section through p, $\sigma\colon U \to \pi^{-1}U$ and $T_p\overline{M} = T_p\sigma(U) \oplus T_p\pi^{-1}(x)$, where $x = \pi(p)$. On $\sigma(U)$, the one-forms $\overline{\alpha}$, $\overline{\alpha}^i$ can be expressed as $\overline{\alpha} = \alpha - fw$, $\overline{\alpha}^i = \alpha^i - r_i w$ where α and α^i are $\overline{\alpha}$ and $\overline{\alpha}^i$ restricted to $T_q\sigma(U)$ and $f(q) = -\overline{\alpha}_q(\hat{\xi}^*_q)$, $r_i(q) = -\overline{\alpha}^i_q(\hat{\xi}^*_q)$ for all $q \in \sigma(U)$. From Lemma 4.2, we have

$$\begin{aligned}\pi^*\,\mathrm{d}F^i - \omega \wedge \mathrm{d}\eta^i = {} & \alpha \wedge \pi^* F^i + \alpha^i \wedge \pi^* F^j - \alpha^k \wedge \pi^* F^k \\ & - f\omega \wedge \pi^* F^i - r_i\omega \wedge \pi^* F^j - r_k\omega \wedge \pi^* F^k \\ & - \omega \wedge \alpha \wedge \pi^*\eta^i - \omega \wedge \alpha^i \wedge \pi^*\eta^j - \omega \wedge \alpha^k \wedge \pi^*\eta^k.\end{aligned}$$

Therefore

$$\begin{aligned}\mathrm{d}F^i &= \sigma^*\alpha \wedge F^i + \sigma^*\alpha^i \wedge F^j - \sigma^*\alpha^k \wedge F^k, \\ \mathrm{d}\eta^i &= \sigma^* f\, F^i + \sigma^* r_i\, F^j - \sigma^* r_k\, F^k + \sigma^*\alpha \wedge \eta^i + \sigma^*\alpha^i \wedge \eta^j - \sigma^*\alpha^k \wedge \eta^k.\end{aligned}$$

Hence M has a trans-Sasakian 3-structure.

Finally, from a) and b), taking Lemma 4.2 into account, we have

$$\alpha\wedge\pi^*\Omega - f\omega\wedge\pi^*\Omega - 2\omega\wedge\alpha\wedge\pi^*\Psi = \pi^*\beta\wedge\pi^*\Omega + \hat{\Omega}\wedge\pi^*\Psi - 2\omega\wedge\pi^*\beta\wedge\pi^*\Psi - a\omega\wedge\pi^*\Omega$$

where $\overline{\alpha} = \alpha - f\omega$ on $\sigma(U)$ as before and β, a are the one-form and the function given in the definition of a trans-Sasakian 3-structure. Then

$$(\beta - \sigma^*\alpha) \wedge \Omega + \hat{\Omega} \wedge \Psi = 0, \qquad (\beta - \sigma^*\alpha) \wedge \Psi + (a - \sigma^* f)\Omega = 0.$$

If $a - \sigma^* f = 0$, then taking Lemma 3.1 into account, we have $\beta - \sigma^*\alpha = 0$. Then $\hat{\Omega} \wedge \Psi = 0$ and using again Lemma 3.1 we have $\hat{\Omega} = 0$.

If $a - \sigma^* f \neq 0$, then

$$0 = -\frac{1}{a - \sigma^* f}(\beta - \sigma^*\alpha) \wedge (\beta - \sigma^*\alpha) \wedge \Psi + \hat{\Omega} \wedge \Psi = \hat{\Omega} \wedge \Psi.$$

Now, taking Lemma 3.1 into account, we have $\hat{\Omega} = 0$. □

Corollary 4.15. *Let M be a connected $(4n+3)$-manifold $(n \geqslant 1)$ with a trans-Sasakian 3-structure, i.e.,*

$$\begin{aligned} \mathrm{d}F^i &= \alpha \wedge F^i + \alpha^i \wedge F^j - \alpha^k \wedge F^k, \\ \mathrm{d}\eta^i &= aF^i + r_i F^j - r^k F^k + \alpha \wedge \eta^i + \alpha^i \wedge \eta^j - \alpha^k \wedge \eta^k \end{aligned}$$

for all (i, j, k) cyclic permutations of $(1, 2, 3)$. Then α is closed and a is constant.

Proof. We consider $M \times \mathbb{R}$ with the projection map ω on the second factor as a connection form. On $M \times \mathbb{R}$ we have a quaternion-Hermitian structure defined as in (4.2). By Theorem 4.14, $M \times \mathbb{R}$ is locally conformal quaternionic Kähler and $\mathrm{d}\overline{\Omega} = 2\overline{\alpha} \wedge \overline{\Omega}$, where $\overline{\alpha} = \pi^*\alpha - a\omega$. Then $\mathrm{d}\alpha = 0$ and $\mathrm{d}a = 0$. □

5. Examples

I. Trivial principal fibre bundles over 3-Sasakian manifolds

In [4] it is shown that any 3-Sasakian homogeneous space is one of the following homogeneous spaces:

$$\frac{\mathrm{Sp}(n)}{\mathrm{Sp}(n-1)} \equiv \mathrm{S}^{4n-1}, \quad \frac{\mathrm{Sp}(n)}{\mathrm{Sp}(n-1) \times \mathbb{Z}_2} \equiv \mathbb{R}P^{4n-1}, \quad \frac{\mathrm{SU}(m)}{\mathrm{S}(\mathrm{U}(m-2) \times \mathrm{U}(1))},$$

$$\frac{\mathrm{SO}(k)}{\mathrm{SO}(k-4) \times \mathrm{Sp}(1)}, \quad \frac{\mathrm{G}_2}{\mathrm{Sp}(1)}, \quad \frac{\mathrm{F}_4}{\mathrm{Sp}(3)}, \quad \frac{\mathrm{E}_6}{\mathrm{SU}(6)}, \quad \frac{\mathrm{E}_7}{\mathrm{Spin}(12)}, \quad \frac{\mathrm{E}_8}{\mathrm{E}(7)},$$

where $n \geqslant 1$, $\mathrm{Sp}(0)$ is the identity group, $m \geqslant 3$ and $k \geqslant 7$.

By Theorem 4.7, we have the following locally conformal hyper-Kähler manifolds:

$$\mathrm{S}^{4n-1} \times \mathrm{S}^1, \quad \mathbb{R}P^{4n-1} \times \mathrm{S}^1, \quad \frac{\mathrm{SU}(m)}{\mathrm{S}(\mathrm{U}(m-2) \times \mathrm{U}(1))} \times \mathrm{S}^1, \quad \frac{\mathrm{SO}(k)}{\mathrm{SO}(k-4) \times \mathrm{Sp}(1)} \times \mathrm{S}^1,$$

$$\frac{\mathrm{G}_2}{\mathrm{Sp}(1)} \times \mathrm{S}^1, \quad \frac{\mathrm{F}_4}{\mathrm{Sp}(3)} \times \mathrm{S}^1, \quad \frac{\mathrm{E}_6}{\mathrm{SU}(6)} \times \mathrm{S}^1, \quad \frac{\mathrm{E}_7}{\mathrm{Spin}(12)} \times \mathrm{S}^1, \quad \frac{\mathrm{E}_8}{\mathrm{E}(7)} \times \mathrm{S}^1,$$

where $n > 1$, $m > 3$ and $k \geqslant 7$. This is an alternative way of obtaining these examples given in [17].

II. Nontrivial principal fibre bundles over a $(4n+3)$-dimensional torus

Let us recall the following well known theorem about classification of principal circle bundles.

Theorem 5.1. ([9, p. 35]) *There is a one-to-one correspondence between equivalence classes of principal circle bundles over a manifold M and the cohomology group $H^2(M,\mathbb{Z})$. Furthermore, given an integral closed two-form $\hat{\Omega}$ on M, there is a principal circle bundle $\pi\colon \overline{M} \to M$ with a connection form ω such that $\hat{\Omega}$ is the curvature of ω ($\pi^*(\hat{\Omega}) = \mathrm{d}\omega$).*

We consider a $(4n+3)$-dimensional torus $\mathbb{T}^{4n+3}$ ($n \geqslant 1$). Let $\{\alpha_1, \alpha_2, \ldots, \alpha_{4n+3}\}$ a basis for one-forms such that each α_i is integral and closed. On $\mathbb{T}^{4n+3}$ we consider the metric tensor field given by $\langle\,\rangle = \sum\limits_{l=1}^{4n+3} \alpha_l \otimes \alpha_l$ and the almost contact metric 3-structure consisting of

– the $(1,1)$ tensor fields

$$\begin{aligned}(5.1)\qquad \varphi_i = \sum_{l=1}^{n}\{&E_{in+l}\otimes\alpha_l - E_l\otimes\alpha_{i+l} + E_{kn+l}\otimes\alpha_{jn+l}\\ &- E_{jn+l}\otimes\alpha_{kn+l} + E_{4n+k}\otimes\alpha_{4n+j} - E_{4n+j}\otimes\alpha_{4n+k}\},\end{aligned}$$

where $\{E_1, \ldots, E_{4n+3}\}$ is the orthonormal frame dual of $\{\alpha_1, \ldots, \alpha_{4n+3}\}$ and (i,j,k) is a cyclic permutattion of $(1,2,3)$;

– the one-forms $\eta^1 = \alpha_{4n+1}$, $\eta^2 = \alpha_{4n+2}$ and $\eta^3 = \alpha_{4n+3}$;

– the vector fields $\xi_1 = E_{4n+1}$, $\xi_2 = E_{4n+2}$ and $\xi_3 = E_{4n+3}$.

Since each α_i is closed, it can be checked that $(\varphi_i, \eta^i, \xi_i, \langle\,\rangle)$ is a hypercosymplectic 3-structure. Hence we can also claim that $\mathbb{T}^{4n+3}$ has a hypernormal 3-structure.

By Theorem 5.1 we have a nontrivial principal circle bundle $\pi\colon \overline{M} \to \mathbb{T}^{4n+3}$ corresponding to $[\hat{\Omega}] \in H^2(\mathbb{T}^{4n+3},\mathbb{Z})$, where

$$(5.2)\qquad \hat{\Omega} = \mathfrak{S}_{ijk}\sum_{l=1}^{n}\{\alpha_l\wedge\alpha_{in+l} - \alpha_{jn+l}\wedge\alpha_{kn+l}\}$$

and $\mathfrak{S}$ denotes the cyclic sum. There is a connection one-form ω on $\overline{M}$ with curvature $\mathrm{d}\omega = \pi^*(\hat{\Omega})$. We will also denote $\pi^*(\hat{\Omega})$ by $\hat{\Omega}$.

We consider on $\overline{M}$ the almost hyper-Hermitian structure $(J_1, J_2, J_3, \langle\,\rangle_0)$ defined as in (4.2) from the connection form ω and the almost contact 3-structure of $\mathbb{T}^{4n+3}$.

Theorem 5.2. *On the $(4n+4)$-dimensional manifold $\overline{M}$ $(n \geqslant 1)$ there is a hypercomplex structure which is not quaternionic semi-Kähler.*

Proof. Since we have a hypernormal 3-structure on $\mathbb{T}^{4n+3}$, we only need to check condition c) of Theorem 4.6, i.e.,

$$\hat{\Omega}(\varphi_i X, \varphi_i Y) = \hat{\Omega}(X, Y) \tag{5.3}$$

for $X, Y \in \mathfrak{X}(\mathbb{T}^{4n+3})$ and $i = 1, 2, 3$. Note that conditions (5.3) are bilinear, so we only have to check those conditions for any pair (E_r, E_s) of the adapted frame $\{E_1, \ldots, E_{4n+3}\}$.

From the expression (5.2) of $\hat{\Omega}$, taking (5.1) into account, we have

$$0 = \hat{\Omega}(E_{4n+j}, E_r) = \hat{\Omega}(\varphi_i E_{4n+j}, \varphi_i E_r), \tag{5.4}$$

where $i, j = 1, 2, 3$ and $r = 1, 2, \ldots, 4n+3$.

From now on $r, s = 1, 2, \ldots, n$, $r \neq s$ and (i, j, k) is a cyclic permutation of $(1, 2, 3)$. From (5.2), taking (5.1) into account, we get

$$\hat{\Omega}(E_r, E_s) = \hat{\Omega}(\varphi_i E_r, \varphi_i E_s) = \hat{\Omega}(E_{in+r}, E_{in+s}) = 0. \tag{5.5}$$

Similarly we have

$$\begin{aligned} \hat{\Omega}(E_r, E_{jn+s}) &= \hat{\Omega}(\varphi_i E_r, \varphi_i E_{jn+s}) = \hat{\Omega}(E_{jn+r}, E_{kn+s}) \\ &= \hat{\Omega}(\varphi_i E_{jn+r}, \varphi_i E_{kn+s}) = 0. \end{aligned} \tag{5.6}$$

Now using again expression (5.2) of $\hat{\Omega}$ and taking (5.1) into account, we have

$$\hat{\Omega}(E_r, E_{in+r}) = \hat{\Omega}(\varphi_i E_r, \varphi_i E_{in+r}) = 1. \tag{5.7}$$

In a similar way we have

$$\hat{\Omega}(E_r, E_{jn+r}) = \hat{\Omega}(\varphi_i E_r, \varphi_i E_{jn+r}) = -1, \tag{5.8}$$

$$\hat{\Omega}(E_r, E_{kn+r}) = \hat{\Omega}(\varphi_i E_r, \varphi_i E_{knr}) = 1, \tag{5.9}$$

$$\hat{\Omega}(E^H_{jn+r}, E_{kn+r}) = \hat{\Omega}(\varphi_i E_{jn+r}, \varphi_i E_{kn+r}) = -1. \tag{5.10}$$

From (5.4), (5.5), (5.6), (5.7), (5.8), (5.9) and (5.10) we can claim that conditions (5.3) are satisfied. Then by Theorem 4.6 the almost hyper-Hermitian structure on $\overline{M}$ is hypercomplex. If $\overline{M}$ were an quaternionic semi-Kähler manifold then by Theorem 4.10, $\hat{\Omega}$ would vanish, which is a contradiction. □

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