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EQUIMORPHY IN VARIETIES OF DISTRIBUTIVE DOUBLE p-ALGEBRAS

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Dedicated to Professor H. A. Priestley

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Abstract. Any finitely generated regular variety $\mathbf{V}$ of distributive double p-algebras is *finitely determined*, meaning that for some finite cardinal $n(\mathbf{V})$, any subclass $S \subseteq \mathbf{V}$ of algebras with isomorphic endomorphism monoids has fewer than $n(\mathbf{V})$ pairwise non-isomorphic members. This result follows from our structural characterization of those finitely generated *almost regular* varieties which are finitely determined. We conjecture that any finitely generated, finitely determined variety of distributive double p-algebras must be almost regular.

Keywords: distributive double p-algebra, variety, endomorphism monoid, equimorphy, categorical universality

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An algebra $A = (L, \vee, \wedge, ^*, ^+, 0, 1)$ of the type $(2, 2, 1, 1, 0, 0)$ is a *distributive double p-algebra* if $(L, \vee, \wedge, 0, 1)$ is a distributive $(0, 1)$-lattice, and * and $^+$ are, respectively, the unary operations of pseudocomplementation and dual pseudocomplementation: the operation * is determined by the requirement that $x \leqslant a^*$ be equivalent to $x \wedge a = 0$, while $y \geqslant a^+$ is to be equivalent to $y \vee a = 1$.

A distributive double p-algebra A is said to be *regular* if $x \vee x^* \geqslant y \wedge y^+$ for all $x, y \in A$. Regular algebras form a variety $\mathbf{R}$.

As shown in [8], the category of all distributive double p-algebras and all their homomorphisms is *universal*, that is, it contains a copy of the category of all graphs,

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and hence also a copy of any category of algebras as a full subcategory, see [18]. The universality implies that for every monoid M there is a proper class $\mathscr{D}_M$ of non-isomorphic distributive double p-algebras A whose endomorphism monoid $\mathrm{End}(A)$ is isomorphic to M. Members of $\mathscr{D}_M$ can also be chosen to be regular, and this is again due to the universality of $\mathbf{R}$, demonstrated in [9].

We say that two algebras are *equimorphic* if their endomorphism monoids are isomorphic. A class $\mathscr{C}$ of algebras is said to be α-*determined* if α is the least cardinal for which any class $\mathscr{E} \subseteq \mathscr{C}$ of pairwise equimorphic algebras with $|\mathscr{E}| = \alpha$ has at least two isomorphic members. Therefore a universal class $\mathscr{C}$ of algebras cannot be α-determined for any cardinal α.

No finitely generated subvariety of $\mathbf{R}$ is universal, see [10], or even rich enough to represent every group as the automorphism group of one of its members [8]. The least nontrivial subvariety of $\mathbf{R}$, the variety $\mathbf{B}$ of Boolean algebras, is 2-determined, see [12], [13] or [19].

We recall that any variety of distributive p-algebras which is α-determined for some α must be either 2-determined or 3-determined, see [1], where a further discussion of other related α-determined classes can also be found.

These results indicate that varieties of distributive double p-algebras may exhibit widely different categorical properties. In the present paper we show, for instance, that every finitely generated subvariety $\mathbf{V} \subseteq \mathbf{R}$ is n-determined for some finite cardinal $n = n(\mathbf{V})$, and that no finite common upper bound of these numbers exists.

To present the general result in its proper context, we need several additional concepts.

The *rudiment* $\mathrm{Rud}(A)$ of a distributive double p-algebra A is the least sublattice of A closed under the formation of relative complements and containing all pseudocomplements and dual pseudocomplements of A, see [10]. We say that an algebra A is *rudimentary* if $\mathrm{Rud}(A) = A$. When directly indecomposable, a rudimentary algebra A is called a *nucleus*. From [10] we recall that every nucleus from any finitely generated variety $\mathbf{V}$ of distributive double p-algebras is finite.

For any distributive double p-algebra A, let $P(A)$ denote the poset of all prime filters of A ordered by the reversed inclusion. Thus, for any finite A, we may identify the poset $P(A)$ with the poset of all join irreducible elements in A. Let $\mathrm{Ext}(P(A)) \subseteq P(A)$ denote the set of all members of $P(A)$ that are minimal or maximal, and let $\mathrm{Mid}(P(A)) = P(A) \setminus \mathrm{Ext}(P(A))$.

Following is one of several characterizations of finitely generated universal varieties of distributive double p-algebras presented in [10].

Theorem [10]. *Let* $\mathbf{V}$ *be a finitely generated variety of distributive double* p-*algebras. Then* $\mathbf{V}$ *is universal if and only if there is a nucleus* $C \in \mathbf{V}$ *such that*

Mid($P(C)$) *contains a three-element order component M such that the identity is the only endomorphism of C extending the identity map of M.*

This characterization suggests that systematic investigation of non-universal finitely generated varieties should center on the properties of their nuclei. The present paper initiates such investigation by examining finitely generated varieties $\mathbf{V}$ for which $\mathrm{Mid}(P(C))$ of any nucleus $C \in \mathbf{V}$ is an antichain. We call such varieties *almost regular*.

Following Beazer [3], for any distributive double p-algebra A, we let Φ_A denote its *determination congruence*, that is, the congruence consisting of all $(a, b) \in A \times A$ with $a^* = b^*$ and $a^+ = b^+$. If A belongs to a finitely generated variety, then Φ_A is the least congruence on A for which A/Φ_A is regular. If A is also directly indecomposable, then A/Φ_A is a finite simple algebra.

Let $B \in \mathbf{V}$ for some finitely generated variety $\mathbf{V}$. For any $p \in P(B)$, let $\mathrm{Ext}(p)$ denote the set of all members of $\mathrm{Ext}(P(B))$ comparable to p. We say that an element $d \in \mathrm{Mid}(P(B))$ is *defective* if $\mathrm{Ext}(d) = \mathrm{Ext}(e)$ for some $e \in \mathrm{Ext}(P(B))$, and let $\mathrm{Def}(P(B))$ denote the set of all defective members of $P(B)$. We recall that $\mathrm{Def}(P(B)) = \emptyset$ for any B which is rudimentary, see [10].

Davey's description [4] of Priestley spaces of subdirectly irreducible algebras shows that a finite algebra B is simple if and only if $P(B)$ is connected and $P(B) = \mathrm{Ext}(P(B))$, while B is subdirectly irreducible but not simple exactly when $P(B)$ is connected and $\mathrm{Mid}(P(B)) = \{b\}$ is a singleton. In the latter case, $P(B/\Phi_B)$ is always isomorphic to $\mathrm{Ext}(P(B))$ and two possibilities arise: either b is non-defective, B is a nucleus, and there is no homomorphism $B/\Phi_B \to B$, or else b is defective and the algebra B/Φ_B is a proper retract of B. Consequently, the rudiment $\mathrm{Rud}(A)$ of an algebra A from a finitely generated variety $\mathbf{V}$ provides no information whatsoever about that fragment of a subdirect decomposition of A which is determined by subdirectly irreducible quotients of A possessing proper retracts. Thus, according to the result of [10] noted earlier, the presence of any combination of subdirectly irreducibles with proper retracts does not affect the universality of a finitely generated variety $\mathbf{V}$ at all. It will be seen that, unlike for universal varieties, α-determinedness of an almost regular variety $\mathbf{V}$ strongly depends on how the two types of subdirectly irreducibles combine in $\mathbf{V}$.

To state our main result, we let $\mathbb{DC}$ denote the class of all those posets $P(A)$ of prime filters of distributive double p-algebras A for which the subposet $\mathrm{Def}(P(A))$ is convex.

Main Theorem. *The following properties of a finitely generated almost regular variety $\mathbf{V}$ of distributive double p-algebras are equivalent:*

(1) $\mathbf{V}$ *is* α*-determined for some cardinal* α*;*
(2) $\mathbf{V}$ *is* n*-determined for some finite cardinal* $n = n(\mathbf{V})$*;*
(3) $\{P(A) \mid A \in \mathbf{V}\} \subseteq \mathbb{DC}$.

Thus, for instance, every finitely generated variety of regular algebras, the group universal variety $\mathbf{S}$ of double Stone algebras, and countably many other almost regular varieties are n-determined for some finite n.

The implication (2) $\Rightarrow$ (1) in the Main Theorem is trivial, while (1) $\Rightarrow$ (3) is proved in the last section, where it is also shown that there is no common finite upper bound of cardinalities $n(\mathbf{V})$ for finitely generated varieties $\mathbf{V} \subseteq \mathbf{R}$.

The remainder of the paper is devoted to showing that (3) $\Rightarrow$ (2). The proof uses Priestley's duality for distributive double p-algebras. Following a section on preliminaries, we begin to build up a supply of 'recognizable' idempotent endomorphisms in Sections 1 to 3, and their collections in Sections 4 and 5. In Section 6, on any equimorphic class $\mathscr{S} \subseteq \mathbb{DC}$ we define nine progressively finer equivalences. Then we show that each of these equivalences decomposes $\mathscr{S}$ into finitely many subclasses, and that any two members of any class of the ninth equivalence are isomorphic.

We hope that the reader will agree that Priestley's duality is a powerful yet delicate tool, and one that is uniquely suited to structural investigations such as those presented here.

Preliminaries

We begin with a brief review of the essentials of Priestley's duality.

Let $(X, \tau, \leqslant)$ be an *ordered topological space*, that is, let (X, τ) be a topological space and $(X, \leqslant)$ a partially ordered set. For any $Z \subseteq X$ write

$$(Z] = \{x \in X \mid \exists z \in Z \quad x \leqslant z\} \quad \text{and} \quad [Z) = \{x \in X \mid \exists z \in Z \quad z \leqslant x\}.$$

A subset Z of X is *decreasing* if $(Z] = Z$, *increasing* if $[Z) = Z$, and *clopen* if it is both τ-open and τ-closed. Any compact ordered topological space $(X, \tau, \leqslant)$ possessing a clopen decreasing set D such that $x \in D$ and $y \notin D$ for any $x, y \in X$ with $x \not\geqslant y$ is called a *Priestley space*.

Following is a well known property of Priestley spaces.

Lemma P.0. *If* F_0 *is a closed subset of a Priestley space* $(X, \tau, \leqslant)$*, then* $[F_0)$ *and* $(F_0]$ *are closed. If* $F_1 \subseteq X$ *is also closed and* $F_0 \cap (F_1] = \emptyset$*, then there is a clopen decreasing set* $D \subseteq X$ *such that* $F_1 \subseteq D$ *and* $F_0 \cap D = \emptyset$*.*

Let $\mathbf{P}$ denote the category of all Priestley spaces and all their continuous order preserving mappings. Clopen decreasing sets of any Priestley space form a distributive (0,1)-lattice, and the inverse image map f^{-1} of any $\mathbf{P}$-morphism f is a (0,1)-homomorphism of these lattices. This gives rise to a contravariant functor $D\colon \mathbf{P} \longrightarrow \mathbf{D}$ into the category $\mathbf{D}$ of all distributive (0,1)-lattices and all their (0,1)-homomorphisms. Conversely, for any lattice $L \in \mathbf{D}$, let $P(L) = (P(L), \tau, \leqslant)$ be the ordered topological space on the set $P(L)$ of all prime filters of L ordered by the reversed inclusion, and such that the sets $\{x \in P(L) \mid A \in x\}$ and $\{x \in P(L) \mid A \notin x\}$ with $A \in L$ form an open subbasis of τ. If $h\colon L \longrightarrow L'$ is a morphism in $\mathbf{D}$ then h^{-1} maps $P(L')$ into $P(L)$ and, according to [15], this determines a contravariant functor $P\colon \mathbf{D} \longrightarrow \mathbf{P}$.

Theorem P.1. (Priestley [15], [16]). *The two composite functors $P \circ D\colon \mathbf{P} \longrightarrow \mathbf{P}$ and $D \circ P\colon \mathbf{D} \longrightarrow \mathbf{D}$ are naturally equivalent to the identity functors of their respective domains. Therefore $\mathbf{D}$ is a category dually isomorphic to $\mathbf{P}$.*

The two simple claims below will also be useful.

Lemma P.2. *Let (X, τ) be a compact 0-dimensional space. Then any collection $\mathscr{U}$ of clopen sets separating points of X is a subbase of τ.*

Proof. Let σ be the coarsest topology on X for which every $U \in \mathscr{U}$ is σ-clopen. Then (X, σ) is a Hausdorff space, and the identity map $(X, \tau) \to (X, \sigma)$ is continuous. Since (X, τ) is compact, both (X, τ) and (X, σ) are compact Hausdorff spaces, and hence $\sigma = \tau$. $\square$

Lemma P.3. *If (X, τ) and (Y, σ) are topological spaces and $f\colon X \longrightarrow Y$ is a mapping such that $f^{-1}(U)$ is open for any $U \in \mathscr{U}$ for some subbase $\mathscr{U}$ of σ, then f is continuous.*

Let $\mathrm{Min}(X)$ and $\mathrm{Max}(X)$ respectively denote the sets of all minimal and maximal elements of a Priestley space $(X, \tau, \leqslant)$, and let $\mathrm{Mid}(X) = X \setminus (\mathrm{Min}(X) \cup \mathrm{Max}(X))$. For any $Y \subseteq X$, denote $\mathrm{Min}(Y) = (Y] \cap \mathrm{Min}(X)$, $\mathrm{Max}(Y) = [Y) \cap \mathrm{Max}(X)$ and $\mathrm{Ext}(Y) = \mathrm{Min}(Y) \cup \mathrm{Max}(Y)$. When $Y = \{y\}$, we write $\mathrm{Min}(y)$ instead of $\mathrm{Min}(\{y\})$, and similarly for Max and Ext. If $(X, \tau, \leqslant)$ is a Priestley space and $Y \subseteq X$ is nonvoid, then the sets $\mathrm{Min}(Y)$ and $\mathrm{Max}(Y)$, and hence also their union $\mathrm{Ext}(Y)$ are nonvoid. In particular, $\mathrm{Min}(x)$, $\mathrm{Max}(x)$ and $\mathrm{Ext}(x)$ are nonvoid for every $x \in X$.

Theorem P.4. (Priestley [17]). *Let $P\colon \mathbf{D} \longrightarrow \mathbf{P}$ be the functor assigning Priestley spaces to distributive (0,1)-lattices, and let $h\colon L \longrightarrow L'$ be a morphism in* $\mathbf{D}$. *Then*

(1) *L is a distributive double p-algebra if and only if $(Y]$ is clopen for every clopen increasing subset Y of $P(L)$ and $[W)$ is clopen for any clopen decreasing set $W \subseteq P(L)$;*
(2) *h is a double p-algebra homomorphism iff $P(h)(\mathrm{Min}(x)) = \mathrm{Min}(P(h)(x))$ and $P(h)(\mathrm{Max}(x)) = \mathrm{Max}(P(h)(x))$ for every $x \in P(L')$;*
(3) *for any distributive double p-algebra L, the sets $\mathrm{Min}(P(L))$ and $\mathrm{Max}(P(L))$ are closed;*
(4) *h is injective if and only if $P(h)\colon\ P(L') \longrightarrow P(L)$ is surjective;*
(5) *h is surjective if and only if $P(h)$ is a homeomorphism and order isomorphism of $P(L')$ onto a closed order subspace $Z \subseteq P(L)$ satisfying $\mathrm{Ext}(Z) \subseteq Z$.*

Definition and notation. The Priestley space $P(A)$ of a distributive double p-algebra A will be called a *dp-space*, the dual of a double p-algebra homomorphism a *dp-map*, and the property from (2) above the *dp-property*.

For any variety $\mathbf{V}$ of distributive double p-algebras, let $P(\mathbf{V})$ denote the category of all dp-spaces of algebras from $\mathbf{V}$ and all dp-maps between them.

For any dp-space X, let $\mathrm{End}(X)$ denote the monoid consisting of all dp-maps $f\colon\ X \to X$ and, for any $f \in \mathrm{End}(X)$, let $\mathrm{Im}(f)$ denote its image. Then $\mathrm{Im}(f) \subseteq X$ is a closed order subspace of X and $\mathrm{Ext}(\mathrm{Im}(f)) \subseteq \mathrm{Im}(f)$ for every $f \in \mathrm{End}(X)$.

We shall also need the following consequence of Lemmas 1.3 and 1.4 of [7].

Lemma P.5. *If X is a dp-space and $f, g \in \mathrm{End}(X)$ are idempotent, then*

(1) *the map $\xi\colon\ \mathrm{End}(\mathrm{Im}(f)) \to f\,\mathrm{End}(X)f$ defined by $\xi(k) = kf$ is an isomorphism of $\mathrm{End}(\mathrm{Im}(f))$ onto $f\,\mathrm{End}(X)f$ with the inverse $\xi^{-1}(h) = fh \restriction \mathrm{Im}(f)$,*
(2) *$\mathrm{Im}(f) \cong \mathrm{Im}(g)$ if and only if there exist $h, k \in \mathrm{End}(X)$ such that $hk = f$, $kh = g$, $hg = fh = h$, and $kf = gk = k$.*

We conclude with a simple but useful claim about partially ordered sets.

Lemma P.6. *For $i = 0, 1$, let $(X_i, \leqslant)$ be posets, and let M_i be monoids of order preserving maps of X_i for which there exists an isomorphism $\psi\colon\ M_0 \longrightarrow M_1$. Let $U \subseteq X_0$ and let $\varphi\colon\ U \longrightarrow X_1$ be a one-to-one mapping such that*

> *elements $u, v \in U$ are comparable in X_0 exactly when $\varphi(u), \varphi(v) \in \varphi(U)$ are comparable in X_1;*

there exists a comparable pair $\{x, y\} \subseteq U$ *such that for every comparable pair* $\{u, v\} \subseteq U$ *there exists an* $f \in M_0$ *satisfying*

$$\{f(x), f(y)\} = \{u, v\} \text{ and } \{\psi(f)(\varphi(x)), \psi(f)(\varphi(y))\} = \{\varphi(u), \varphi(v)\}.$$

Then the bijection φ *of* U *onto* $\varphi(U) \subseteq X_1$ *is either an order isomorphism or an order anti-isomorphism.*

Proof. We may assume that $x \leqslant y$. Then either $\varphi(x) \leqslant \varphi(y)$ or $\varphi(x) \geqslant \varphi(y)$. For any $u \leqslant v$ in U, there is an $f \in M_0$ such that $f(x) = u$, $f(y) = v$ and $\{\varphi(u), \varphi(v)\} = \{\psi(f)(\varphi(x)), \psi(f)(\varphi(y))\}$ is a comparable pair. Since $\psi(f)$ preserves order, we have $\varphi(u) \leqslant \varphi(v)$ when $\varphi(x) \leqslant \varphi(y)$, and $\varphi(u) \geqslant \varphi(v)$ when $\varphi(x) \geqslant \varphi(y)$, so that φ either preserves or reverses the order. But ψ is an isomorphism, so that the bijection $\varphi^{-1}\colon \varphi(U) \to U$ preserves or reverses the order as well. $\square$

1. Basic idempotent *dp*-endomorphisms

Definitions. Let $\mathbb{FG}$ denote the class of all *dp*-spaces X for which the algebra $D(X)$ belongs to some finitely generated variety $\mathbf{V}$.

For any $X \in \mathbb{FG}$, let $\mathrm{Rud}(X)$ denote the *dp*-space of the rudiment of the distributive double *p*-algebra $D(X)$. We say that $X \in \mathbb{FG}$ is *rudimentary* if $\mathrm{Rud}(X) \cong X$. A rudimentary *dp*-space X is called a *nucleus* if the algebra $D(X)$ is directly indecomposable. We recall that, for $X \in \mathbb{FG}$, the algebra $D(X)$ is directly indecomposable exactly when X is order connected.

Any maximal order connected subset $C \subseteq X \in \mathbb{FG}$ is closed in X, see [4] or [11]. Any such C will be called a *component* of X, and the set of all components of X will be denoted by $\mathbb{C}(X)$.

For any $Y \subseteq X$, write $K(Y) = \bigcup\{C \in \mathbb{C}(X) \mid C \cap Y \neq \emptyset\}$. Clearly, $K(Y) \subseteq X$ is the union of all components intersected by Y.

Finally, let $\mathbb{AR}$ be the subclass of $\mathbb{FG}$ formed by all dp-spaces X for which $\mathrm{Mid}(\mathrm{Rud}(X))$ is an antichain. Any such space will be called *almost regular.*

Lemma 1.1 [10]. *Let* $k_X\colon X \longrightarrow \mathrm{Rud}(X)$ *denote the Priestley dual of the inclusion map of the algebraic rudiment* $D(\mathrm{Rud}(X))$ *of the algebra* $D(X)$ *into* $D(X)$ *itself. Then* $k_X(x) = k_X(x')$ *if and only if* $\mathrm{Ext}(x) = \mathrm{Ext}(x')$.

Furthermore, for every dp-map $f\colon X \longrightarrow X'$ *there exists a uniquely determined dp-map* $\mathrm{Rud}(f)\colon \mathrm{Rud}(X) \longrightarrow \mathrm{Rud}(X')$ *such that* $\mathrm{Rud}(f)k_X = k_{X'}f$.

For any rudimentary $R \in \mathbb{FG}$, *the set* $\mathbb{C}(R)$ *consists of finite nuclei and has only finitely many isomorphism classes.*

From Lemma 1.1 it follows that $X \in \mathbb{FG}$ if and only if all components of $\mathrm{Rud}(X)$ are finite and only finitely many of them are non-isomorphic.

Definitions. For any $y \in X \in \mathbb{FG}$, set

$$E(y) = \{x \in \mathrm{Mid}(X) \mid \mathrm{Ext}(x) = \mathrm{Ext}(y)\}.$$

Let $X \in \mathbb{FG}$. Any $x \in \mathrm{Mid}(X)$ with $k_X(x) \in \mathrm{Ext}(\mathrm{Rud}(X))$ will be called *defective*. According to Lemma 1.1, this means that $x \in E(z)$ for some $z \in \mathrm{Ext}(X)$.

Let $C \in \mathbb{C}(X)$. If $|C| > 1$ and $E(u) \cap E(z) \neq \emptyset$ for some $z \in \mathrm{Min}(C)$ and some $u \in \mathrm{Max}(C)$, then $\mathrm{Ext}(C) = \{z, u\}$ and $E(x) = \mathrm{Mid}(C)$ for all $x \in C$. In this case, any element $x \in \mathrm{Mid}(C)$ is called *doubly defective*. If $|\mathrm{Ext}(C)| > 2$ and $x \in E(z)$ for some $z \in \mathrm{Min}(C)$, then $x \notin E(u)$ for every $u \in \mathrm{Max}(C)$, and we say that x is *min-defective*. A *max-defective* element is defined dually. Any $x \in C$ such that $x \notin E(z)$ for every $z \in \mathrm{Ext}(C)$ is *non-defective*.

For any $X \in \mathbb{FG}$, let $\mathrm{Def}(X) \subseteq \mathrm{Mid}(X)$ denote the set of all defective elements of X.

Finally, let $\mathbb{DC}$ consist of all spaces $X \in \mathbb{FG}$ for which the set $\mathrm{Def}(X)$ is convex.

Lemma 1.2 [10]. *Let $X \in \mathbb{FG}$. If $Y \subseteq X$ is closed then $K(Y)$ is closed, if Y is clopen decreasing or clopen increasing then $K(Y)$ is clopen. If $z \in X$ is non-defective, then $E(z)$ is closed.*

The claim below is of central importance, and may be of independent interest. In algebraic terms, it says that any directly indecomposable image of a rudimentary distributive double p-algebra R from a finitely generated variety $\mathbf{V}$ is a retract of a direct factor of R.

Lemma 1.3. *If $X \in \mathbb{FG}$ is rudimentary and $C \in \mathbb{C}(X)$, then there exist a clopen set $D = K(D)$ containing C and an idempotent $g \in \mathrm{End}(D)$ with $\mathrm{Im}(g) = C$.*

P r o o f. If the component C is a singleton $\{c\}$, then the constant mapping g with $g(X) = \{c\}$ fulfils all requirements.

If the component C has more than one element, we proceed analogously to the proof of Lemma 4.1 in [10], as follows.

Since $\mathrm{Min}(C) \cap \mathrm{Max}(X) = \emptyset$, $\mathrm{Max}(C) \cap \mathrm{Min}(X) = \emptyset$, and C is finite by Lemma 1.1, for every $z \in \mathrm{Min}(C)$ there is a clopen decreasing set dA_z with $dA_z \cap C = \{z\}$ and $dA_z \cap \mathrm{Max}(X) = \emptyset$. Furthermore, for every $u \in \mathrm{Max}(C)$ there is a clopen increasing set iA_u with $iA_u \cap C = \{u\}$, $iA_u \cap \mathrm{Min}(X) = \emptyset$, and such that $iA_u \cap dA_z = \emptyset$ for every $z \in \mathrm{Min}(C)$.

For any $z \in \mathrm{Min}(C)$ and $u \in \mathrm{Max}(C)$, set

$dX_z = dA_z \setminus [\bigcup\{dA_v \mid v \in \mathrm{Min}(C) \setminus \{z\}\}),$

$iX_u = iA_u \setminus (\bigcup\{iA_t \mid t \in \mathrm{Max}(C) \setminus \{u\}\}].$

Since $(X, \leqslant, \tau)$ is a dp-space and C is finite, dX_z is clopen decreasing and $z \in dX_z \subseteq dA_z$ for every $z \in \mathrm{Min}(C)$, while iX_u is clopen increasing and $u \in iX_u \subseteq iA_u$ for every $u \in \mathrm{Max}(C)$. Hence $\{dX_z \mid z \in \mathrm{Min}(C)\} \cup \{iX_u \mid u \in \mathrm{Max}(C)\}$ is a family of pairwise disjoint sets.

Next we set, for every $z \in \mathrm{Min}(C)$ and every $u \in \mathrm{Max}(C)$,

$dZ_z = dX_z \setminus [\bigcup\{(iX_t] \mid t \in \mathrm{Max}(C) \setminus \mathrm{Max}(z)\}),$ and

$iZ_u = iX_u \setminus (\bigcup\{[dX_v) \mid v \in \mathrm{Min}(C) \setminus \mathrm{Min}(u)\}].$

Again, the finiteness of C and the fact that $(X, \leqslant, \tau)$ is a dp-space imply that dZ_z is clopen decreasing and $z \in dZ_z \subseteq dX_z$ for every $z \in \mathrm{Min}(C)$, while iZ_u is clopen increasing and $u \in iZ_u \subseteq iX_u$ for every $u \in \mathrm{Max}(C)$. Hence $\{dZ_z \mid z \in \mathrm{Min}(C)\} \cup \{iZ_u \mid u \in \mathrm{Max}(C)\}$ is a family of pairwise disjoint sets. Furthermore,

(cZ) if $p < q$ for some $p \in dZ_z$ and $q \in iZ_u$, then $z < u$.

Indeed, $p \in (iX_u]$ because $q \in iZ_u \subseteq iX_u$, and the definition of dZ_z shows that, for $p \in dZ_z$, this is possible only when $u \in \mathrm{Max}(z)$.

In the next step, for every $z \in \mathrm{Min}(C)$ and every $u \in \mathrm{Max}(C)$ we set

$dB_z = \bigcap\{(iZ_t] \mid t \in \mathrm{Max}(z)\} \cap dZ_z,$ and

$iB_u = \bigcap\{[dZ_v) \mid v \in \mathrm{Min}(u)\} \cap iZ_u.$

It is clear that dB_z is clopen decreasing, that $z \in dB_z \subseteq dZ_z \subseteq dA_z$ and hence $dB_z \cap C = \{z\}$ for every $z \in \mathrm{Min}(C)$, and that iB_u is clopen increasing such that $u \in iB_u \subseteq iZ_u \subseteq iA_u$ and hence $iB_u \cap C = \{u\}$ for every $u \in \mathrm{Max}(C)$. Therefore $\{dB_z \mid z \in \mathrm{Min}(C)\} \cup \{iB_u \mid u \in \mathrm{Max}(C)\}$ consists of pairwise disjoint sets. The property (cB) below then follows from (cZ), while (BZ) follows from the definition of dB_z and iB_u.

(cB) if $z \in \mathrm{Min}(C)$ and $u \in \mathrm{Max}(C)$ are such that $p < q$ for some $p \in dB_z$ and $q \in iB_u$, then $z < u$.

(BZ) if $z \in \mathrm{Min}(C)$ and $u \in \mathrm{Max}(C)$ are such that $z < u$, then $dB_z \subseteq (iZ_u]$ and $iB_u \subseteq [dZ_z)$.

Set $D_0 = \bigcup\{dB_z \mid z \in \mathrm{Min}(C)\}$, $D_1 = \bigcup\{iB_z \mid z \in \mathrm{Max}(C)\}$. Since D_0 is clopen and decreasing, the decreasing set $D_2 = X \setminus [D_0)$ is clopen, and $\mathrm{Min}(D_2) = \mathrm{Min}(X) \setminus D_0$. Similarly, $D_3 = X \setminus (D_1]$ is clopen increasing and such that $\mathrm{Max}(D_3) = \mathrm{Max}(X) \setminus D_1$. Hence $K(D_2) \cup K(D_3)$ is clopen, and so is the set $D_4 = X \setminus (K(D_2) \cup K(D_3))$. From the definition of D_4 it follows that $\mathrm{Min}(x) \subseteq D_0$ and $\mathrm{Max}(x) \subseteq D_1$ for every $x \in D_4$, and that $D_4 \supseteq C$. Set

$dD_z = dB_z \cap D_4$ for every $z \in \mathrm{Min}(C)$,

$iD_u = iB_u \cap D_4$ for every $u \in \mathrm{Max}(C)$.

Clearly, the set dD_z is clopen decreasing and $z \in dD_z \subseteq dB_z$ for every $z \in \mathrm{Min}(C)$, and iD_u is clopen increasing and $u \in iD_u \subseteq iB_u$ for every $u \in \mathrm{Max}(C)$. Hence the family $\{dD_z \mid z \in \mathrm{Min}(C)\} \cup \{iD_u \mid u \in \mathrm{Max}(C)\}$ consists of pairwise disjoint sets, and (cB) implies that

(cD) if $z \in \mathrm{Min}(C)$ and $u \in \mathrm{Max}(C)$ are such that $p < q$ for some $p \in dD_z$ and $q \in iD_u$ then $z < u$.

These sets also have a strong converse property.

(DD) If $z \in \mathrm{Min}(C)$, $u \in \mathrm{Max}(C)$ and $z < u$, then $dD_z \subseteq (iD_u]$ and $iD_u \subseteq [dD_z)$.

To justify the first conclusion of (DD), let $x \in dD_z = dB_z \cap D_4$. From (BZ) it follows that $x \leqslant y$ for some $y \in iZ_u$. Then $y \in D_4$ and, since iZ_u is increasing, we may assume that $y \in \mathrm{Max}(x)$. But $\mathrm{Max}(x) \subseteq D_1$ and $iZ_u \cap iB_t = \emptyset$ for all $t \in \mathrm{Max}(C) \setminus \{u\}$, so that $y \in iB_u \cap D_4 \cap \mathrm{Max}(X)$. This proves the first claim in (DD). The remainder follows by a dual argument.

For any $Z \subseteq \mathrm{Min}(C)$ and $U \subseteq \mathrm{Max}(C)$ define

$dD_Z = \bigcup\{dD_z \mid z \in Z\}$ and $iD_U = \bigcup\{iD_z \mid z \in U\}$;

$Q(Z) = (\bigcap\{[dD_z) \mid z \in Z\}) \cap (D_4 \setminus [dD_{\mathrm{Min}(C)\setminus Z}))$ or, equivalently,

$y \in Q(Z) \Leftrightarrow y \in D_4$ and $Z = \{z \in \mathrm{Min}(C) \mid \mathrm{Min}(y) \cap dD_z \neq \emptyset\}$;

$R(U) = (\bigcap\{(iD_u] \mid u \in U\}) \cap (D_4 \setminus (iD_{\mathrm{Max}(C)\setminus U}])$ or, equivalently,

$y \in R(U) \Leftrightarrow y \in D_4$ and $U = \{u \in \mathrm{Max}(C) \mid \mathrm{Max}(y) \cap iD_u \neq \emptyset\}$;

$S(Z, U) = Q(Z) \cap R(U)$.

Since we are working in a *dp*-space and because C is finite, all these sets are clopen. Since $\{dD_z \mid z \in \mathrm{Min}(C)\}$ and $\{iD_u \mid u \in \mathrm{Max}(C)\}$ are disjoint families, the (possibly empty) sets $S(Z, U)$ are pairwise disjoint.

If $c \in S(Z, U) \cap C$ then, since $dD_z \cap C = \{z\}$ for $z \in \mathrm{Min}(C)$ and $iD_u \cap C = \{u\}$ for $u \in \mathrm{Max}(C)$, we have $Z = \mathrm{Min}(c)$ and $U = \mathrm{Max}(c)$. Hence $c \in S(\mathrm{Min}(c), \mathrm{Max}(c)) = S_c$ and, because C is rudimentary, $S_c \cap C = \{c\}$. Therefore, for any $Z \subseteq \mathrm{Min}(C)$ and $U \subseteq \mathrm{Max}(C)$, either $S(Z, U) \cap C = \emptyset$ or $S(Z, U) = S_c$ for some $c \in C$.

Since each of the finitely many sets $S(Z, U)$ is closed, each set

$$K(S(Z, U)) = \bigcup\{C' \in \mathbb{C}(X) \mid C' \subseteq D_4 \text{ and } C' \cap S(Z, U) \neq \emptyset\}$$

is closed, and the set

$$D_5 = \bigcup\{K(S(Z, U)) \mid Z \subseteq \mathrm{Min}(C), U \subseteq \mathrm{Max}(C), S(Z, U) \cap C = \emptyset\}$$

is closed as well. Clearly, $D_5 = K(D_5)$, $D_5 \cap C = \emptyset$, and a component C' of D_4 is contained in $D_4 \setminus D_5$ if and only if $C \cap K(S(Z, U)) = \emptyset$ implies $C' \cap K(S(Z, U)) = \emptyset$ for every $Z \subseteq \mathrm{Min}(C)$ and every $U \subseteq \mathrm{Max}(C)$.

For $Z_1 \subseteq Z_2 \subseteq \mathrm{Min}(C)$ and $U_2 \subseteq U_1 \subseteq \mathrm{Max}(C)$, and only for such sets, write $T(Z_1, Z_2, U_1, U_2) = K(S(Z_1, U_1) \cap (S(Z_2, U_2)])$. Then

$$T(Z_1, Z_2, U_1, U_2) = \bigcup \{C' \in \mathbb{C}(X) \mid \exists x \in S(Z_1, U_1), \\ \exists y \in S(Z_2, U_2) \ x \leqslant y \in C' \subseteq D_4\}$$

is a closed set, so that the finite union

$$D_6 = \bigcup \{T(Z_1, Z_2, U_1, U_2) \mid c \in S(Z_1, U_1) \cap C, d \in S(Z_2, U_2) \cap C \Longrightarrow c \nleqslant d\}$$

is also closed. Clearly $D_6 = K(D_6)$ and $D_6 \cap C = \emptyset$.

Therefore $D_7 = D_4 \setminus (D_5 \cup D_6)$ is open, $D_7 \supseteq C$ and $K(D_7) = D_7$. Since $D_7 \cap D_5 = \emptyset$, the open sets $H_c = S_c \cap D_7$ form a decomposition of D_7 and satisfy $H_c \cap C = \{c\}$ for every $c \in C$.

We may thus define a mapping $h\colon D_7 \longrightarrow D_7$ by the requirement that $h(y) = c$ for all $y \in H_c$ and $c \in C$. Then h is idempotent, $\mathrm{Im}(h) = C$, and h is continuous because $\mathrm{Im}(h)$ is finite and $h^{-1}\{c\} = H_c$ is open for every $c \in \mathrm{Im}(h)$. From $D_7 \cap D_6 = \emptyset$ it follows that $x \leqslant y$ for some $x \in H_c$ and $y \in H_d$ only when $c \leqslant d$ in C, and this shows that h preserves the order.

Next we prove that h preserves extremal elements. If $x \in \mathrm{Min}(D_7)$, then $x \in dD_z$ for a unique $z \in \mathrm{Min}(C)$, so that $x \in Q(\{z\})$, and we need only show that $x \in R(\mathrm{Max}(z))$. But (DD) implies that $x \in (iD_u]$ for all $u \in \mathrm{Max}(z)$ and from (cD) it follows that $x \notin (iD_t]$ for each $t \in \mathrm{Max}(C) \setminus \mathrm{Max}(z)$. Hence $x \in H_z$ and $h(x) = z$ follows. This also shows that $h(dD_z) = \{z\}$ for every $z \in \mathrm{Min}(C)$. Analogously we find that $h(iD_u) = \{u\}$ for all $u \in \mathrm{Max}(C)$.

Let $y \in D_7$ be arbitrary and $h(y) = c \in C$. Then $y \in H_c \subseteq Q(\mathrm{Min}(c))$, so that $\mathrm{Min}(c) = \{z \in \mathrm{Min}(C) \mid \mathrm{Min}(y) \cap dD_z \neq \emptyset\}$. From $h(dD_z) = \{z\}$ for $z \in \mathrm{Min}(C)$ it then follows that $h(\mathrm{Min}(y)) = \mathrm{Min}(c) = \mathrm{Min}(h(y))$. Analogously, $h(\mathrm{Max}(y)) = \mathrm{Max}(h(y))$ for any $y \in D_7$.

Since C is closed decreasing and D_7 is open decreasing, there exists a clopen decreasing set D_8 with $C \subseteq D_8 \subseteq D_7$. Then $D = K(D_8)$ is clopen, and $D \subseteq D_7$ because D_7 is also increasing. The restriction g of h to D is the required idempotent *dp*-map. $\square$

Theorem 1.4. *Let $X \in \mathbb{FG}$ be rudimentary and let $\mathscr{C} \subseteq \mathbb{C}(X)$ be a finite set containing an isomorphic copy of every member of $\mathbb{C}(X)$. Let $\mathscr{D} \subseteq \mathbb{C}(X)$ be disjoint with $\mathscr{C}$ and finite. For every $D \in \mathscr{D}$, let a dp-map $\varphi_D\colon D \to C \in \mathscr{C}$ be given. Then there exists an idempotent $f \in \mathrm{End}(X)$ with $\mathrm{Im}(f) = \bigcup \mathscr{C}$ and $f \restriction D = \varphi_D$ for every $D \in \mathscr{D}$.*

P r o o f. Let $\mathscr{C}' = \mathscr{C} \cup \mathscr{D}$. Since $\mathscr{C}'$ is finite, Lemma 1.3 implies the existence of a family $\{Z_C \mid C \in \mathscr{C}'\}$ of disjoint clopen sets such that $C \subseteq Z_C = K(Z_C)$, and of idempotent dp-maps $g_C\colon Z_C \longrightarrow Z_C$ with $\mathrm{Im}(g_C) = C$ for every $C \in \mathscr{C}'$. Thus $Y = X \setminus (\bigcup\{Z_C \mid C \in \mathscr{C}'\})$ is clopen in X and hence compact. Again by Lemma 1.3, for every component $D \subseteq Y$ there exists an idempotent dp-map $g_D\colon Z_D \longrightarrow Z_D$ with $\mathrm{Im}(g_D) = D$ defined on a clopen set Z_D satisfying $D \subseteq Z_D = K(Z_D) \subseteq Y$. Since Y is compact, we may assume that $Y = \bigcup\{Z_D \mid D \in \mathscr{D}'\}$ for some finite $\mathscr{D}' \subseteq \mathbb{C}(Y)$. Clearly $\mathscr{D}' \cap \mathscr{D} = \emptyset$. Since all $Z_D = K(Z_D)$ with $D \in \mathscr{D}'$ are clopen, we may also assume that they are pairwise disjoint. For each $D \in \mathscr{D}'$ choose a dp-map $\varphi_D\colon D \to C \in \mathscr{C}$ arbitrarily. Then a mapping $f\colon X \longrightarrow X$ defined by

$$f(y) = \begin{cases} g_C(y) & \text{for all } y \in Z_C \text{ with } C \in \mathscr{C}, \\ \varphi_D g_D(y) & \text{for all } y \in Z_D \text{ with } D \in \mathscr{D} \cup \mathscr{D}' \end{cases}$$

is the required idempotent dp-map. □

Definition. For $X \in \mathbb{AR}$ and any $C \in \mathbb{C}(X)$, we define the *Stone nucleus* $\mathrm{Nuc}(C)$ of C by

$$\mathrm{Nuc}(C) = \begin{cases} \mathrm{Rud}(C) & \text{if } |\operatorname{Ext}(C)| \neq 2, \\ \operatorname{Ext}(C) & \text{if } |\operatorname{Ext}(C)| = 2. \end{cases}$$

It is clear that in the latter case C represents a double Stone algebra.

Observe that if $C \in \mathbb{FG}$ is a Stone nucleus then every $x \in \mathrm{Mid}(C)$ is non-defective and $E(x) = \{x\}$.

To show some important properties of Stone nuclei, first we recall from Lemma 1.1 that for any $X \in \mathbb{FG}$, the surjective mapping $k_X\colon X \longrightarrow \mathrm{Rud}(X)$ satisfies $k_X(x) = k_X(y)$ exactly when $\operatorname{Ext}(x) = \operatorname{Ext}(y)$. In particular, k_C maps $\operatorname{Ext}(C)$ bijectively onto $\operatorname{Ext}(k_C(C))$ with only one exception: if $C \in \mathbb{C}(X)$ is such that $|\operatorname{Ext}(C)| = 2$, then $k_C(C) = \mathrm{Rud}(C)$ is a singleton.

If, on the other hand, $C \in \mathbb{AR}$ has more than two extremal elements, then any mapping $h_C\colon \mathrm{Rud}(C) \longrightarrow C$ such that $k_C h_C = 1_{\mathrm{Rud}(C)}$ and $h_C k_C(z) = z$ for every $z \in \operatorname{Ext}(C)$ is a dp-map. Indeed, its continuity follows from the finiteness of $\mathrm{Rud}(C)$, we have $h_C(\operatorname{Ext}(t)) = \operatorname{Ext}(u) = \operatorname{Ext}(h_C(t))$ for every $t = k_C(u) \in \mathrm{Rud}(C)$, and h_C preserves order because $\mathrm{Mid}(\mathrm{Rud}(C))$ is an antichain. Furthermore, if $x \in \mathrm{Mid}(C)$ is non-defective, then $x \in \mathrm{Im}(h_C)$ for some left inverse h_C of k_C.

A Stone nucleus $\mathrm{Nuc}(C) = \{z, u\}$ of a component C with $\mathrm{Min}(C) = \{z\}$ and $\mathrm{Max}(C) = \{u\}$ has a similar property: there is a surjective dp-map $l_C\colon C \longrightarrow \{z, u\}$ because for some clopen decreasing set $A \subseteq C$ we have $z \in A$ and $u \in C \setminus A$. The injection h_C of $\{z, u\}$ into C is a dp-map for which $l_C h_C$ is the identity of $\mathrm{Nuc}(C)$.

From these observations it follows that

(A) for any order connected $C \in \mathbb{AR}$ and any subspace $N \subseteq C$ isomorphic to $\mathrm{Nuc}(C)$, there is an idempotent $f_N \in \mathrm{End}(C)$ with $\mathrm{Im}(f_N) = N$.

Let $X \in \mathbb{AR}$. If $C \in \mathbb{C}(X)$ and an idempotent $f \in \mathrm{End}(X)$ satisfy $\mathrm{Im}(f) \cap C \neq \emptyset$, then $f(C) \subseteq C$ and hence $\mathrm{Ext}(C) \subseteq \mathrm{Im}(f)$. Hence $\mathrm{Ext}(f(t)) = \mathrm{Ext}(t)$ for all $t \in C$. When $|\,\mathrm{Ext}(C)| > 2$, this implies that $k_C f h_C = k_C h_C$ is the identity endomorphism of $\mathrm{Nuc}(C)$. Thus $N = f h_C(\mathrm{Nuc}(C)) \subseteq \mathrm{Im}(f) \cap C$ is a *dp*-subspace of C isomorphic to $\mathrm{Nuc}(C)$. For any C with $|\,\mathrm{Ext}(C)| \leqslant 2$ it is clear that $N = \mathrm{Ext}(C) \subseteq \mathrm{Im}(f)$ is isomorphic to $\mathrm{Nuc}(C)$. Therefore

(B) if $X \in \mathbb{AR}$, then for any idempotent $f \in \mathrm{End}(X)$ and for any $C \in \mathbb{C}(X)$, either $f(C) \cap C = \emptyset$ or $f(C) \cap C$ contains a *dp*-subspace N isomorphic to $\mathrm{Nuc}(C)$.

If $C, D \in \mathbb{C}(X)$ and $\mathrm{Nuc}(C) \cong \mathrm{Nuc}(D)$, then, by (A), there exists a *dp*-map $g\colon D \longrightarrow C$ with finite image. Thus for every *dp*-map $f\colon C \longrightarrow D$ there exists some finite n such that $(gf)^n$ is idempotent. Hence the foregoing observations can be extended as follows:

(C) if $X \in \mathbb{AR}$ and if $C, D \in \mathbb{C}(X)$ with $\mathrm{Nuc}(C) \cong \mathrm{Nuc}(D)$ then for every *dp*-endomorphism f of X with $f(C) \subseteq D$ there exists a *dp*-subspace $N \subseteq C$ with $N \cong \mathrm{Nuc}(C)$ such that f is one-to-one on N; hence $\mathrm{Im}(f) \cap D$ contains a *dp*-subspace N' isomorphic to $\mathrm{Nuc}(D)$.

Even though observations (A), (B), and (C) deal only with connected *dp*-spaces, they also inform us that the notion of Stone nucleus might be quite useful.

Definitions. For a given Stone nucleus N and a *dp*-space $X \in \mathbb{FG}$, write

$$\mathbb{C}_N(X) = \{C \in \mathbb{C}(X) \mid \mathrm{Nuc}(C) \cong N\}$$

and

$$\mathbb{C}_{(2)}(X) = \bigcup\{\mathbb{C}_N(X) \mid |\mathbb{C}_N(X)| \geqslant 2\}.$$

A family $\mathscr{C} \subseteq \mathbb{C}(X)$ of components of a *dp*-space $X \in \mathbb{FG}$ is a *Stone plot* of X if for every $C' \in \mathbb{C}(X)$ there exists a component $C \in \mathscr{C}$ with $\mathrm{Nuc}(C') \cong \mathrm{Nuc}(C)$.

Clearly, any $X \in \mathbb{FG}$ has a finite Stone plot.

Theorem 1.5. *Let $X \in \mathbb{AR}$, let $\mathscr{C}$ be a finite Stone plot of X, and let $N_C \subseteq C$ be a dp-subspace isomorphic to* $\mathrm{Nuc}(C)$ *for every component $C \in \mathscr{C}$. Let $\mathscr{D} \subseteq \mathbb{C}(X)$ be disjoint with $\mathscr{C}$ and finite, and let a dp-map $\varphi_D\colon D \to \bigcup\{N_C \mid C \in \mathscr{C}\}$ be given for every $D \in \mathscr{D}$.*

Then there exists an idempotent $f \in \mathrm{End}(X)$ *such that* $\mathrm{Im}(f) = \bigcup\{N_C \mid C \in \mathscr{C}\}$ *and* $f \restriction D = \varphi_D$ *for every* $D \in \mathscr{D}$.

P r o o f. Let $k\colon X \longrightarrow X'$ be the *dp*-map of X onto its rudiment $X' = \mathrm{Rud}(X)$, see Lemma 1.1. For $j = 1, 2$, denote $\mathscr{C}_j = \{C \in \mathscr{C} \mid |\mathrm{Ext}(C)| = j\}$ and $\mathscr{C}'_j = \{k(C) \mid C \in \mathscr{C}_j\}$, and write $\mathscr{C}' = \{k(C) \mid C \in \mathscr{C}\}$, $\mathscr{D}' = \{k(D) \mid D \in \mathscr{D}\}$. Let $\mathscr{D}_2$ denote the set of all $D \in \mathscr{D}$ such that $\varphi_D\colon D \to C \in \mathscr{C}_2$ and $\mathscr{D}'_2 = \{k(D) \mid D \in \mathscr{D}_2\}$. Clearly, if $D \in \mathscr{D}_2$ then $|\mathrm{Ext}(D)| \geqslant 2$.

Since every $D \in \mathscr{D}$ is connected, so is $\mathrm{Im}(\varphi_D)$. Hence there is a unique $C \in \mathscr{C}$ such that $\mathrm{Im}(\varphi_D) \subseteq N_C \subseteq C$. By Lemma 1.1, there exists a unique *dp*-map $\varphi'_D\colon k(D) = D' \to k(C)$ with $k\varphi_D = \varphi'_{D'}k$.

Since the set $\mathscr{C}' \cup \mathscr{D}' \subseteq \mathbb{C}(X')$ is finite and $k(\mathrm{Min}(X) \cap \mathrm{Max}(X))$ is closed, for every $C' \in \mathscr{C}'_2 \cup \mathscr{D}'_2$ there exists a clopen set $V_{C'} \subseteq X'$ with $V_{C'} \cap k(\mathrm{Max}(X) \cap \mathrm{Min}(X)) = \emptyset$, $K(V_{C'}) = V_{C'} \supseteq C'$, and such that C' is the only member of $\mathscr{C}'_2 \cup \mathscr{D}'_2$ intersected by $V_{C'}$. Write $X'_2 = \bigcup\{V_{C'} \mid C' \in \mathscr{C}'_2 \cup \mathscr{D}'_2\}$ and $X'_1 = X' \setminus X'_2$. By Theorem 1.4, there exist idempotents $f'_1 \in \mathrm{End}(X'_1)$ and $f'_2 \in \mathrm{End}(X'_2)$ such that $\mathrm{Im}(f'_1) = \bigcup(\mathscr{C}' \setminus \mathscr{C}'_2)$, $\mathrm{Im}(f'_2) = \bigcup \mathscr{C}'_2$ and $f'_1 \restriction D' = \varphi'_{D'}$ for every $D' \in \mathscr{D}' \setminus \mathscr{D}'_2$, $f'_2 \restriction D' = \varphi'_{D'}$ for every $D' \in \mathscr{D}'_2$. Then $f' = f'_1 \cup f'_2 \in \mathrm{End}(X')$ is idempotent.

Since $\mathrm{Im}(f')$ is finite, there exists a clopen set $B \subseteq X'$ such that $B = K(B)$ and $B \cap \mathrm{Im}(f') = \bigcup \mathscr{C}'_2$. Hence $A = (f'k)^{-1}(B)$ is clopen, $A = K(A)$ and $A \cap (\bigcup \mathscr{C}'_1) = \emptyset$, so that there is a clopen decreasing set $A_0 \subseteq A$ with $\mathrm{Min}(A) \subseteq A_0$ and $\mathrm{Max}(A) \cap A_0 = \emptyset$ and such that, for every $D \in \mathscr{D}_2$, $A_0 \cap D = \varphi_D^{-1}(\mathrm{Min}(X)) \cap D$—see Lemma P.0. For every $C \in \mathscr{C} \setminus \mathscr{C}_2$, choose $h_C\colon k(C) \to C$ so that $\mathrm{Im}(h_C) = N_C$ and $h_C k$ is the identity of N_C. Then $h_C \varphi'_D k = \varphi_D$ for any $D \in \mathscr{D} \setminus \mathscr{D}_2$. For $C \in \mathscr{C}_2$ denote $\mathrm{Min}(C) = \{y_C\}$, $\mathrm{Max}(C) = \{z_C\}$ and define a mapping f by

$$f(x) = \begin{cases} h_C f'k(x) & \text{if } f'k(x) \in C' \in \mathscr{C}' \setminus \mathscr{C}'_2, \\ z_C & \text{if } f'k(x) \in C' \in \mathscr{C}'_2 \text{ and } x \notin A_0, \\ y_C & \text{if } f'k(x) \in C' \in \mathscr{C}'_2 \text{ and } x \in A_0. \end{cases}$$

Since A contains only components with at least two extremals we obtain that $f \in \mathrm{End}(X)$ is idempotent. From the choice of $D' \in \mathscr{D}'$, $\varphi'_{D'}$ and A_0 it follows that $f \restriction D = \varphi_D$ for any $D \in \mathscr{D}$. □

The observation below supplements Theorem 1.5.

Statement 1.6. *Let* $X \in \mathbb{FG}$, *and let* $f \in \mathrm{End}(X)$ *be an idempotent such that* $\mathrm{Im}(f)$ *intersects only finitely many components of* X. *Then for every* $g \in \mathrm{End}(X)$ *with* $\mathrm{Im}(g) \subseteq \mathrm{Im}(f)$ *there exists an idempotent* $h \in \mathrm{End}(X)$ *such that*

$$\mathrm{Im}(h) = \bigcup\{\mathrm{Im}(f) \cap C \mid C \in \mathbb{C}(X),\ C \cap \mathrm{Im}(g) \neq \emptyset\}.$$

P r o o f. Denote $\mathscr{C} = \{C \in \mathbb{C}(X) \mid C \cap \mathrm{Im}(g) \neq \emptyset\}$ and define a mapping h as follows:

$$h(x) = \begin{cases} gf(x) & \text{for } x \in f^{-1}(X \setminus \bigcup \mathscr{C}), \\ f(x) & \text{for } x \in f^{-1}(\bigcup \mathscr{C}). \end{cases}$$

The set of all components intersecting $\mathrm{Im}(f)$ is finite, so that $f^{-1}(C)$ is clopen for every $C \in \mathbb{C}(X)$, and hence $h \in \mathrm{End}(X)$ because f and g are *dp*-maps. Since f is idempotent and $\mathrm{Im}(g) \subseteq \mathrm{Im}(f)$, the *dp*-map h is idempotent and $\mathrm{Im}(h) = \bigcup\{\mathrm{Im}(f) \cap C \mid C \in \mathbb{C}(X),\ \ \mathrm{Im}(g) \cap C \neq \emptyset\}$. $\square$

Theorem 1.7. *Let $X \in \mathbb{FG}$, and let $g \in \mathrm{End}(X)$ be an idempotent such that $\mathrm{Im}(g)$ is finite. Let $x \in \mathrm{Im}(g)$ and $y \in g^{-1}\{x\}$ satisfy $x < y$. If $F_1 \subseteq g^{-1}\{x\} \setminus \mathrm{Min}(X)$ is a closed set with $y \in F_1$ and such that*

(t) $\qquad v \in \mathrm{Im}(g) \cap \mathrm{Mid}(X),\ v \neq x \text{ and } v \notin [y) \Rightarrow g^{-1}\{v\} \cap [F_1) = \emptyset,$

and if $F_0 \subseteq g^{-1}\{x\}$ is another closed set with $x \in F_0$ and $[F_1) \cap F_0 = \emptyset$, then there is an idempotent $f \in \mathrm{End}(X)$ with $\mathrm{Im}(f) = \mathrm{Im}(g) \cup \{y\}$, $F_0 \subseteq f^{-1}\{x\}$, $F_1 \subseteq f^{-1}\{y\}$, and such that $f^{-1}\{z\} = g^{-1}\{z\}$ for all $z \in \mathrm{Im}(g) \setminus \{x\}$.

P r o o f. The set $G = \bigcup\{g^{-1}\{v\} \mid v \in \mathrm{Mid}(X) \setminus ([y) \cup \{x\})\}$ is closed because g is a *dp*-map and $\mathrm{Im}(g)$ is finite. From the hypothesis and from (t) it then follows that $[F_1)$ and the closed set $F_0 \cup G \cup \mathrm{Min}(X)$ are disjoint. Since X is a Priestley space, there is a clopen increasing set $U \supseteq [F_1)$ disjoint with $F_0 \cup G \cup \mathrm{Min}(X)$. Since $x \notin \mathrm{Max}(X)$, we have $g^{-1}\{x\} \cap \mathrm{Max}(X) = \emptyset$, so that the set $Y = g^{-1}\{x\} \cap U$ is contained in $\mathrm{Mid}(X)$, and is increasing in $g^{-1}\{x\}$.

Set

$$f(t) = \begin{cases} y & \text{for all } t \in Y, \\ g(t) & \text{for all } t \in X \setminus Y. \end{cases}$$

Then f is idempotent with $\mathrm{Im}(f) = \mathrm{Im}(g) \cup \{y\}$, $F_0 \subseteq f^{-1}\{x\}$, $F_1 \subseteq f^{-1}\{y\}$ and $f^{-1}\{z\} = g^{-1}\{z\}$ for all $z \neq x$. Since Y is clopen and g is continuous, the map f is continuous as well. Since $g \in \mathrm{End}(X)$ is idempotent and $x \in \mathrm{Im}(g)$, we have $g(z) = z$ for all $z \in \mathrm{Ext}(K(x))$. But then $\mathrm{Ext}(y) = \mathrm{Ext}(x)$ follows from $y \in K(x)$ and $g(y) = x$. Moreover, $f(z) = g(z)$ for all $z \in \mathrm{Ext}(X)$ because $Y \subseteq \mathrm{Mid}(X)$. These two facts imply that $f(\mathrm{Ext}(t)) = \mathrm{Ext}(f(t))$ for all $t \in X$.

To show that f preserves order, it is enough to consider comparable $t \in Y$ and $t' \in X \setminus Y$. For such elements we have $g(t) = x$, $f(t) = y$ and $f(t') = g(t')$. If $t' < t$, then $f(t') = g(t') \leqslant g(t) = x < y = f(t)$ because g preserves order. Suppose that $t < t'$. Since U is increasing and $t \in U$, we have $t' \in U$, and from $Y = g^{-1}\{x\} \cap U$ it then follows that $x = g(t) < g(t')$. In particular, $g(t') \notin \mathrm{Min}(X)$.

If $g(t') \in \mathrm{Mid}(X)$ and $f(t) \nleqslant f(t')$, then $y \nleqslant g(t')$ and hence $t' \in G$, which contradicts the fact that $G \cap U = \emptyset$. In the remaining case we have $g(t') \in \mathrm{Max}(x)$, and $f(t) = y < g(t') = f(t')$ follows from $\mathrm{Max}(x) = \mathrm{Max}(y)$. $\square$

Lemma 1.8. *Let $X \in \mathbb{AR}$, let $h \in \mathrm{End}(X)$ be idempotent, and let $\mathscr{D}_0, \mathscr{D}_1 \subseteq \mathbb{C}(X)$ be finite disjoint sets such that $\mathrm{Im}(h) \cap D = \emptyset$ for every $D \in \mathscr{D}_0 \cup \mathscr{D}_1$. For each $D \in \mathscr{D}_0 \cup \mathscr{D}_1$, let φ_D be a dp-map defined on D and factorizing through $\mathrm{Nuc}(D)$, and such that $\varphi_D(D) \subseteq \mathrm{Im}(h)$ for all $D \in \mathscr{D}_0$ and $\varphi_D(D) \subseteq D$ for each $D \in \mathscr{D}_1$.*

Then there is an idempotent $f \in \mathrm{End}(X)$ such that $f \restriction D = \varphi_D$ for every $D \in \mathscr{D}_0 \cup \mathscr{D}_1$ and $\mathrm{Im}(f) = \mathrm{Im}(h) \cup \bigcup\{\mathrm{Im}(\varphi_D) \mid D \in \mathscr{D}_1\}$.

Proof. Since $\mathscr{D} = \mathscr{D}_0 \cup \mathscr{D}_1 \subseteq \mathbb{C}(X)$ is finite, from Lemma P.0 and Lemma 1.3 it follows that there is a family $\{V_D | D \in \mathscr{D}\}$ of mutually disjoint, clopen, increasing and decreasing sets satisfying $V_D \supseteq D$ and $V_D \cap \mathrm{Im}(h) = \emptyset$ for every $D \in \mathscr{D}$, and such that there exists a surjective *dp*-map $f_D \colon V_D \to \mathrm{Nuc}(D)$. It can be also assumed that, for any $x, y \in D$, we have $f_D(x) = f_D(y)$ exactly when $\varphi_D(x) = \varphi_D(y)$. For any $D \in \mathscr{D}$, let $g_D \colon \mathrm{Nuc}(D) \to X$ be a *dp*-map for which $g_D f_D = \varphi_D$. Then the map f defined by

$$f(t) = \begin{cases} g_D f_D(t) & \text{for } t \in V_D \text{ with } D \in \mathscr{D}, \\ h(t) & \text{for all other } t \end{cases}$$

satisfies our claim. $\square$

Remark 1.9. To formulate a more practical condition that is equivalent to $P(\mathbf{V}) \subseteq \mathbb{DC}$ for a finitely generated variety $\mathbf{V}$, suppose that $X \in \mathbb{FG}$ contains elements $x, y, z \in \mathrm{Mid}(X)$ such that x is min-defective, y is max-defective, z is non-defective, and $[x) \cap E(z) \neq \emptyset \neq (y] \cap E(z)$. We claim that there exists a surjective *dp*-map $h \colon X \to Y$ for which $h\{x, y, z\} \subseteq \mathrm{Mid}(Y)$, $h(x) < h(z) < h(y)$ and $h(z)$ is non-defective. To prove this claim, we observe that x, y, z must belong to the same component C of X, and that $E(z)$ is closed, see Lemma 1.2. Thus the set $E(z)$ and all singletons in $X \setminus E(z)$ form a closed decomposition θ of X such that X/θ is Hausdorff. The surjective map $h \colon X \to X/\theta = Y$ is therefore continuous and induces an order on Y such that Y is a *dp*-space and $h \colon X \to Y$ is a *dp*-map. Clearly $h(x) < h(z) < h(y)$ and $h(z)$ is non-defective. Therefore $Y \notin \mathbb{DC}$ and $D(Y)$ is isomorphic to a subalgebra of $D(X)$.

Thus if $\mathbf{V} \subseteq \mathbb{DC}$ is a variety and $X \in P(\mathbf{V})$, then $[x) \cap E(z) = \emptyset$ or $(y] \cap E(z) = \emptyset$ for any min-defective x, max-defective y and non-defective z in X.

Throughout this and subsequent sections, we restrict our attention to *dp*-spaces from $\mathbb{AR}$.

It is easy to see that if $f \in \mathrm{End}(X)$ is idempotent then, for any $g \in \mathrm{End}(X)$, $fg = g$ exactly when $\mathrm{Im}(g) \subseteq \mathrm{Im}(f)$.

Notation. For any $f, g \in \mathrm{End}(X)$, we write $g \lesssim f$ instead of $fg = g$ whenever f is idempotent. When g is also idempotent, we write $g \leqslant f$, while $g < f$ means that $g, f \in \mathrm{End}(X)$ are idempotents and $\mathrm{Im}(g)$ is a proper subset of $\mathrm{Im}(f)$.

For any idempotent $f \in \mathrm{End}(X)$, let $[f]$ be the set of all idempotents $g \in \mathrm{End}(X)$ satisfying $g \leqslant f$ and $f \leqslant g$. Hence $g \in [f]$ means that $f, g \in \mathrm{End}(X)$ are idempotents and $\mathrm{Im}(f) = \mathrm{Im}(g)$. We say that such idempotents are *equivalent*.

If $f\colon X \longrightarrow Y$ is a *dp*-map and $C \in \mathbb{C}(X)$, then $f(C) \subseteq D$ for a uniquely determined $D \in \mathbb{C}(Y)$. From the fact that a Stone nucleus of any component is its retract, see (A), it follows that there exists a *dp*-map $f'\colon \mathrm{Nuc}(C) \longrightarrow \mathrm{Nuc}(D)$. Conversely, if C and D are connected and if there is a *dp*-map $h\colon \mathrm{Nuc}(C) \longrightarrow \mathrm{Nuc}(D)$ then (A) again implies the existence of a *dp*-map $f\colon C \longrightarrow D$.

Definition. A subspace S of $X \in \mathbb{AR}$ is called a *Stone kernel* of X if it satisfies these three conditions:

(r1) for every $C \in \mathbb{C}(X)$ there exists a $D \in \mathbb{C}(X)$ with $S \cap D \neq \emptyset$ and $\mathrm{Nuc}(D) \cong \mathrm{Nuc}(C)$,

(r2) if $C_0, C_1 \in \mathbb{C}(X)$ are distinct and $S \cap C_0 \neq \emptyset \neq S \cap C_1$, then $\mathrm{Nuc}(C_0) \not\cong Nuc(C_1)$,

(r3) if $C \in \mathbb{C}(X)$ and $S \cap C \neq \emptyset$ then $S \cap C$ is isomorphic to $\mathrm{Nuc}(C)$.

It is clear that for any Stone kernel S of any $X \in \mathbb{AR}$, the set $\{C \in \mathbb{C}(X) \mid S \cap C \neq \emptyset\}$ is a minimal Stone plot of X.

Definition. Any idempotent $f \in \mathrm{End}(X)$ such that $\mathrm{Im}(f)$ is a Stone kernel of X will be called an *r-map*.

An isomorphism $\psi\colon \mathrm{End}(X) \to \mathrm{End}(Y)$ is an *R-isomorphism* if for any $g \in \mathrm{End}(X)$, $\psi(g)$ is an r-map if and only if g is an r-map.

Statement 2.1. *Let $X, Y \in \mathbb{AR}$. Then*

(1) *if $S \subseteq X$ is a Stone kernel of X then S is finite and there exists an r-map $f \in \mathrm{End}(X)$ with $\mathrm{Im}(f) = S$;*

(2) *if $f \in \mathrm{End}(X)$ is an r-map and $g \in \mathrm{End}(X)$ is idempotent, then g is an r-map if and only if $\mathrm{Im}(f)$ is isomorphic to $\mathrm{Im}(g)$;*

(3) *if* $\psi\colon \operatorname{End}(X) \longrightarrow \operatorname{End}(Y)$ *is an isomorphism such that* $\psi(f)$ *is an* r*-map for some* r*-map* $f \in \operatorname{End}(X)$*, then* ψ *is an* R*-isomorphism;*

(4) *if* $f_0, f_1, \dots, f_{n-1} \in \operatorname{End}(X)$ *are* r*-maps, then there exist* r*-maps* $g_0, g_1, \dots, g_{n-1} \in \operatorname{End}(X)$ *such that* $g_i \in [f_i]$ *and* $g_i g_j = g_i$ *for any* $i, j \in \{0, 1, \dots, n-1\}$*; if, moreover,* $f_0(\operatorname{Im}(f_i)) = \operatorname{Im}(f_0)$ *for all* $i \in \{1, \dots, n-1\}$*, then* $g_0 = f_0$ *may be chosen;*

(5) *if* f_i*,* g_i *are* r*-maps such that* $f_i g_i = f_i$ *and* $g_i f_i = g_i$ *for* $i = 0, 1$*,* $f_0 \in [f_1]$*, and* $f_0(z) = f_1(z)$ *for all* $z \in \operatorname{Im}(g_0) \cap \operatorname{Im}(g_1)$*, then there exist* r*-maps* h_0 *and* h_1 *such that* $h_i \in [g_i]$*,* $h_i h_{1-i} = h_i$ *and* $f_i = f_{1-i} h_{1-i} g_i$ *for* $i = 0, 1$*;*

(6) *if* $f \in \operatorname{End}(X)$ *is an* r*-map and for every* $x \in \operatorname{Mid}(X) \cap \operatorname{Im}(f)$ *an element* $v_x \in E(x)$ *is given, then the mapping* g *defined for* $y \in X$ *by*

$$g(y) = \begin{cases} f(y) & \text{if } f(y) \in \operatorname{Ext}(X), \\ v_{f(y)} & \text{if } f(y) \in \operatorname{Mid}(X) \end{cases}$$

is an r*-map of* X*;*

(7) *for every* $x \in X \setminus \operatorname{Def}(X)$ *there exists an* r*-map* $f \in \operatorname{End}(X)$ *with* $x \in \operatorname{Im}(f)$*;*

(8) *if* $x, y \in X$ *are such that either* $\operatorname{Nuc}(K(x)) \ncong \operatorname{Nuc}(K(y))$*, or* $K(x) = K(y)$ *and* $\operatorname{Ext}(x) \neq \operatorname{Ext}(y)$*, then there exists an* r*-map* $f \in \operatorname{End}(X)$ *with* $f(x) \neq f(y)$*;*

(9) *if* $f \in \operatorname{End}(X)$ *is an idempotent and* $f \geqslant g$ *for some* r*-map* $g \in \operatorname{End}(X)$*, then for every* $x \in \operatorname{Im}(f) \setminus \operatorname{Def}(X)$ *there exists an* r*-map* $g_x \in \operatorname{End}(X)$ *with* $g_x \leqslant f$ *and* $g_x(x) = x$*;*

(10) *if* $f, g \in \operatorname{End}(X)$ *are* r*-maps and* $h \in \operatorname{End}(X)$*, then* $h(\operatorname{Im}(f)) = \operatorname{Im}(g)$ *if and only if* $ghf = hf$ *and every idempotent* $g' \in \operatorname{End}(X)$ *with* $g' < g$ *satisfies* $g'hf \neq hf$*.*

Proof. (1) follows from the definition of a Stone kernel and Theorem 1.5.

(2) is a consequence of the definition of an r-map.

(3) follows from (2) and Lemma P.5.

If $f, g \in \operatorname{End}(X)$ are r-maps and $C \in \mathbb{C}(X)$ is such that $\operatorname{Im}(f) \cap C \neq \emptyset \neq \operatorname{Im}(g) \cap C$, then $(f \restriction C)(g \restriction C) = f \restriction C$ and $(g \restriction C)(f \restriction C) = g \restriction C$. Both statements of (4) then follow by Theorem 1.5 because the image of any r-map intersects only finitely many components of X.

For $i = 0, 1$, denote $\mathscr{D}_i = \{C \in \mathbb{C}(X) \mid \operatorname{Im}(g_i) \cap C = \emptyset \neq \operatorname{Im}(g_{1-i}) \cap C\}$. Then $\mathscr{D}_i$ is finite and, by Lemma 1.8, there is an r-map $h_i \in [g_i]$ such that $h_i(x) = g_i f_{1-i}(x)$ for all $x \in \mathscr{D}_i$. A direct calculation verifies the required expressions, and (5) is proved.

Since $\{v_x \mid x \in \operatorname{Im}(f) \cap \operatorname{Mid}(X)\} \cup \operatorname{Ext}(\operatorname{Im}(f))$ is a subspace of $X \in \mathbb{AR}$ isomorphic to $\operatorname{Im}(f)$, claim (6) follows from (1).

(7) follows from Theorem 1.5 and (6).

(8) follows from the definition of an r-map.

We turn to (9) now. If $f \in \operatorname{End}(X)$ is an idempotent such that $g \leqslant f$ for some r-map g of X, then any Stone kernel of $\operatorname{Im}(f)$ is isomorphic to any Stone kernel of X. By (7) applied to $\operatorname{Im}(f)$, for every $x \in \operatorname{Im}(f) \setminus \operatorname{Def}(X)$ there is an r-map $g'_x \in \operatorname{End}(\operatorname{Im}(f))$ with $x \in \operatorname{Im}(g'_x)$. But then $g_x = g'_x f \in \operatorname{End}(X)$ is an r-map with $x \in \operatorname{Im}(g_x)$, and $g_x \leqslant f$. Thus (9) is proved.

One implication in (10) is clear, and the other follows from Statement 1.6, (C), and the definition of an r-map. $\square$

Definition and notation. An idempotent $f \in \operatorname{End}(X)$ is called a *dr-map* if there exists exactly one equivalence class $[g]$ of r-maps with $g < f$, and $h \in [g]$ for any idempotent $h \in \operatorname{End}(X)$ with $g \leqslant h < f$. For a dr-map f and an r-map $g < f$, we shall use $r(f)$ to denote any member of $[g]$ for which $r(f)f = r(f)$.

Lemma 2.2. *Let* $x \in \operatorname{Def}(X)$, *and let* $f \in \operatorname{End}(X)$ *be an r-map such that* $\operatorname{Im}(f) \cap K(x) \neq \emptyset$. *Then there exists a dr-map* $g \in \operatorname{End}(X)$ *with* $\operatorname{Im}(g) = \operatorname{Im}(f) \cup \{x\}$ *and* $fg = f$ *exactly when, for every* $y \in \operatorname{Mid}(X) \setminus \operatorname{Def}(X)$,

$$E(y) \cap ((x] \cap [x)) \neq \emptyset \textit{ implies } \operatorname{Im}(f) \cap E(y) \cap ((x] \cap [x)) \neq \emptyset.$$

Moreover, for any $z \in \operatorname{Def}(X)$ *we can assume that* $g(z) \neq g(x)$, *except when*

x and z are min-defective and $x \leqslant z$, or
x and z are max-defective and $z \leqslant x$.

Finally, an idempotent $f \in \operatorname{End}(X)$ *is a dr-map if and only if* $\operatorname{Im}(f) = \operatorname{Im}(g) \cup \{x\}$ *for some r-map g and some* $x \in \operatorname{Def}(X)$.

Proof. Assume that x is defective and $f \in \operatorname{End}(X)$ is an r-map satisfying the hypothesis. Then the assumptions of Theorem 1.7 or of its dual are satisfied by f, $F_1 = \{x\}$, and $F_0 = \{u, z\} \cap f^{-1}(u)$ with $f(x) = u$, and Theorem 1.7 or its dual gives an idempotent $g \in \operatorname{End}(X)$ with $g(z) \neq g(x)$, $\operatorname{Im}(g) = \operatorname{Im}(f) \cup \{x\}$ and $fg = f$. Clearly, g is a dr-map. The converse is clear.

Let $f \in \operatorname{End}(X)$ be a dr-map. Then $\operatorname{Im}(r(f)) \subset \operatorname{Im}(f)$ for an r-map $r(f)$. Let $x \in \operatorname{Im}(f) \setminus \operatorname{Im}(r(f))$. If x were non-defective then, by Statement 2.1(9), there would exist an r-map g' with $x \in \operatorname{Im}(g')$ and $g' < f$. But then $g' \notin [r(f)]$—a contradiction. Therefore x must be defective. For every non-defective $z \in \operatorname{Mid}(X)$ such that $E(z) \cap ((x] \cup [x)) \neq \emptyset$ we have $\operatorname{Im}(f) \cap E(z) \cap ((x] \cup [x)) \neq \emptyset$ because $f(E(z)) \subseteq E(z)$. By Statement 2.1(6), there is an r-map $g < f$ such that $\operatorname{Im}(g) \cap E(z) \cap ((x] \cup [x)) \neq \emptyset$ whenever $E(z) \cap ((x] \cup [x)) \neq \emptyset$. We then apply the first part of the proof to obtain a dr-map $g' \in \operatorname{End}(X)$ with $\operatorname{Im}(g') = \operatorname{Im}(g) \cup \{x\}$. Thus $g < g' \leqslant f$, so that $g' \in [f]$ and $\operatorname{Im}(f) = \operatorname{Im}(g')$. The converse implication is clear. $\square$

Notation. For any dr-map $f \in \mathrm{End}(X)$, let $d(f)$ denote the defective element $x \in \mathrm{Im}(f)$.

The statement below summarizes properties of dr-maps.

Statement 2.3. *Let $X, Y \in \mathbb{AR}$. Then:*

(1) *for every $x \in \mathrm{Def}(X)$, there is a dr-map f such that $d(f) = x$;*
(2) *if $x, y \in X$ are min-defective then $x \leqslant y$ if and only if for any dr-maps $f, g \in \mathrm{End}(X)$ with $d(f) = x$, $d(g) = y$ and every r-map $h \in \mathrm{End}(X)$, we have $hfg \neq fg$;*
(3) *if $x, y \in X$ are max-defective then $y \leqslant x$ if and only if for any dr-maps $f, g \in \mathrm{End}(X)$ with $d(f) = x$, $d(g) = y$ and every r-map $h \in \mathrm{End}(X)$, we have $hfg \neq fg$;*
(4) *if $f, g \in \mathrm{End}(X)$ are dr-maps, then $d(f) = d(g)$ if and only if $hf'g' \neq f'g'$ and $hg'f' \neq g'f'$ for all $f' \in [f]$, $g' \in [g]$ and every r-map $h \in \mathrm{End}(X)$;*
(5) *for any defective $x \in X$ and any $y \in X$ with $x \neq y$ there exists a dr-map $f \in \mathrm{End}(X)$ with $f(x) \neq f(y)$;*
(6) *if $\psi\colon \mathrm{End}(X) \longrightarrow \mathrm{End}(Y)$ is an R-isomorphism then*
(a) *for every $g \in \mathrm{End}(X)$, g is a dr-map if and only if $\psi(g)$ is a dr-map, and*
(b) *for any two dr-maps $g_0, g_1 \in \mathrm{End}(X)$, $d(g_0) = d(g_1)$ exactly when $d(\psi(g_0)) = d(\psi(g_1))$;*
(7) *if $x_0, x_1 \in \mathrm{Def}(X)$ are such that $\mathrm{Nuc}(K(x_0)) \cong \mathrm{Nuc}(K(x_1))$ and if there exist r-maps $f_i \in \mathrm{End}(X)$ for $i = 0, 1$ such that $\mathrm{Nuc}(K(x_0)) \cong \mathrm{Nuc}(K(f_0(x_0)))$, $f_0[x_0) = f_1[x_1)$ and $f_0(x_0] = f_1(x_1]$, then there exist dr-maps $g_i \in \mathrm{End}(X)$ with $d(g_i) = x_i$, $g_i g_{1-i} = g_i$, and $f_i \restriction K(x_i) = (f_{1-i} g_{1-i}) \restriction K(x_i)$ for $i = 0, 1$;*
(8) *if $f \in \mathrm{End}(X)$ is a dr-map and for every non-defective $x \in \mathrm{Mid}(X) \cap \mathrm{Im}(f)$ an element $v_x \in E(x)$ is given such that $v_x \leqslant d(f)$ whenever $x \leqslant d(f)$, $v_x \geqslant d(f)$ whenever $x \geqslant d(f)$ then the mapping g defined for $y \in X$ by*

$$g(y) = \begin{cases} f(y) & \text{if } f(y) \in \mathrm{Ext}(X) \cup \mathrm{Def}(X), \\ v_{f(y)} & \text{if } f(y) \in \mathrm{Mid}(X) \setminus \mathrm{Def}(X) \end{cases}$$

is a dr-map of X with $d(f) = d(g)$;
(9) *an $x \in \mathrm{Def}(X)$ is doubly defective if and only if for every dr-map $f \in \mathrm{End}(X)$ with $d(f) = x$ there exist two distinct r-maps $g_i \in \mathrm{End}(X)$ with $g_i f = g_i$ and $g_i \leqslant f$ for $i = 0, 1$;*
(10) *if $f \in \mathrm{End}(X)$ is an idempotent such that $f \geqslant g$ for some r-map $g \in \mathrm{End}(X)$, then for every $x \in \mathrm{Im}(f) \cap \mathrm{Def}(X)$ there exists a dr-map $g_x \in \mathrm{End}(X)$ with $g_x \leqslant f$ and $d(g_x) = x$;*

(11) *if* $f, g \in \mathrm{End}(X)$ *are dr-maps and* $h \in \mathrm{End}(X)$, *then* $h(\mathrm{Im}(f)) = \mathrm{Im}(g)$ *if and only if* $ghf = hf$ *and* $g'hf \neq hf$ *for every idempotent* $g' \in \mathrm{End}(X)$ *with* $g' < g$.

Proof. From Lemma 2.2 and Statement 2.1(6) we obtain (1).

Next we turn to (2), (3) and (4). If $f \in \mathrm{End}(X)$ is a dr-map then, for any $g \in \mathrm{End}(X)$, either $d(f) \in \mathrm{Im}(fg)$ and hence $hfg \neq fg$ for any r-map $h \in \mathrm{End}(X)$, or else $d(f) \notin \mathrm{Im}(fg)$ and $r(f)fg = fg$ for any r-map $r(f) \in \mathrm{End}(X)$. Thus if $g \in \mathrm{End}(X)$ is a dr-map such that either $d(f) = d(g)$, or $d(f) \leqslant d(g)$ are min-defective, or $d(f) \geqslant d(g)$ are max-defective, then $hfg \neq fg$ for any r-map $h \in \mathrm{End}(X)$. Conversely, if either x, y are min-defective and $x \nleqslant y$, or x, y are max-defective and $x \ngeqslant y$, or x, y are doubly defective and $x \neq y$ then, by Lemma 2.2 there exists a dr-map $f \in \mathrm{End}(X)$ with $d(f) = x \neq f(y)$. Then for any dr-map $g \in \mathrm{End}(X)$ with $d(g) = y$ we have $r(f)fg = fg$ for any r-map $r(f) \in \mathrm{End}(X)$. This completes the proof of (2), (3) and (4).

Let $x \in \mathrm{Def}(X)$ and $y \neq x$. If the Stone nuclei of $K(x)$ and $K(y)$ are not isomorphic then (5) follows from Statement 2.1(8). If $\mathrm{Nuc}(K(x)) \cong \mathrm{Nuc}(K(y))$, then there is an r-map g such that $g(K(x)) \subseteq K(y)$ by Statement 2.1(4), and g maps $\mathrm{Ext}(K(x))$ bijectively onto $\mathrm{Ext}(K(y))$. Assume that $g(y) = g(x)$. If $y \in \mathrm{Ext}(X)$, then $x = f(x) \neq f(y)$ for any dr-map f with $d(f) = x$. If $y \in \mathrm{Mid}(X)$ then x, y are both min-defective (or max-defective or doubly defective), and there exists a dr-map f with $f(x) \neq f(y)$, by Lemma 2.2. If $g(y) \neq g(x)$, then the existence of such an f is clear. This proves (5).

If ψ is an R-isomorphism, then (a) in (6) follows from the definition of a dr-map, and (b) in (6) is a consequence of (a) and (4).

Let $x_0, x_1 \in \mathrm{Def}(X)$. By Statement 2.1(4) and 2.1(6), for $i = 0, 1$ there exist r-maps $h_i \in \mathrm{End}(X)$ with $h_i f_i = h_i$, $f_i h_i = f_i$ and such that, for every $y \in \mathrm{Mid}(X)$, $E(y) \cap ((x_i] \cup [x_i)) \neq \emptyset$ implies $\mathrm{Im}(h_i) \cap E(y) \cap ((x_i] \cup [x_i)) \neq \emptyset$. By Statement 2.1(5), there exist r-maps $g'_i \in \mathrm{End}(X)$ such that $g'_i g'_{1-i} = g'_i$, $g'_i \in [h_i]$, and $f_i g'_i h_{1-i} = f_{1-i}$. Lemma 2.2 then supplies dr-maps $g_i \in \mathrm{End}(X)$ with $d(g_i) = x_i$ and $g'_i = g'_i g_i$, and (7) follows.

(8) follows from Statement 2.1(6) and Lemma 2.2.

To prove (9), let $f \in \mathrm{End}(X)$ be a dr-map with $d(f) = x$. If $g \in \mathrm{End}(X)$ is an r-map with $gf = g$ and $g < f$, then $\mathrm{Im}(g) = \mathrm{Im}(f) \setminus \{x\}$ and hence for any $u \in X$, $g(u) = gf(u) = f(u)$ whenever $f(u) \neq x$, and $g(u) = gf(u) = g(x)$ whenever $f(u) = x$. Since $g(u) = u$ for any $u \in \mathrm{Ext}(x)$, we conclude that if x is min-defective and $\{y\} = \mathrm{Min}(x)$ then $g(x) = y$, if x is max-defective and $\{z\} = \mathrm{Max}(x)$ then $g(x) = z$. If x is doubly defective and $\{y\} = \mathrm{Min}(x)$, $\{z\} = \mathrm{Max}(x)$ then $g(x) = y$ or $g(x) = z$, and both cases occur. Thus (9) is proved.

Let $f \in \mathrm{End}(X)$ be an idempotent such that $f > g$ for some r-map $g \in \mathrm{End}(X)$ and let $x \in \mathrm{Im}(f) \cap \mathrm{Def}(X)$. Then any Stone kernel of $\mathrm{Im}(f)$ is isomorphic to any Stone kernel of X. Since any $z \in \mathrm{Im}(f)$ is defective in $\mathrm{Im}(f)$ exactly when it is defective in X, from (1) we obtain a dr-map $g'_x \in \mathrm{End}(\mathrm{Im}(f))$ with $d(g'_x) = x$. But then $g_x = g'_x f \in \mathrm{End}(X)$ is a dr-map with $d(g_x) = x$ and $g_x \leqslant f$. This proves (10).

Let $f, g \in \mathrm{End}(X)$ be dr-maps and $h \in \mathrm{End}(X)$. If $h(\mathrm{Im}(f)) = \mathrm{Im}(g)$ then it is clear that the condition in (11) is satisfied. Conversely, assume that the condition holds. Then $\mathrm{Im}(hf) \subseteq \mathrm{Im}(g)$ and, by Statement 1.6, $\mathrm{Im}(hf)$ and $\mathrm{Im}(g)$ intersect the same components. From (C) it follows that $\mathrm{Im}(r(g)) \subseteq \mathrm{Im}(hf)$. From $\mathrm{Im}(hf) \subseteq \mathrm{Im}(g)$ and $r(g)hf \neq hf$ we then obtain that $d(g) \in \mathrm{Im}(hf)$. Thus $h(\mathrm{Im}(f)) = \mathrm{Im}(g)$, and (11) is proved. □

Theorem 2.4. *A space $X \in \mathbb{AR}$ is finite if and only if $\mathrm{End}(X)$ is finite.*

Proof. Clearly, if X is finite then $\mathrm{End}(X)$ is finite. Conversely, if X is infinite then for every $x \in X$ there exists an r-map or a dr-map $f_x \in \mathrm{End}(X)$ with $x \in \mathrm{Im}(f_x)$. Since $\mathrm{Im}(f_x)$ is finite, it follows that the set $\{f_x \mid x \in X\} \subseteq \mathrm{End}(X)$ is infinite. □

Definition. An idempotent $f \in \mathrm{End}(X)$ is a *br-map* if and only if f is $\leqslant$-maximal amongst idempotents with the property

(q) $$g_0, g_1 < f,\ g_0 g_1 = g_0,\ g_1 g_0 = g_1 \text{ imply } g_0 = g_1.$$

First we prove a technical lemma.

Lemma 2.5. *If $f \in \mathrm{End}(X)$ is an idempotent satisfying* (q)*, then* $\mathrm{Im}(f)$ *satisfies* (r2) *and the following condition:*

(b1) *If $x, y \in \mathrm{Im}(f) \cap \mathrm{Mid}(X)$ are distinct and $\mathrm{Ext}(x) = \mathrm{Ext}(y)$, then x, y are both either min-defective or max-defective, and*

$$((x] \cup [x)) \cap (\mathrm{Mid}(\mathrm{Im}(f)) \setminus \mathrm{Def}(X)) \neq ((y] \cup [y)) \cap (\mathrm{Mid}(\mathrm{Im}(f)) \setminus \mathrm{Def}(X)).$$

Proof. Assume that $f \in \mathrm{End}(X)$ satisfies (q).

First we observe that any idempotents $h_0, h_1 \in \mathrm{End}(\mathrm{Im}(f))$ satisfying $h_i h_{1-i} = h_i$ for $i = 0, 1$ must coincide. Indeed, if $h_0 \neq h_1$ then the maps $g_i = h_i f \in \mathrm{End}(X)$ would be distinct idempotents satisfying $g_i g_{1-i} = g_i$ for $i = 0, 1$—a contradiction.

This observation and Statement 2.1(4) imply that any two Stone kernels of $\mathrm{Im}(f)$ coincide, so that there is exactly one equivalence class $[f']$ of r-maps of $\mathrm{Im}(f)$. Therefore distinct components intersecting $\mathrm{Im}(f)$ must have non-isomorphic Stone nuclei, and this proves (r2).

We turn to (b1). Suppose that $x, y \in \mathrm{Im}(f) \cap \mathrm{Mid}(X)$ are distinct and such that $\mathrm{Ext}(x) = \mathrm{Ext}(y)$. If x, y are non-defective, then Statement 2.1(6) implies the existence of two distinct Stone kernels of $\mathrm{Im}(f)$, contradicting the previous paragraph. Thus x and y are defective. If

$$((x] \cup [x)) \cap (\mathrm{Mid}(\mathrm{Im}(f)) \setminus \mathrm{Def}(X)) = ((y] \cup [y)) \cap (\mathrm{Mid}(\mathrm{Im}(f)) \setminus \mathrm{Def}(X)),$$

then we apply Theorem 1.7 or its dual to f', $F_1 = \{x, y\}$, and x or y, to obtain dr-maps f_0, f_1 of $\mathrm{Im}(f)$ such that $r(f_0) = r(f_1) = f'$, $d(f_0) = x$, $d(f_1) = y$, and $f_1(x) = y$, $f_0(y) = x$. Thus $f_i f_{1-i} = f_i$ and $f_0 \neq f_1$ – a contradiction to the initial observation. Therefore

$$((x] \cup [x)) \cap (\mathrm{Mid}(\mathrm{Im}(f)) \setminus \mathrm{Def}(X)) \neq ((y] \cup [y)) \cap (\mathrm{Mid}(\mathrm{Im}(f)) \setminus \mathrm{Def}(X)),$$

which also shows that x and y cannot be doubly defective. From $\mathrm{Ext}(x) = \mathrm{Ext}(y)$ it then follows that x and y are both either min-defective or max-defective. This demonstrates (b1). □

Statement 2.6. *Let $X, Y \in \mathbb{AR}$. Then:*

(1) *there exists a br-map $f \in \mathrm{End}(X)$;*
(2) *the image $\mathrm{Im}(f)$ of any br-map f is finite;*
(3) *for any br-map $f \in \mathrm{End}(X)$ there exists exactly one equivalence class $[g]$ of r-maps $g \leqslant f$;*
(4) *if $\psi\colon \mathrm{End}(X) \to \mathrm{End}(Y)$ is an isomorphism then f is a br-map if and only if $\psi(f)$ is a br-map.*

P r o o f. Let $g \in \mathrm{End}(X)$ be an r-map. Let $\mathscr{H}$ denote the set of all classes of idempotents $h \in \mathrm{End}(X)$ satisfying (q) and $h \geqslant g$. Then $\mathscr{H} \neq \emptyset$ because $g \in \mathscr{H}$. If $h \in \mathscr{H}$ then $\mathrm{Im}(h) \setminus \mathrm{Def}(X) = \mathrm{Im}(g)$ by Lemma 2.5 and, moreover, $|\mathrm{Im}(h) \setminus \mathrm{Im}(g)| < 2^{|\mathrm{Im}(g)|+1}$. Therefore any chain in $\mathscr{H}$ with respect to $\leqslant$ has the length at most $2^{|\mathrm{Im}(g)|+1}$ and thus $\mathscr{H}$ has a maximal element $[f]$. Any maximal element of $\mathscr{H}$ is a br-map, and (1) is proved.

(2) follows from Lemma 2.5.

To prove (3), consider a br-map $f \in \mathrm{End}(X)$. First we prove that $\mathrm{Im}(f)$ satisfies (r1). To do so, suppose that there is a component D such that $\mathrm{Nuc}(C) \not\cong \mathrm{Nuc}(D)$ for every $C \in \mathbb{C}(X)$ with $\mathrm{Im}(f) \cap C \neq \emptyset$. Then Lemma 1.8 implies the existence of an idempotent h with $\mathrm{Im}(h) = \mathrm{Im}(f) \cup N$ for some dp-subspace $N \subseteq D$ isomorphic to $\mathrm{Nuc}(D)$.

Suppose that $g_0, g_1 < h$ satisfy $g_i g_{1-i} = g_i$ for $i = 0, 1$. Since $\mathrm{Nuc}(D) \not\cong \mathrm{Nuc}(C)$ for every $C \in \mathbb{C}(X)$ intersecting $\mathrm{Im}(f)$, either $\mathrm{Im}(g_i) \cap D = N$ or $\mathrm{Im}(g_i) \cap D = \emptyset$ for

$i = 0, 1$. In the second case $g_i \leqslant f$ for $i = 0, 1$, and hence $g_0 = g_1$ because f satisfies (q). In the first case, we have $g_0(x) = g_1(x)$ for all $x \in g_0^{-1}(N)$. If $g_i f g_i(D) \nsubseteq \mathrm{Im}(f)$, then $g_i f g_i(D) = N$ follows from (C) because $g_i \leqslant h$, and hence $g_i f(N) = N$. But then $\mathrm{Nuc}(K(f(N))) \cong N$, a contradiction. Therefore $g'_i = g_i f g_i \leqslant f$ is an idempotent, $g'_i g'_{1-i} = g'_i$ for $i = 0, 1$, and $g_0 = g_1$ exactly when $g'_0 = g'_1$. But f satisfies (q), and $g_0 = g_1$ follows. Therefore (q) holds for $h > f$, in contradiction to the maximality of f. This shows that f satisfies (r1).

Since $f \in \mathrm{End}(X)$ is an idempotent, (B) implies that for any component $C \in \mathbb{C}(X)$ either $C \cap \mathrm{Im}(f) = \emptyset$ or $C \cap \mathrm{Im}(f)$ contains a *dp*-subspace isomorphic to $\mathrm{Nuc}(C)$. Since f also satisfies (r1), there exists a Stone kernel S of X contained in $\mathrm{Im}(f)$ and hence, by Statement 2.1(1), there exists an r-map $g \leqslant f$. The unicity of the equivalence class of r-maps contained in $\mathrm{Im}(f)$ then follows from Lemma 2.5. This proves (3).

From the definition of a br-map we immediately obtain (4). □

3. $2r$-maps

In this section, we introduce $2r$-maps—idempotent endomorphisms reflecting specific relations of two r-maps. We begin with the definition of a supremum of a finite set of idempotents.

Definition and notation. Let $\mathscr{A} \subseteq \mathrm{End}(X)$ be a finite set of idempotents. We shall write $h = \sup \mathscr{A}$ to denote any idempotent $h \in \mathrm{End}(X)$ satisfying $h \geqslant f$ for every $f \in \mathscr{A}$, and such that $k \geqslant h$ for every idempotent $k \in \mathrm{End}(X)$ satisfying $k \geqslant f$ for all $f \in \mathscr{A}$.

It is clear that any idempotent $h \in \mathrm{End}(X)$ with $\mathrm{Im}(h) = \bigcup\{\mathrm{Im}(f) \mid f \in \mathscr{A}\}$ is a supremum of a given finite set $\mathscr{A} \subseteq \mathrm{End}(X)$ of idempotents.

Definition and notation. An idempotent $f \in \mathrm{End}(X)$ is called a *$2r$-map* if there exist non-equivalent r-maps $g_0, g_1 < f$ such that $f = \sup\{g_0, g_1\}$ and $g \in [g_0] \cup [g_1]$ for any r-map $g < f$.

For any $2r$-map f we denote

$$\Delta f = \Delta(\mathrm{Im}(g_0), \mathrm{Im}(g_1)) = (\mathrm{Im}(g_0) \setminus \mathrm{Im}(g_1)) \cup (\mathrm{Im}(g_1) \setminus \mathrm{Im}(g_0)).$$

Lemma 3.1. *If f is a $2r$-map, then exactly one of the following two cases occurs:*

(1) *there exist distinct $C_0, C_1 \in \mathbb{C}(X)$ with isomorphic Stone nuclei such that $\Delta f \subseteq C_0 \cup C_1$ and $\Delta f \cap C_i \cong \mathrm{Nuc}(C_i)$ for $i = 0, 1$,*
(2) *there exist distinct non-defective $x_0, x_1 \in \mathrm{Mid}(X)$ such that $x_1 \in E(x_0)$ and $\Delta f = \{x_0, x_1\}$.*

In either case, if $g_0, g_1 \leqslant f$ are non-equivalent r-maps then $\mathrm{Im}(f) = \mathrm{Im}(g_0) \cup \mathrm{Im}(g_1)$.

Proof. Let f be a $2r$-map, and let $g_0, g_1 < f$ be r-maps with $[g_0] \neq [g_1]$.

First suppose that $\mathrm{Im}(g_0) \cap C_0 \neq \emptyset = \mathrm{Im}(g_1) \cap C_0$ for some $C_0 \in \mathbb{C}(X)$. Since $\mathrm{Im}(g_1)$ is a Stone kernel of X, there must exist a $C_1 \in \mathbb{C}(X) \setminus \{C_0\}$ with $\mathrm{Nuc}(C_0) \cong \mathrm{Im}(g_1) \cap C_1$, and $\mathrm{Im}(g_0) \cap C_1 = \emptyset$ because g_0 satisfies (r2). Also, the dp-subspace $S = (\mathrm{Im}(g_0) \cap C_0) \cup (\mathrm{Im}(g_1) \setminus C_1)$ is a Stone kernel of X, and thus there exists an r-map g_2 with $\mathrm{Im}(g_2) = S \subseteq \mathrm{Im}(f)$, by Statement 2.1(1). Clearly $g_2 \notin [g_1]$ and, since f is a $2r$-map, this implies that $g_2 \in [g_0]$. But then $\mathrm{Im}(g_1) \setminus C_1 = S \setminus C_0 = \mathrm{Im}(g_0) \setminus C_0$. By Theorem 1.5, there exists an idempotent dp-map $h \in \mathrm{End}(X)$ with $\mathrm{Im}(h) = \mathrm{Im}(g_0) \cup \mathrm{Im}(g_1)$. But then $\mathrm{Im}(f) = \mathrm{Im}(g_0) \cup \mathrm{Im}(g_1)$ because $f = \sup\{g_0, g_1\}$. Hence $\Delta f \subseteq C_0 \cup C_1$, and $\Delta f \cap C_i = \mathrm{Im}(g_i) \cap C_i$ is isomorphic to $\mathrm{Nuc}(C_i)$ for $i = 0, 1$. This describes the first case and proves that $\mathrm{Im}(f) = \mathrm{Im}(g_0) \cup \mathrm{Im}(g_1)$ in this case.

We may thus assume that $\mathrm{Im}(g_0)$ and $\mathrm{Im}(g_1)$ intersect the same components of X. Then $\mathrm{Ext}(\mathrm{Im}(g_0)) = \mathrm{Ext}(\mathrm{Im}(g_1))$, and hence $\Delta f \subseteq \mathrm{Mid}(\mathrm{Im}(g_0) \cup \mathrm{Im}(g_1))$. Since f is a $2r$-map and g_0, g_1 are r-maps, Statement 2.1(6) implies that, for $i = 0, 1$, there exists exactly one $x_i \in \mathrm{Im}(g_i) \setminus \mathrm{Im}(g_{1-i})$, and $x_1 \in E(x_0)$. Then $\Delta f = \{x_0, x_1\}$. This concludes the proof of the first statement.

To prove the second statement in case of $\Delta f = \{x_0, x_1\}$, we set $g = g_0 f$, and note that $gf = g \in [g_0]$. Define $h\colon X \longrightarrow X$ by

$$h(t) = \begin{cases} x_1 & \text{for } t \in f^{-1}\{x_1\}, \\ g(t) & \text{for } t \in X \setminus f^{-1}\{x_1\}. \end{cases}$$

It is clear that h is an idempotent with $\mathrm{Im}(h) = \mathrm{Im}(g_0) \cup \mathrm{Im}(g_1)$. Since $f^{-1}\{x_1\} \subseteq g^{-1}\{x_0\}$ and these two sets are clopen and convex, and from the choice of g, it follows that $h \in \mathrm{End}(X)$. But then $\mathrm{Im}(f) = \mathrm{Im}(g_0) \cup \mathrm{Im}(g_1)$ because $f = \sup\{g_0, g_1\}$. $\square$

Definition. We now specify five types of $2r$-maps as follows:

c2r-map—this is any $2r$-map f such that Δf is a disjoint union of two isomorphic Stone nuclei,

p2r-map—this is any $2r$-map f such that Δf consists of two non-defective elements from $\mathrm{Mid}(X)$,

t2r-map—this is any $2r$-map f for which there exist an $h \in \mathrm{End}(X)$ and an r-map $g < f$ such that $hg < f$ is an r-map, $h^2 g = g$ and $hg \notin [g]$,

n2r-map—this is any $2r$-map f which is not a $t2r$-map.

e2r-map—this is any $n2r$-map f—with its non-equivalent r-maps $g_0, g_1 < f$—such that for every r-map $g \notin [g_0] \cup [g_1]$ for which there exist $n2r$-maps $f_0 > g, g_0$ and $f_1 > g, g_1$, and for all $g_0' \in [g_0]$, $g_1' \in [g_1]$ and $h \in \mathrm{End}(X)$ such that hg_0', hg_1' are equivalent r-maps, we have $hg \lesssim hg_0'$.

We proceed to interpret these properties in structural terms.

Lemma 3.2. *A $2r$-map f is a $t2r$-map if and only if f is either a $c2r$-map or a $p2r$-map for which $\Delta f = \{x_0, x_1\}$ is an antichain.*

Definition. Any $t2r$-map f for which $\Delta f \subseteq \mathrm{Mid}(X)$ is an antichain will be called a *$pt2r$-map*.

P r o o f o f L e m m a 3.2. Let f be a $t2r$-map, and let $g < f$ and h be as in the definition above. If f is not a $c2r$-map then, by Lemma 3.1, there is an $x_0 \in \mathrm{Mid}(X)$ such that $\Delta f = \{x_0, x_1\} \subseteq E(x_0)$. If $x_0 \in \mathrm{Im}(g)$, then $h(x_0) = x_1$ because hg is an r-map satisfying $hg \notin [g]$, and $h(x_1) = x_0$ because $h^2 g = g$. Since h preserves order, the set $\{x_0, x_1\}$ must be an antichain.

To prove the converse, let f be a $c2r$-map, and let C_0, C_1 be distinct components with $\Delta f \subseteq C_0 \cup C_1$. Let $g_0, g_1 < f$ be non-equivalent r-maps with $\mathrm{Im}(g_i) \cap C_i \neq \emptyset$. By Statement 2.1(4), we can assume that $g_i g_{1-i} = g_i$ for $i = 0, 1$. Define a mapping h by

$$h(x) = \begin{cases} g_1 f(x) & \text{if } x \in f^{-1}(C_0), \\ g_0 f(x) & \text{if } x \in f^{-1}(C_1), \\ f(x) & \text{if } x \in X \setminus f^{-1}(C_0 \cup C_1). \end{cases}$$

Since the image $\mathrm{Im}(f)$ of $f \in \mathrm{End}(X)$ is finite, the map h is continuous, and $h \in \mathrm{End}(X)$ follows. Clearly $hg_i = g_{1-i}$ for $i = 0, 1$, and hence f is a $t2r$-map.

Let f be a $2r$-map for which $\Delta f = \{x_0, x_1\} \subseteq E(x_0)$ is an antichain, and let $g_0, g_1 < f$ be non-equivalent r-maps with $x_i \in \mathrm{Im}(g_i)$ for $i = 0, 1$. Then $g_i g_{1-i} = g_i$ for $i = 0, 1$. Formal replacement of C_i by $\{x_i\}$ in the above definition of h defines a dp-map h' because $\{x_0, x_1\} \subseteq E(x_0)$ is an antichain. From $hg_i = g_{1-i}$ for $i = 0, 1$ it then follows that f is a $t2r$-map. $\square$

Remark. Thus any $c2r$-map is a $t2r$-map, and any $n2r$-map is a $p2r$-map.

Corollary 3.3. *Let X have distinct components C_i with isomorphic Stone nuclei $N_i \subseteq C_i$ for $i = 0, 1$. Then for every r-map g_0 with $\mathrm{Im}(g_0) \cap C_0 = N_0$ there exists a $c2r$-map f with $\mathrm{Im}(f) = \mathrm{Im}(g_0) \cup N_1$.*

P r o o f. Since g_0 is an r-map, $C_1 \cap \mathrm{Im}(g_0) = \emptyset$. By Lemma 1.8, there is an idempotent $f \in \mathrm{End}(X)$ with $\mathrm{Im}(f) = \mathrm{Im}(g_0) \cup N_1$. But then $\mathrm{Im}(f) \setminus N_1$ and $\mathrm{Im}(f) \setminus N_0$ are the only two Stone kernels contained in $\mathrm{Im}(f)$, and hence f is a $c2r$-map. $\square$

Lemma 3.4. *A $2r$-map f is an $n2r$-map if and only if $\Delta f = \{x, y\} \subseteq \mathrm{Mid}(X)$ is a 2-element chain with $y \in E(x)$.*

Furthermore, if $x \in \mathrm{Mid}(X)$ is non-defective and $y \in E(x)$ is such that $x < y$ then, for any closed sets $E_0, E_1 \subseteq E(x)$ such that $x \in E_0$, $y \in E_1$, $(E_0] \cap [E_1) = \emptyset$ and for every r-map g with $x \in \mathrm{Im}(g)$, there exists an $n2r$-map f with $\mathrm{Im}(f) = \mathrm{Im}(g) \cup \{y\}$ and $E_0 \subseteq f^{-1}\{x\}$, $E_1 \subseteq f^{-1}\{y\}$.

Proof. Since an $n2r$-map f is not a $t2r$-map, the set $\Delta f = \{x, y\}$ must be a 2-element chain, by Lemmas 3.1 and 3.2.

The second statement follows immediately from Theorem 1.7. $\square$

Next we give a sufficient condition for the existence of a $pt2r$-map f with a given Δf. We note that, in general, the requirement that $\Delta f = \{x, y\}$ be an antichain does not suffice.

Lemma 3.5. *Let $x \in X \in \mathbb{AR}$ be non-defective and such that $E(x)$ is an antichain, and let $E_0, E_1 \subseteq E(x)$ be closed disjoint sets with $y \in E_0$ and $x \in E_1$. Then for any r-map g with $x \in \mathrm{Im}(g)$, there is a $t2r$-map $f > g$ such that $\Delta f = \{x, y\}$, $E_0 \subseteq f^{-1}\{y\}$ and $E_1 \subseteq f^{-1}\{x\}$.*

Proof. Since $g(E(x)) = \{x\}$, we have $E_0 \cup E_1 \subseteq g^{-1}\{x\} = Z$. The set Z is clopen, convex and $Z \cap K(x) = E(x)$.

There is a clopen decreasing set $U \subseteq X$ such that $E_0 \subseteq U$ and $E_1 \subseteq Z \setminus U$. The set $V = Z \cap U \cap (Z \setminus U]$ is then closed and, since $Z \cap K(x) = E(x)$ is an antichain, we must have $V \cap K(x) = \emptyset$. The union of components $S = K(V)$ and the component $K(x)$ are closed, by Lemma 1.2, and $S \cap K(x) = \emptyset$. Hence there is a clopen decreasing set T' such that $T' \cap S = \emptyset$ and $K(x) \subseteq T'$. By Lemma 1.2, the union of components $T = K(T')$ is clopen, $E(x) \subseteq T$ and $T \cap S = \emptyset$. Then the set $U' = T \cap U$ is clopen and decreasing, and such that $W = U' \cap Z$ is decreasing in Z, $E_0 \subseteq W$ and $E_1 \cap W = \emptyset$. We claim that W is also increasing in Z. Indeed, if $u < v$ for some $u \in W$ and $v \in Z \setminus W$, then $u, v \in T \cap Z$, $u \in U$ and $v \notin U$, so that $u \in V \subseteq S$—a contradiction because $T \cap S = \emptyset$. Hence W is clopen, increasing and decreasing in Z, $E_0 \subseteq W$, and $E_1 \cap W = \emptyset$. Set

$$f(t) = \begin{cases} y & \text{for } t \in W, \\ g(t) & \text{for } t \in X \setminus W. \end{cases}$$

Then $f \in \mathrm{End}(X)$ because $\mathrm{Im}(g)$ is finite. Since $\mathrm{Im}(f)$ contains no Stone kernels other than $\mathrm{Im}(g)$ and $(\mathrm{Im}(g) \setminus \{x\}) \cup \{y\}$, we conclude that f is a $2r$-map. By Lemma 3.2, the idempotent f is a $t2r$-map with $\Delta f = \{x, y\}$, $E_0 \subseteq f^{-1}\{y\}$ and $E_1 \subseteq f^{-1}\{x\}$. $\square$

Statements 2.1(8), 2.3(5), Lemmas 3.4 and 3.5, and Corollary 3.3 give the following claim.

Corollary 3.6. *If $X \in \mathbb{AR}$ then for every pair of distinct points $u, v \in X$ there exists either an r-map or a dr-map or a $2r$-map $f \in \mathrm{End}(X)$ with $f(u) \neq f(v)$.*

Since the set $E(x) \subseteq \mathrm{Mid}(X)$ is closed whenever $x \in \mathrm{Mid}(X)$ is non-defective, see Lemma 1.2, every such x is comparable to a minimal and a maximal element of $E(x)$.

Next we characterize $e2r$-maps.

Lemma 3.7. *Let f be an $n2r$-map of an $X \in \mathbb{AR}$ with $\Delta f = \{x, y\}$ and $x < y$. Then f is an $e2r$-map if and only if x is minimal in $E(x)$ and y is maximal in $E(x)$.*

Proof. Let f be an $n2r$-map with $\Delta f = \{x, y\} \subseteq E(x)$, and let $g_0, g_1 < f$ be r-maps such that $x \in \mathrm{Im}(g_0)$ and $y \in \mathrm{Im}(g_1)$. Hence $g_0(y) = g_0(x) = x$ and $g_1(y) = g_1(x) = y$.

If x is not minimal in $E(x)$, then $E(x)$ contains a chain $z < x < y$. Statement 2.1(6) and Lemma 3.4 supply an r-map $g \notin [g_0] \cup [g_1]$ with $z \in \mathrm{Im}(g)$ and $n2r$-maps $f_i > g, g_i$ such that $\mathrm{Im}(f_i) = \mathrm{Im}(g_i) \cup \{z\}$ for $i = 0, 1$. Hence $f_0(y) = x$. Set $h = f_0$. Then $hg_1, g_0 \in [hg_0]$, $hg = g$, and $hg_0hg = g_0hg \neq hg$ because $g_0hg(z) \neq z = hg(z)$, and hence f is not an $e2r$-map. A dual argument applies when y is not maximal in $E(x)$.

For the converse, suppose that $x < y$ are extremal in $E(x)$. Let g be an r-map and let $f_i > g, g_i$ be $n2r$-maps for $i = 0, 1$. Then $\Delta f_0 = \{x, z\}$ and $\Delta f_1 = \{y, z\}$ for some $z \in E(x)$ comparable to both x and y, and this is possible only when $x < z < y$.

Let $h \in \mathrm{End}(X)$ be such that hg_0 and hg_1 are equivalent r-maps. By Statement 2.1(2), h is one-to-one on $\mathrm{Im}(g_i)$ for $i = 0, 1$. Thus $h(x) = h(z) = h(y)$. Since $\mathrm{Im}(g) \setminus \{z\} = \mathrm{Im}(g_0) \setminus \{x\}$, we obtain $\mathrm{Im}(hg) = \mathrm{Im}(hg_0)$ and hence $hg \lesssim hg_0$. Therefore f is an $e2r$-map. $\square$

Next we prove a statement concerning comparability of doubly defective points.

Lemma 3.8. *Let $x, y \in X$ be doubly defective. Then x, y are comparable if and only if there exist dr-maps $f, g \in \mathrm{End}(X)$ satisfying $d(f) = x$, $d(g) = y$ and $r(f) = r(g)$, and for any such maps there exists a $k = \sup\{f, g\}$ such that*

(1) *if $h \leqslant k$ is a dr-map, then $h \in [f] \cup [g]$;*
(2) *if $h \leqslant k$ is an r-map, then $h \in [r(f)]$;*

and $hg \neq f$ for every $h \in \mathrm{End}(X)$ with $hf = g$.

Moreover, if x, y are comparable, then $\mathrm{Im}(k) = \mathrm{Im}(f) \cup \mathrm{Im}(g)$.

Proof. If $x, y \in X$ are comparable and doubly defective, and if $f \in \mathrm{End}(X)$ is a dr-map with $d(f) = x$, then, by Lemma 2.2, there exists a dr-map $g \in \mathrm{End}(X)$ with $d(g) = y$ and $r(f) = r(g)$ and, by Theorem 1.7 or its dual, there exists an idempotent

$k \in \mathrm{End}(X)$ with $\mathrm{Im}(k) = \mathrm{Im}(f) \cup \{y\}$. Then $k = \sup\{f, g\}$ and thus any dr-map $h \leqslant k$ belongs to $[f] \cup [g]$, and any r-map $h \leqslant k$ belongs to $[r(f)]$. If $h\colon X \longrightarrow X$ is a mapping such that $hf = g$ and $hg = f$, then $h(x) = y$ and $h(y) = x$. Hence h cannot preserve ordering, and therefore h is not a dp-map. Thus both statements hold.

Conversely, assume that $f, g \in \mathrm{End}(X)$ are dr-maps such that $d(f) = x$, $d(g) = y$, $r(f) = r(g)$ where $x, y \in X$ are doubly defective, and that there exists a $k = \sup\{f, g\}$ satisfying both conditions. By Statement 2.1(9), $\mathrm{Im}(k) \setminus \mathrm{Def}(X) = \mathrm{Im}(r(f))$ and, by Statement 2.3(10), $\mathrm{Im}(k) \cap \mathrm{Def}(X) = \{x, y\}$. Thus $\mathrm{Im}(k) = \mathrm{Im}(f) \cup \mathrm{Im}(g)$ and $K(x) = K(y)$. If x, y are incomparable, we define

$$h(t) = \begin{cases} k(t) & \text{for } t \in X \setminus k^{-1}\{x, y\}, \\ y & \text{for } t \in k^{-1}\{x\}, \\ x & \text{for } t \in k^{-1}\{y\} \end{cases}$$

Obviously, $h \in \mathrm{End}(X)$, and $fk, gk \in \mathrm{End}(X)$ are dr-maps with $d(fk) = x$, $d(gk) = y$, $r(fk) = r(gk)$, $k = \sup\{fk, gk\}$, and $hfk = gk$ and $hgk = fk$—a contradiction. Thus x and y must be comparable. □

Definition and notation. Any dp-map k satisfying conditions of Lemma 3.8 will be called an *ndr-map*. For any ndr-map k, write $\Delta k = \{x, y\}$.

Statement 3.9. *Let $X \in \mathbb{AR}$. Then:*

(1) *if $f_0, f_1 \in \mathrm{End}(X)$ are p2r-maps satisfying $\mathrm{Nuc}(K(\Delta f_0)) \cong \mathrm{Nuc}(K(\Delta f_1))$ and such that $g(\Delta f_0) \subseteq E(\Delta f_1)$ for some dp-map g, then there exists a $k \in \mathrm{End}(X)$ with $k(z) = z$ for all $z \in \mathrm{Ext}(\mathrm{Im}(f_1))$ and $k(\mathrm{Im}(f_0)) = \mathrm{Im}(f_1)$ whenever f_0 is a pt2r-map or f_1 is an n2r-map;*

(2) *if $f_0, f_1 \in \mathrm{End}(X)$ are ndr-maps, then $k(\mathrm{Im}(f_0)) = \mathrm{Im}(f_1)$ for some $k \in \mathrm{End}(X)$;*

(3) *if $f_0, f_1 \in \mathrm{End}(X)$ are both either p2r-maps or ndr-maps, and if $h \in \mathrm{End}(X)$, then $h(\mathrm{Im}(f_0)) = \mathrm{Im}(f_1)$ (and hence also $h(\Delta f_0) = \Delta f_1$) if and only if $f_1 h f_0 = h f_0$ and $k h f_0 \neq h f_0$ for every $k \in \mathrm{End}(X)$ with $k < f_1$;*

(4) *if $f_i \in \mathrm{End}(X)$ are c2r-maps such that $\mathrm{Nuc}(K(x_0)) \cong \mathrm{Nuc}(K(x_1))$ for $x_i \in \Delta f_i$ with $i = 0, 1$, then there exist $h_i \in [f_i]$ with $h_i h_{1-i} = h_i$ for $i = 0, 1$.*

(5) *Let f be an n2r-map or a pt2r-map for which $E(\Delta f)$ is an antichain. If $g, g_0, g_1 \in \mathrm{End}(X)$ are r-maps such that $g < f$, $\mathrm{Im}(g_i) \cap E(\Delta f) \neq \emptyset$ and $g_i g = g_i$, $g g_i = g$ for $i = 0, 1$, then $\mathrm{Im}(g_0) \cap E(\Delta f) = \mathrm{Im}(g_1) \cap E(\Delta f)$ if and only if $f' g_0 = f' g_1$ for all $f' \in [f]$ such that $f' g_i$ are r-maps for $i = 0, 1$.*

P r o o f. Let $f_0, f_1 \in \mathrm{End}(X)$ be either $p2r$-maps with $\mathrm{Nuc}(K(\Delta f_0)) \cong \mathrm{Nuc}(K(\Delta f_1))$ or arbitrary ndr-maps. By Statement 2.1(4), there exist an $f_0' \in [f_0]$

and an r-map $h \in \mathrm{End}(X)$ such that $hf_0' \leqslant f_1$ is an r-map. It is clear that $hf_0'(z) = z$ for all $z \in \mathrm{Ext}(\mathrm{Im}(f_1))$. In case when f_0, f_1 are $p2r$-maps and $\mathrm{Nuc}(K(\Delta f_0)) \cong \mathrm{Nuc}(K(\Delta f_1))$, Lemma 1.8 and the existence of a dp-map g with $g(\Delta f_0) \subseteq E(\Delta f_1)$ allow us to assume that $h(\Delta f_0) \subseteq E(\Delta f_1)$ as well.

Suppose that $\Delta f_i = \{x_i, y_i\}$, where either x_i and y_i are incomparable or $x_i < y_i$ for $i = 0, 1$. To prove (1) and (2), define a mapping k by

$$k(u) = \begin{cases} hf_0'(u) & \text{if } f_0'(u) \notin \Delta f_0, \\ x_1 & \text{if } f_0'(u) = x_0, \\ y_1 & \text{if } f_0'(u) = y_0. \end{cases}$$

Since $\mathrm{Im}(f_0)$ is finite and f_0' and h are dp-maps, we conclude that k is continuous, has the dp-property, and $k(\mathrm{Im}(f_0)) = \mathrm{Im}(f_1)$. Furthermore, $k(z) = z$ for all $z \in \mathrm{Ext}(\mathrm{Im}(f_1))$ follows from a similar property of hf_0'. Also, k preserves order except in case when $x_0 < y_0$ and x_1, y_1 are incomparable. Thus (1) and (2) are proved.

To prove (3), we first observe that $h(\mathrm{Im}(f_0)) = \mathrm{Im}(f_1)$ clearly implies the condition in (3). To prove the converse, note that $f_1hf_0 = hf_0$ implies that $\mathrm{Im}(hf_0) \subseteq \mathrm{Im}(f_1)$. If $g_0, g_1 < f_1$ are non-equivalent r-maps, then $\mathrm{Im}(g_0) \subseteq h(\mathrm{Im}(f_0))$ or $\mathrm{Im}(g_1) \subseteq h(\mathrm{Im}(f_0))$, by Statement 1.6 and (C). By the hypothesis, $g_ihf_0 \neq hf_0$ for $i = 0, 1$, so that $\mathrm{Im}(g_{1-i}) \subseteq h(\mathrm{Im}(f_0))$ for $i = 0, 1$. But then $\mathrm{Im}(f_1) = \mathrm{Im}(g_0) \cup \mathrm{Im}(g_1) \subseteq h(\mathrm{Im}(f_0))$, and (3) is proved.

Now we turn to (4). Assume that $C_i, D_i \in \mathbb{C}(X)$ are such that $\mathrm{Nuc}(C_0) \cong \mathrm{Nuc}(C_1)$ and the $c2r$-maps $f_i \in \mathrm{End}(X)$ satisfy $\Delta f_i \subseteq C_i \cup D_i$ for $i = 0, 1$.

With no loss of generality we may assume that $C_i \neq D_j$ for $i, j \in \{0, 1\}$. By Statement 2.1(4), there exist r-maps $g_i \in \mathrm{End}(X)$ with $g_i < f_i$, $C_i \cap \mathrm{Im}(g_i) \neq \emptyset$ and $g_ig_{1-i} = g_i$ for $i = 0, 1$. By Lemma 1.8, we may assume that $g_i(C_0 \cup C_1 \cup D_0 \cup D_1) \subseteq C_i$ for $i = 0, 1$. Also by Lemma 1.8, there exists a $c2r$-map $h_0 \in [f_0]$ with $g_0h_0 = g_0$ and $h_0(D_1) \subseteq D_0$. Then for any $u \in D_0 \cap \mathrm{Im}(h_0)$ there exists exactly one $v_u \in D_1 \cap \mathrm{Im}(f_1)$ with $h_0(v_u) = u$. This enables us to define $h_1\colon\ X \longrightarrow X$ by

$$h_1(x) = \begin{cases} h_0(x) & \text{for } x \in X \setminus h_0^{-1}(D_0), \\ v_u & \text{for } x \in h_0^{-1}(u) \text{ and } u \in D_0. \end{cases}$$

Since h_0 is a $c2r$-map and $\mathrm{Nuc}(D_1) \cong \mathrm{Nuc}(D_0) \cong \mathrm{Im}(f_1) \cap D_1$ we obtain that $h_1 \in \mathrm{End}(X)$ and $h_1 \in [f_1]$. Clearly, $h_ih_{1-i} = h_i$ for $i = 0, 1$, and (4) is proved.

Assume that either f is an $n2r$-map, or f is a $pt2r$-map and $E(\Delta f)$ is an antichain. Let $g, g_0, g_1 \in \mathrm{End}(X)$ be r-maps such that $g < f$, $\mathrm{Im}(g_i) \cap E(\Delta f) \neq \emptyset$, and $g_ig = g_i$, $gg_i = g$ for $i = 0, 1$. For $f' \in [f]$, the maps $f'g_i$ are r-maps exactly when $f' \restriction (\mathrm{Im}(g_0) \setminus E(\Delta f)) = g \restriction (\mathrm{Im}(g_0) \setminus E(\Delta f))$ and $f' \restriction (\mathrm{Im}(g_1) \setminus E(\Delta f)) =$

$g \restriction (\mathrm{Im}(g_1) \setminus E(\Delta f))$. Thus if $\mathrm{Im}(g_0) \cap E(\Delta f) = \mathrm{Im}(g_1) \cap E(\Delta f)$ then necessarily $f'g_0 = f'g_1$.

Conversely, if $\mathrm{Im}(g_0) \cap E(\Delta f) \neq \mathrm{Im}(g_1) \cap E(\Delta f)$ then $\{u\} = \mathrm{Im}(g_0) \cap E(\Delta f) \neq \mathrm{Im}(g_1) \cap E(\Delta f) = \{v\}$ and, by Lemmas 3.4 or 3.5, there exists an $f' \in [f]$ such that $gf' = g$ and $f'(u) \neq f'(v)$. Then $f'g_i$ are r-maps and $f'(u) = f'g_0(u) \neq f'g_1(u) = f'(v)$. This proves (5). $\square$

Lemma 3.10. *Let $C_0, C_1 \in \mathbb{C}(X)$ be such that $\mathrm{Nuc}(C_0) \cong \mathrm{Nuc}(C_1)$. For $i = 0, 1$, let $x_i \in C_i$ be min-defective and $y_i \in C_i$ max-defective elements such that, for any $z \in \mathrm{Mid}(X) \setminus \mathrm{Def}(X)$,*

$$E(z) \cap [x_i) \neq \emptyset \neq E(z) \cap (y_i] \text{ only when } E(z) \cap [x_i) \cap (y_i] \neq \emptyset.$$

For $i = 0, 1$, denote $\{u_i\} = \mathrm{Min}(x_i)$ and $\{v_i\} = \mathrm{Max}(y_i)$, and suppose that there exists an r-map g with $g(u_0) = u_1$, $g(v_0) = v_1$ and such that

$$z \in ([x_0) \cup (y_0]) \setminus \mathrm{Def}(X) \text{ implies } g(z) \in [x_1) \cup (y_1].$$

Then there exists an $h \in \mathrm{End}(X)$ with $h(x_0) = x_1$, $h(y_0) = y_1$ if and only if $x_0 \nleqslant y_0$ or $x_1 \leqslant y_1$.

P r o o f. By the hypothesis $g(x_i) = g(u_i) = u_1$ and $g(y_i) = g(v_i) = v_1$ for $i = 0, 1$. Furthermore, by Statement 2.1(6), we may assume that for any $z \in \mathrm{Mid}(X) \setminus \mathrm{Def}(X)$,

$$g(z) \in [x_1) \text{ whenever } E(z) \cap ([x_0) \cup [x_1)) \neq \emptyset \text{ and}$$

$$g(z) \in (y_1] \text{ whenever } E(z) \cap ((y_0] \cup (y_1]) \neq \emptyset.$$

Theorem 1.7 applied to g, $F_1 = \{x_0, x_1\}$ and x_1 gives rise to a dr-map $g' \in \mathrm{End}(X)$ such that $gg' = g$ and $g'(x_0) = g'(x_1) = x_1$. If $x_0 \nleqslant y_0$ or $x_1 \leqslant y_1$ then the order dual of Theorem 1.7, applied to g', $F_1 = \{y_0, y_1\}$ and y_1 this time, yields a dp-map $h \in \mathrm{End}(X)$ with $g'h = g'$ and $h(y_0) = h(y_1) = y_1$. Thus $h(x_0) = x_1$ and $h(y_0) = y_1$. Conversely, if there exists an $h \in \mathrm{End}(X)$ with $h(x_0) = x_1$ and $h(y_0) = y_1$ then either $x_0 \nleqslant y_0$ or $x_1 \leqslant y_1$ because h preserves order. $\square$

Statement 3.11. *Let $X, Y \in \mathbb{AR}$, and let $\psi\colon \mathrm{End}(X) \longrightarrow \mathrm{End}(Y)$ be an R-isomorphism. Then, for any $g \in \mathrm{End}(X)$:*

(1) *g is a $2r$-map if and only if $\psi(g)$ is a $2r$-map;*
(2) *g is a $t2r$-map if and only if $\psi(g)$ is a $t2r$-map;*
(3) *g is an $n2r$-map if and only if $\psi(g)$ is an $n2r$-map;*
(4) *g is an $e2r$-map if and only if $\psi(g)$ is an $e2r$-map;*

(5) *g is an ndr-map if and only if $\psi(g)$ is an ndr-map;*

(6) *if $f_0, f_1 \in \mathrm{End}(X)$ are p2r-maps (or dr-maps, or ndr-maps) and $h \in \mathrm{End}(X)$, then $h(\mathrm{Im}(f_0)) = \mathrm{Im}(f_1)$ if and only if $\psi(h)(\mathrm{Im}(\psi(f_0))) = \mathrm{Im}(\psi(f_1))$.*

Proof. The first five claims follow from the respective definitions, while (6) is a consequence of Statements 2.3(11) and 3.9(3). □

Definition. Let $X, Y \in \mathbb{AR}$. An R-isomorphism $\psi\colon \mathrm{End}(X) \to \mathrm{End}(Y)$ is called a *C-isomorphism* if for any $f \in \mathrm{End}(X)$, the endomorphism $\psi(f)$ is a $c2r$-map exactly when f is a $c2r$-map.

Statement 3.12. *Let $X, Y \in \mathbb{AR}$ and let $\psi\colon \mathrm{End}(X) \longrightarrow \mathrm{End}(Y)$ be an R-isomorphism such that*

> *for any Stone nucleus N with $|\mathbb{C}_N(X)| > 1$, there exists a $c2r$-map $f_N \in \mathrm{End}(X)$ with $\Delta f_N \subseteq \bigcup \mathbb{C}_N(X)$, such that $\psi(f_N) \in \mathrm{End}(Y)$ is a $c2r$-map;*
>
> *for any Stone nucleus N with $|\mathbb{C}_N(Y)| > 1$, there exists a $c2r$-map $f_N \in \mathrm{End}(Y)$ with $\Delta f_N \subseteq \bigcup \mathbb{C}_N(Y)$, and such that $\psi^{-1}(f_N) \in \mathrm{End}(X)$ is a $c2r$-map.*

Then, for any $h \in \mathrm{End}(X)$,

(1) *h is a $c2r$-map if and only if $\psi(h)$ is a $c2r$-map;*

(2) *h is a $pt2r$-map if and only if $\psi(h)$ is a $pt2r$-map,*

and hence ψ is a C-isomorphism.

Proof. If $h \in \mathrm{End}(X)$ is a $c2r$-map, then there is a unique Stone nucleus N for which $\Delta h \subseteq \bigcup \mathbb{C}_N(X)$. By the first hypothesis, we have a $c2r$-map f_N with $\Delta f_N \subseteq \bigcup \mathbb{C}_N(X)$ for which $\psi(f_N)$ is a $c2r$-map. By Statement 3.9(4), there exist $f' \in [f_N]$ and $h' \in [h]$ such that $h'f' = h'$ and $f'h' = f'$. From $\psi(f') \in [\psi(f_N)]$ it follows that $\psi(f')$ is a $c2r$-map. By Lemma P.5(2), $\mathrm{Im}(\psi(h')) \cong \mathrm{Im}(\psi(f'))$, so that $\psi(h')$ and hence also $\psi(h)$ are $c2r$-maps. The converse in (1) follows by symmetry, and (2) is a consequence of (1) and Statement 3.11(2). □

Statement 3.13. *Let $X, Y \in \mathbb{AR}$ and let $\psi\colon \mathrm{End}(X) \to \mathrm{End}(Y)$ be a C-isomorphism. Let $f \in \mathrm{End}(X)$ be a $p2r$-map such that either f is an $n2r$-map or else both $E(\Delta f)$ and $E(\Delta\psi(f))$ are antichains. If g_0 and g_1 are r-maps such that $\mathrm{Im}(g_i) \cap E(\Delta f) \neq \emptyset$ and $\mathrm{Im}(\psi(g_i)) \cap E(\Delta\psi(f)) \neq \emptyset$ for $i = 0, 1$, then $\mathrm{Im}(g_0) \cap \mathrm{Im}(g_1) \cap E(\Delta f) \neq \emptyset$ if and only if $\mathrm{Im}(\psi(g_0)) \cap \mathrm{Im}(\psi(g_1)) \cap E(\Delta\psi(f)) \neq \emptyset$.*

Proof. If f is a $p2r$-map satisfying the hypothesis, then there exist an r-map $g < f$ and maps $g_i' \in [g_i]$ such that $g_i'g = g_i'$ and $gg_i' = g$ for $i = 0, 1$, by Statement 2.1(4). But then the conclusion follows from Statement 3.9(5). □

This section investigates relations between $2r$-maps, and combines $p2r$-maps into suitable collections preserved by C-isomorphisms.

Definition. Let f_0, f_1 be $2r$-maps and let $g < f_0, f_1$ be an r-map. We say that f_0, f_1 are *independent over* g if $h = \sup\{f_0, f_1\}$ exists and there are exactly four distinct equivalence classes of r-maps below h and also exactly four distinct equivalence classes of $2r$-maps below h.

Following is a structural description of independence.

Lemma 4.1. *Let g be an r-map and let $f_0, f_1 > g$ be $2r$-maps. Then f_0, f_1 are independent over g if and only if $\Delta f_0 \cap \Delta f_1 = \emptyset$.*

If $\Delta f_0 \cap \Delta f_1 = \emptyset$, then there is an idempotent $h \in \mathrm{End}(X)$ with $\mathrm{Im}(f_0) \cup \mathrm{Im}(f_1) = \mathrm{Im}(h)$, and hence $h = \sup\{f_0, f_1\}$. If, in addition, $\Delta f_i = \{x_i, y_i\} \subseteq \mathrm{Mid}(X)$ for $i = 0, 1$ and $\{x_0, x_1\} \subseteq \mathrm{Im}(g)$, then we may assume that

$$h(t) = \begin{cases} f_0(t) & \text{for } t \in X \setminus (f_0^{-1}(E(x_1)) \cap f_1^{-1}(E(x_1))), \\ f_1(t) & \text{for } t \in f_0^{-1}(E(x_1)) \cap f_1^{-1}(E(x_1)). \end{cases}$$

P r o o f. We begin with the second claim. Assume that $\Delta f_0 \cap \Delta f_1 = \emptyset$. If Δf_0 or Δf_1 is a union of Stone nuclei then the claim follows from Lemma 1.8. By Lemma 3.1, in the remaining case $\Delta f_i = \{x_i, y_i\} \subseteq E(x_i) \subseteq \mathrm{Mid}(X)$ for $i = 0, 1$. If, say, $x_0, x_1 \in \mathrm{Im}(g)$, then $\mathrm{Ext}(x_0) \neq \mathrm{Ext}(x_1)$ because g is an r-map and $\Delta f_0 \cap \Delta f_1 = \emptyset$.

Set $E = f_0^{-1}(E(x_1)) \cap f_1^{-1}(E(x_1))$, and write

$$h(t) = \begin{cases} f_1(t) & \text{for } t \in E, \\ f_0(t) & \text{for } t \in X \setminus E. \end{cases}$$

From $f_0(E) = \{x_1\}$ and $f_1(E) \subseteq E(x_1)$ it follows that $h \in \mathrm{End}(X)$ because $f_0, f_1 \in \mathrm{End}(X)$ have finite images. Since f_0 and f_1 are idempotents and $f_1(E) = \Delta f_1$, the dp-map h is idempotent and $\mathrm{Im}(h) = (\mathrm{Im}(f_0) \setminus \{x_1\}) \cup \Delta f_1 = \mathrm{Im}(f_0) \cup \mathrm{Im}(f_1)$. This completes the proof of the second statement.

Let $g, g_i < f_i$ be r-maps such that $g_i \notin [g]$ for $i = 0, 1$. For $i = 0, 1$, write $J_i = \mathrm{Im}(g) \setminus \mathrm{Im}(g_i)$ and $K_i = \mathrm{Im}(g_i) \setminus \mathrm{Im}(g)$, and denote $L = \mathrm{Im}(g) \cap \mathrm{Im}(g_0) \cap \mathrm{Im}(g_1)$. Then either $J_i \cong K_i$ are Stone nuclei, or J_i and K_i are non-defective points for $i = 0, 1$.

We show that $\Delta f_0 \cap \Delta f_1 = \emptyset$ implies that $f_0, f_1 > g$ are independent over g.

The idempotent h defined in the first part of the proof satisfies $\mathrm{Im}(h) = \mathrm{Im}(f_0) \cup \mathrm{Im}(f_1)$, and hence no Stone kernels other than $\mathrm{Im}(g)$, $\mathrm{Im}(g_0)$, $\mathrm{Im}(g_1)$, and $K_0 \cup K_1 \cup L$

are contained in $\mathrm{Im}(h)$. Similarly, no images of $2r$-maps other than $\mathrm{Im}(f_0)$, $\mathrm{Im}(f_1)$ and $J_i \cup K_0 \cup K_1 \cup L = \mathrm{Im}(g_i) \cup (\mathrm{Im}(f_{1-i}) \setminus \mathrm{Im}(g))$ with $i = 0, 1$ are contained in $\mathrm{Im}(h)$. Therefore $f_0, f_1 > g$ are independent over g.

To prove the converse, let $f_0, f_1 > g$ be independent $2r$-maps with $\Delta f_0 \cap \Delta f_1 \neq \emptyset$. Since $\Delta f_0 \cap \Delta f_1 = (K_0 \cap K_1) \cup (J_0 \cap J_1)$ and because $\mathrm{Im}(g) \supseteq J_0 \cup J_1$ is a Stone kernel of X, this is possible only when $J_0 \cap J_1 \neq \emptyset$, see Lemma 3.1.

Suppose that $J_0 \neq J_1$. Since g is an r-map, one of the sets Δf_i, say Δf_0, is the union of two disjoint Stone nuclei while the other $\Delta f_1 = \{x_0, x_1\} \subseteq E(x_0)$ with a non-defective $x_0 \in \mathrm{Mid}(X)$ and $x_0 \in \Delta f_0$. By Lemma 3.1, $(\Delta f_0 \setminus \mathrm{Im}(g)) \cap \mathrm{Im}(f_1) = \emptyset$, and thus we may apply Lemma 1.8 to f_1 to obtain an idempotent $h \in \mathrm{End}(X)$ with $\mathrm{Im}(h) = \mathrm{Im}(f_0) \cup \mathrm{Im}(f_1)$. But then only $\mathrm{Im}(g)$, $\mathrm{Im}(g_0)$ and $\mathrm{Im}(g_1)$ are distinct Stone kernels contained in $\mathrm{Im}(h)$, a contradiction.

Therefore $J_0 = J_1$.

Suppose that $\mathrm{Im}(f_0)$ and $\mathrm{Im}(f_1)$ do not intersect the same components. Then, by Lemma 3.1, J_0 is a Stone nucleus and $K_0 \cap K_1 = \emptyset$, and Lemma 1.8 implies the existence of an idempotent $h \in \mathrm{End}(X)$ with $\mathrm{Im}(h) = \mathrm{Im}(f_0) \cup \mathrm{Im}(f_1)$. But then $\mathrm{Im}(h)$ contains only three distinct Stone kernels, namely $\mathrm{Im}(g)$, $\mathrm{Im}(g_0)$ and $\mathrm{Im}(g_1)$. This contradiction shows that $\mathrm{Im}(f_0)$ and $\mathrm{Im}(f_1)$ must intersect the same components.

Since $f_0, f_1 > g$ are independent over g, a supremum $h = \sup\{f_0, f_1\}$ exists. Since there are only four distinct equivalence classes of r-maps below h, the image of h intersects only finitely many components. Thus, by Statement 1.6, $\mathrm{Im}(h)$ and $\mathrm{Im}(f_0)$ intersect the same components.

Since $\mathrm{Im}(f_0) \cup \mathrm{Im}(f_1)$ contains at least three distinct Stone kernels, and $\mathrm{Im}(h)$ contains exactly four, from Statement 2.1(6) and from the fact that $\mathrm{Im}(h)$ and $\mathrm{Im}(f_0)$ intersect the same components it follows that $\mathrm{Im}(h) \setminus (\mathrm{Im}(f_0) \cup \mathrm{Im}(f_1) \cup \mathrm{Def}(X))$ has at most one element. We claim that

(m) if $z \in \mathrm{Im}(h) \setminus (\mathrm{Im}(f_0) \cup \mathrm{Im}(f_1) \cup \mathrm{Def}(X))$, then no order preserving idempotent $f\colon E(z) \cap \mathrm{Im}(h) \to E(z) \cap \mathrm{Im}(h)$ satisfies $\mathrm{Im}(f) = (\mathrm{Im}(f_0) \cup \mathrm{Im}(f_1)) \cap E(z)$.

Indeed, if f is such a map, then for any r-map g' of $\mathrm{Im}(h)$, the mapping

$$k(t) = \begin{cases} f(h(t)) & \text{for } t \in h^{-1}(E(z)), \\ h(t) & \text{for } t \in h^{-1}(\mathrm{Mid}(X) \setminus (\mathrm{Def}(X) \cup E(z))), \\ g'(h(t)) & \text{for } t \in h^{-1}(\mathrm{Ext}(X) \cup \mathrm{Def}(X)) \end{cases}$$

satisfies $\mathrm{Im}(k) = \mathrm{Im}(f_0) \cup \mathrm{Im}(f_1)$. Also, $k \in \mathrm{End}(X)$ is an idempotent because g' maps the clopen set $h^{-1}(\mathrm{Ext}(X) \cup \mathrm{Def}(X))$ into itself. Therefore $f_0, f_1 < k < h = \sup\{f_0, f_1\}$—a contradiction.

Suppose that $J_0 = J_1$ is a Stone nucleus. Since $\mathrm{Im}(h)$ contains exactly four distinct Stone kernels, from Statement 2.1(6) it follows that $K_0 \setminus K_1 = \{u_0\}$ and $K_1 \setminus K_0 = \{u_1\}$ are singletons, $u_1 \in E(u_0)$, and $\mathrm{Im}(h) \setminus (\mathrm{Im}(f_0) \cup \mathrm{Im}(f_1) \cup \mathrm{Def}(X)) = \{z\}$ is a singleton such that $z \in \mathrm{Mid}(K(J_0)) \setminus \mathrm{Def}(X)$ or $z \in E(u_0)$. The first case contradicts (m) because there certainly is an idempotent order preserving $f\colon E(z) \cap \mathrm{Im}(h) \to E(z) \cap \mathrm{Im}(h)$ with $\mathrm{Im}(f) = \mathrm{Im}(g) \cap E(z) = (\mathrm{Im}(f_0) \cup \mathrm{Im}(f_1)) \cap E(z)$.

Thus $z \in E(u_0)$. By Statement 2.1(6) and Corollary 3.3, there exists a $c2r$-map f_2 with $\mathrm{Im}(f_2) = (\mathrm{Im}(f_0) \setminus \{u_0\}) \cup \{z\}$. Since $u_1, z \in E(u_0)$ and because there are at most four $2r$-maps below h, Lemma 3.4 implies that the subposet $E(u_0) \cap \mathrm{Im}(h) = \{u_0, u_1, z\}$ contains at most one comparable pair. But then there exists an idempotent order preserving mapping of $E(u_0) \cap \mathrm{Im}(h)$ into itself whose image is $\{u_0, u_1\}$, in contradiction to (m).

Now let $J_0 = J_1 = \{x\}$ be a singleton. Then $K_i = \{y_i\}$ are singletons and $y_i \in E(x)$ for $i = 0, 1$. Since $\mathrm{Im}(h)$ contains exactly four distinct Stone kernels, the set $\mathrm{Im}(h) \cap E(x) = \{x, y_0, y_1, z\} = T$ must have four elements. By (m), there is no idempotent $f\colon T \to T$ with $\mathrm{Im}(f) = \{x, y_0, y_1\}$, and this implies that z is comparable to at least two other, incomparable members of T, and z is extremal in $E(x)$. Since $f_i(E(x)) = \Delta f_i = \{x, y_i\}$ for $i = 0, 1$, if x and y_i are in the same component of $E(x)$, then y_i is comparable to x. It follows that y_0, y_1 are incomparable and that $\{y_0, z\}$, $\{y_1, z\}$ are comparable pairs. So, if z is comparable to x, then T has five comparable pairs, and hence there are five non-equivalent $n2r$-maps whose images are contained in $\mathrm{Im}(h)$. Thus z is not comparable to x and two cases arise. First, x is comparable to both y_0 and y_1, in which case the map which sends z to x and leaves all other elements of T fixed is an order preserving idempotent—a contradiction with (m). In the second case, x is incomparable to all other members of T, and there exist five $2r$-maps whose images intersect $\mathrm{Im}(h) \cap E(x)$ in sets $\{y_i, x\}$, $\{y_i, z\}$ with $i = 0, 1$ and $\{x, z\}$.

Therefore $\Delta f_0 \cap \Delta f_1 = \emptyset$ for any $2r$-maps f_0, f_1 independent over g. □

Corollary 4.2. *For every r-map g of $X \in \mathbb{AR}$ there are only finitely many $2r$-maps $f_j > g$ that are pairwise independent over g.*

P r o o f. The claim follows from Lemma 4.1 and the finiteness of $\mathrm{Im}(g)$. □

Lemma 4.3. *Let $\{f_0, f_1, \dots f_n\}$ be a set of pairwise independent $n2r$-maps or $pt2r$-maps over an r-map $g \in \mathrm{End}(X)$. Then there exists an $h = \sup\{f_0, f_1, \dots, f_n\}$ with $\mathrm{Im}(h) = \bigcup\{\mathrm{Im}(f_i) \mid i = 0, 1, \dots, n\}$. Furthermore, the supremum h may be selected so that $h \restriction E(x_i) = f_i \restriction E(x_i)$ for any $i = 0, 1, \dots, n$ and $x_i \in \Delta f_i$.*

P r o o f. We proceed by induction on n. For $n = 1$, the statement follows from Lemma 4.1. Assume that it is true for $n - 1$. By the induction hypothesis, there exists an $h' = \sup\{f_0, f_1, \ldots, f_{n-1}\}$ with $\mathrm{Im}(h') = \bigcup\{\mathrm{Im}(f_i) \mid i = 0, 1, \ldots, n-1\}$ and $h' \restriction E(x_i) = f_i \restriction E(x_i)$ for all $x_i \in \Delta f_i$ with $i = 0, \ldots, n-1$. Denote $\Delta f_n = \{x, y\}$. Then, because $f_0, \ldots, f_n$ are pairwise independent over g, the set $\mathrm{Im}(h') \cap E(x)$ is a singleton and for any $z \in \Delta f_i$ with $i = 0, \ldots, n-1$ we have $E(z) \cap E(x) = \emptyset$. Let $E = (h')^{-1}(E(x)) \cap f_n^{-1}(E(x))$. Define

$$h(t) = \begin{cases} h'(t) & \text{for } t \in X \setminus E, \\ f_n(t) & \text{for } t \in E. \end{cases}$$

Since $h'(E), f_n(E) \subseteq E(x)$ and because $f_n, h' \in \mathrm{End}(X)$ are idempotents with finite images, we deduce that $h \in \mathrm{End}(X)$ is idempotent. From $f_n(E) = \Delta f_n$ it follows that $\mathrm{Im}(h) = (\mathrm{Im}(h') \setminus E(x)) \cup \Delta f_n = \mathrm{Im}(h') \cup \mathrm{Im}(f_n) = \bigcup\{\mathrm{Im}(f_i) \mid i = 0, 1, \ldots n\}$. Thus $h = \sup\{f_0, f_1, \ldots, f_n\}$. Since $E(x) \subseteq E$, we have $h \restriction E(x) = f_n \restriction E(x)$. $\square$

We say that r-maps g and g' are *close* if $\mathrm{Im}(g)$ and $\mathrm{Im}(g')$ intersect the same components of X.

Lemma 4.4. *Let $f_0, f_1 \in \mathrm{End}(X)$ be idempotent. Then:*

(1) *Two r-maps f_0, f_1 are close if and only if $ff_1 \lesssim ff_0$ for every $c2r$-map f such that ff_0 is idempotent and, vice versa, $ff_0 \lesssim ff_1$ for every $c2r$-map f for which ff_1 is idempotent.*

(2) *If $f_i \geqslant g_i$ for some r-map g_i for $i = 0, 1$, then $\mathrm{Im}(f_0)$ and $\mathrm{Im}(f_1)$ intersect the same components of X if and only if for $i = 0, 1$ and for any r-map $r_i \leqslant f_i$ there is an r-map $r_{1-i} \leqslant f_{1-i}$ close to r_i.*

Proof. Let f_0, f_1 be r-maps. If $\mathrm{Im}(f_0)$, $\mathrm{Im}(f_1)$ do not intersect the same components, then there are distinct $C_0, C_1 \in \mathbb{C}(X)$ with $\mathrm{Nuc}(C_0) \cong \mathrm{Nuc}(C_1)$ and $C_i \cap (\mathrm{Im}(f_i) \setminus \mathrm{Im}(f_{1-i})) \neq \emptyset$ for $i = 0, 1$. By Corollary 3.3, there is a $c2r$-map $f > f_0$ with $\Delta f \subseteq C_0 \cup C_1$. Then $ff_0 = f_0$ is idempotent and $ff_1 \neq ff_0ff_1$ because $C_1 \cap \mathrm{Im}(ff_1) \neq \emptyset$.

Conversely, suppose that $\mathrm{Im}(f_0)$ and $\mathrm{Im}(f_1)$ intersect the same components of X, and let f be a $c2r$-map. Then $f(E(x))$ is a singleton, and $f(E(x) \cap \mathrm{Im}(f_0)) = f(E(x) \cap \mathrm{Im}(f_1))$ for every $x \in \mathrm{Mid}(X) \setminus \mathrm{Def}(X)$ implies that $\mathrm{Im}(ff_0) = \mathrm{Im}(ff_1)$. Therefore (1) holds.

Since $g_i \leqslant f_i$ for some r-map $g_i \in \mathrm{End}(X)$ for $i = 0, 1$, from Statement 2.1(9) it follows that a component $C \in \mathbb{C}(X)$ intersects $\mathrm{Im}(f_i)$ if and only if there exists an r-map $r_i < f_i$ with $C \cap \mathrm{Im}(r_i) \neq \emptyset$. The remainder follows from the definition of closeness of r-maps. $\square$

Definition. We say that an r-map g is *nice* whenever

(n1) every $x \in \mathrm{Im}(g) \cap \mathrm{Mid}(X)$ is extremal in $E(x)$, (n2) if $x \in \operatorname{Im}(g) \cap \operatorname{Mid}(X)$ and $E(x)$ is not an antichain, then x is comparable to some $z \in E(x) \setminus \{x\}$.

A finite non-empty collection $\mathscr{F}$ of $p2r$-maps independent over a nice r-map g is *proper* if it satisfies these three conditions:

(p1) if $x \in \operatorname{Im}(g) \cap \operatorname{Mid}(X)$ and $E(x) \neq \{x\}$ is not an antichain, then $x \in \Delta f$ for some $e2r$-map $f \in \mathscr{F}$,

(p2) if $x \in \operatorname{Im}(g) \cap \operatorname{Mid}(X)$ and $E(x) \neq \{x\}$ is an antichain, then $x \in \Delta f$ for some $pt2r$-map $f \in \mathscr{F}$,

(p3) each member of $\mathscr{F}$ is of the type described in (p1) or (p2).

Notation. For a given r-map g, let $e(g)$ denote the maximal number of mutually independent $e2r$-maps over g, let $n(g)$ denote the maximal number of mutually independent $n2r$-maps over g, and let $p(g)$ be the maximal number of all mutually independent $p2r$-maps over g. Then $p(g) \geqslant n(g) \geqslant e(g) \geqslant 0$, and these numbers are finite because of Corollary 4.2.

Lemma 4.5. *For any nice r-map g, there exists a proper collection $\mathscr{F}$ of $p2r$-maps over g.*

Secondly, a collection $\mathscr{F}$ of independent $e2r$- or $pt2r$-maps over a nice r-map g is proper if and only if

(1) *$\mathscr{F}$ contains $e(g)$ distinct $e2r$-maps, and*

(2) *$\mathscr{F}$ contains $p(g) - e(g)$ distinct $pt2r$-maps.*

P r o o f. Let g be a nice r-map, and let G denote the set of all $x \in \operatorname{Im}(g) \cap \operatorname{Mid}(X)$ with $E(x) \neq \{x\}$.

Let $x \in G$. If $E(x)$ is not an antichain, then by (n2) there is, say, some $z \in E(x)$ comparable with x, and by (n1) and Lemma 3.7 there exists an $e2r$-map $f_x > g$ with $x \in \Delta f_x$. If $E(x)$ is an antichain, then by Lemma 3.5 there exists a $pt2r$-map $f_x > g$ with $x \in \Delta f \subseteq E(x)$. Any collection $\mathscr{F} = \{f_x \mid x \in G\}$ of such $p2r$-maps is independent and satisfies (p1)-(p3).

It is straightforward to verify that a collection $\mathscr{F}$ of independent $p2r$-maps over a nice r-map g is proper exactly when it satisfies (1) and (2). $\square$

Notation. For r-maps $g, g' \in \operatorname{End}(X)$, let $V(g', g)$ consist of all r-maps $h \in \operatorname{End}(X)$ close to g', and such that $\operatorname{Im}(h) \cap C = \operatorname{Im}(g) \cap C$ for any $C \in \mathbb{C}(X)$ with $\operatorname{Im}(g) \cap \operatorname{Im}(g') \cap C \neq \emptyset$.

Lemma 4.6. *There exists an r-map $g \in \operatorname{End}(X)$ such that $e(g) \geqslant n(g')$ for every r-map $g' \in \operatorname{End}(X)$. Any such g is nice.*

Secondly, let $g \in \mathrm{End}(X)$ be a nice r-map and let $g' \in \mathrm{End}(X)$ be an r-map. Then there exists an r-map $g_0 \in V(g', g)$ such that $e(g_0) \geqslant n(h)$ for every $h \in V(g', g)$. Any such g_0 is nice.

P r o o f. Both statements follow from the definition of a nice r-map, Statement 2.1(6) and Lemma 3.7. $\square$

Lemma 4.7. *Let $f_0, f_1 \in \mathrm{End}(X)$ be either pt2r-maps or n2r-maps. Then:*

(1) *if there are $f_0' \in [f_0]$, $f_1' \in [f_1]$ and an r-map g such that $gf_0'f_1' = f_0'f_1'$ or $gf_1'f_0' = f_1'f_0'$, then $\Delta f_0 \neq \Delta f_1$;*

(2) *if $gf_0'f_1' \neq f_0'f_1'$ and $gf_1'f_0' \neq f_1'f_0'$ for every r-map g and for all $f_0' \in [f_0]$, $f_1' \in [f_1]$, then $\Delta f_0 \neq \Delta f_1$ only when f_0 and f_1 are pt2r-maps with $\Delta f_0 \cup \Delta f_1 \subseteq E(x)$ for some $E(x)$ which is not an antichain.*

P r o o f. If $\Delta f_0 = \Delta f_1$ then $\Delta f_0 \subseteq \mathrm{Im}(f_0'f_1') \cap \mathrm{Im}(f_1'f_0')$ for every $f_0' \in [f_0]$ and $f_1' \in [f_1]$. But $\Delta f_0 \nsubseteq \mathrm{Im}(g)$ for any r-map g, so that $gf_0'f_1' \neq f_0'f_1'$ and $gf_1'f_0' \neq f_1'f_0'$. This proves (1).

To prove (2), assume that $\Delta f_0 \neq \Delta f_1$. Then there exist $x_i \in \mathrm{Mid}(X)$ such that $x_i \in \Delta f_i \setminus \Delta f_{1-i}$ for $i = 0, 1$. If $E(x_0) \neq E(x_1)$, then $f_i'(\Delta f_{1-i})$ is a singleton for some $f_i' \in [f_i]$, $i = 0, 1$, and hence $gf_i'f_{1-i}' = f_i'f_{1-i}'$ for some r-map g and any $f_{i-1}' \in [f_{1-i}]$. Hence we may assume that $\Delta f_0 \cup \Delta f_1 \subseteq E(x)$. If f_i is an $n2r$-map or $E(x)$ is an antichain then, by Lemmas 3.4 or 3.5, there exists an $f_i' \in \mathrm{End}(X)$ such that $f_i'(\Delta f_{1-i})$ is a singleton, so that $gf_i'f_{1-i}' = f_i'f_{1-i}'$ for some r-map g and any $f_{1-i}' \in [f_{1-i}]$. Thus both f_i must be pt2r-maps and $E(x)$ cannot be an antichain. $\square$

Notation. Given an idempotent $h \in \mathrm{End}(X)$ with finite $\mathrm{Im}(h)$ and a non-defective $x \in \mathrm{Im}(h)$, we define a map h_x by

$$h_x(t) = \begin{cases} x & \text{for } t \in h^{-1}(E(x)), \\ h(t) & \text{for } t \in X \setminus h^{-1}(E(x)). \end{cases}$$

Then h_x is an idempotent whose image is finite, and the fineteness of $\mathrm{Im}(h)$ implies that $h_x \in \mathrm{End}(X)$.

Let $\mathscr{F}$ be a proper collection of $p2r$-maps over a nice r-map $g \in \mathrm{End}(X)$ and let $f \in \mathscr{F}$. Denote $\Delta f = \{x, y\}$ and $h = \sup \mathscr{F}$. Let $S(\mathscr{F}, g, f)$ denote the family of all those idempotents $k \in \mathrm{End}(X)$ for which

(s1) $h(C) \subseteq C$ implies $k(C) \subseteq C$ for every $C \in \mathbb{C}(X)$;

(s2) $h'kh \in [h_x] \cup [h_y]$ for every $h' \in [h]$.

Lemma 4.8. *Let $\mathscr{F}$ be a proper collection of $p2r$-maps over a nice r-map $g \in \operatorname{End}(X)$ and let $f \in \mathscr{F}$. Then for every $k \in S(\mathscr{F}, g, f)$ there exists an r-map $k' \in \operatorname{End}(X)$ with $k' < k$.*

Proof. Since any $f' \in \mathscr{F}$ is either an $n2r$-map or a $pt2r$-map, from Lemma 4.3 it follows that $h = \sup \mathscr{F}$ has the image $\operatorname{Im}(h) = \bigcup\{\operatorname{Im}(f') \mid f' \in \mathscr{F}\}$, so that the components $C \in \mathbb{C}(X)$ intersecting $\operatorname{Im}(h)$ form a minimal Stone plot. Since $k \in S(\mathscr{F}, g, f)$ is idempotent, from (C) it follows that $\operatorname{Im}(k)$ contains a Stone kernel. Statement 2.1(1) then completes the proof. $\square$

The claim below follows immediately.

Statement 4.9. *Let $X, Y \in \mathbb{AR}$ and let $\psi\colon \operatorname{End}(X) \longrightarrow \operatorname{End}(Y)$ be an R-isomorphism. Then:*

(1) *if $f_0, f_1 \in \operatorname{End}(X)$ are $2r$-maps and $f_0, f_1 > g$ for an r-map g, then f_0, f_1 are independent over g if and only if $\psi(f_0)$, $\psi(f_1)$ are independent over $\psi(g)$;*
(2) *$e(\psi(g)) = e(g)$, $n(\psi(g)) = n(g)$ for any r-map $g \in \operatorname{End}(X)$.*

If ψ is also a C-isomorphism, then

(3) *$p(\psi(g)) = p(g)$ for any r-map $g \in \operatorname{End}(X)$;*
(4) *if $g \in \operatorname{End}(X)$ and $\psi(g) \in \operatorname{End}(Y)$ are nice r-maps, then a collection $\mathscr{F}$ of $p2r$-maps is a proper collection over g if and only if $\psi(\mathscr{F}) = \{\psi(f) \mid f \in \mathscr{F}\}$ is a proper collection over $\psi(g)$;*
(5) *if $g \in \operatorname{End}(X)$ and $\psi(g) \in \operatorname{End}(Y)$ are nice r-maps, then*

$$\psi(S(\mathscr{F}, g, f)) = S(\psi(\mathscr{F}), \psi(g), \psi(f))$$

for every proper collection $\mathscr{F}$ of $p2r$-maps over g and for every $f \in \mathscr{F}$;
(6) *if $f, g \in \operatorname{End}(X)$ are r-maps, then f and g are close if and only if $\psi(f)$ and $\psi(g)$ are close;*
(7) *if $f_0, f_1 \in \operatorname{End}(X)$ are idempotents such that $f_i > g_i$ for some r-maps $g_0, g_1 \in \operatorname{End}(X)$, then $\operatorname{Im}(f_0)$ and $\operatorname{Im}(f_1)$ intersect the same components of X if and only if $\operatorname{Im}(\psi(f_0))$ and $\operatorname{Im}(\psi(f_1))$ intersect the same components of Y;*
(8) *if $f \in \operatorname{End}(X)$ is either a $pt2r$-map such that $E(\Delta f)$ is an antichain or an $n2r$-map then, for any $p2r$-map $f' \in \operatorname{End}(X)$, we have $\Delta f = \Delta f'$ if and only if $\Delta\psi(f) = \Delta\psi(f')$.*

Theorem 4.10. *Let $\mathscr{F}$ be a proper collection of $p2r$-maps over a nice r-map g. Let $f \in \mathscr{F}$, and let $\Delta f = \{x, y\}$ with $x \in \operatorname{Im}(g)$. Denote $h = \sup \mathscr{F}$. Then*

(1) *$k(z) = z$ for every $z \in \operatorname{Im}(h) \setminus \{x, y\}$ and every $k \in S(\mathscr{F}, g, f)$,*

(2) *$k(x) = k(y) \in E(x)$ for every $k \in S(\mathscr{F}, g, f)$,*

(3) *for every $z \in E(x)$ there exists a $k \in S(\mathscr{F}, g, f)$ with $k(x) = z$,*

(4) *for $k_1, k_2 \in S(\mathscr{F}, g, f)$, we have $k_1(x) = k_2(x)$ if and only if $k_1 g = k_2 g$.*

Thus the map $\eta_f \colon \{kg \mid k \in S(\mathscr{F}, g, f)\} \longrightarrow E(x)$ given by $\eta_f(kg) = kg(x)$ is a bijection.

Proof. Since $h'kh \in [h_x] \cup [h_y]$ by (s2), the map $h'kh$ is idempotent for any $k \in S(\mathscr{F}, g, f)$, and $\mathrm{Im}(h'kh) = \mathrm{Im}(h) \setminus \{y\}$ or $\mathrm{Im}(h'kh) = \mathrm{Im}(h) \setminus \{x\}$ for any $h' \in [h]$. Therefore

(a) $h'k(t) = t$ for all $t \in \mathrm{Im}(h) \setminus \{x, y\}$ and any $h' \in [h]$.

Let $C \in \mathbb{C}(X)$ be such that $g(C) \subseteq C$. Then $h(C) \subseteq C$.

First, for any $c \in \mathrm{Im}(h) \cap \mathrm{Mid}(C)$ with $E(c) \neq \{c\}$, we have $\mathrm{Im}(h) \cap E(c) = \Delta f'$ for some $f' \in \mathscr{F}$ because of (p1), (p2) and Lemma 4.3. Since members of $\mathscr{F}$ are independent over the r-map g, the $2r$-map $f' \in \mathscr{F}$ with $E(c) \cap \mathrm{Im}(h) = \Delta f'$ is uniquely determined.

Next we show that $k(c) = c$ for every $c \in (\mathrm{Im}(h) \cap C) \setminus \{x, y\}$. By (a), for every such c and for all $h' \in [h]$ we already have $h'k(c) = c$. Thus $k(c) = c$ for all $c \in \mathrm{Ext}(C)$ and also for all $c \in \mathrm{Mid}(C)$ with $E(c) = \{c\}$. If $c \in \mathrm{Mid}(\mathrm{Im}(h) \cap C)$ and $E(c) \neq \{c\}$, then, as shown above, $\mathrm{Im}(h) \cap E(c) = \Delta f' = \{u, v\}$ for a unique $f' \in \mathscr{F}$. But then $f' \neq f$ since $c \notin \{x, y\} = \Delta f$ and because $\mathscr{F}$ consists of independent $2r$-maps. Since k is the identity on $\mathrm{Ext}(C)$, we must have $k(u), k(v) \in E(c)$. If $E(c)$ is not an antichain then u and v are comparable extremal elements of $E(c)$ because of (n1), (n2) and (p2). If $k(u) \neq u$ then, by Lemma 3.4, there is a $2r$-map $f'' \in [f']$ such that $f''\{k(u), k(v)\} = \{v\}$. If $E(c)$ is an antichain, then such an f'' exists because of Lemma 3.5. But then, in either case, Lemma 4.3 implies the existence of an $h' \in [h]$ with $h'\{k(u), k(v)\} = \{v\} \subset \Delta f'$. Whence $h'kh \notin [h_x] \cup [h_y]$—a contradiction with (s2). This shows that $k(u) = u$ and, symmetrically, $k(v) = v$. Whence $k(c) = c$ for every $c \in (\mathrm{Im}(h) \cap C) \setminus \{x, y\}$, and the proof of (1) is complete.

To prove (2), suppose that $k(x) \neq k(y)$. Then by Lemmas 3.4 and 3.5, there exists an $\hat{f} \in [f]$ with $\hat{f}(k(x)) \neq \hat{f}(k(y))$ and, by Lemma 4.3, there exists an $h' \in [h]$ with $h'k(x) \neq h'k(y)$—a contradiction because $h'kh \notin [h_x] \cup [h_y]$ again. This proves (2).

Let $z \in E(x)$. Define

$$k(u) = \begin{cases} h(u) & \text{for } u \in X \setminus h^{-1}\{x, y\}, \\ z & \text{for } u \in h^{-1}\{x, y\}. \end{cases}$$

Then $k \in S(\mathscr{F}, g, f)$, and this proves (3).

To prove (4), we note that, by (1), $k_1(z) = z = k_2(z)$ for all $z \in \mathrm{Im}(h) \setminus \{x, y\}$, and hence for all $z \in \mathrm{Im}(g) \setminus \{x\}$. Since $g(x) = x$, we have $k_1 g = k_2 g$ if and only if $k_1 g(x) = k_2 g(x)$.

From the above it follows that η_f is a bijection. $\square$

Statement 4.11. *Let $X, Y \in \mathbb{AR}$, and let $\psi\colon \operatorname{End}(X) \longrightarrow \operatorname{End}(Y)$ be a C-isomorphism. Let $g \in \operatorname{End}(X)$ and $\psi(g) \in \operatorname{End}(Y)$ be nice r-maps, and let $\mathscr{F}$ be a proper collection over g. For any $f \in \mathscr{F}$ and all $z \in E(\Delta f)$, write*

$$\nu_f(z) = \eta_{\psi(f)}(\psi(\eta_f^{-1}(z))).$$

Then $\nu_f\colon E(\Delta f) \longrightarrow E(\Delta\psi(f))$ is a bijection such that

(1) *$\nu_f(x) \in \operatorname{Im}(\psi(g))$ for the element $x \in \operatorname{Im}(g) \cap \Delta f$;*

(2) *elements $u, v \in E(\Delta f)$ are comparable if and only if $\nu_f(u), \nu_f(v) \in E(\Delta\psi(f))$ are comparable.*

Proof. By Theorem 4.10 and Statement 4.9(5), the map ν_f is a correctly defined bijection of $E(\Delta f)$ onto $E(\Delta\psi(f))$. Since for any $k \in S(\mathscr{F}, g, f)$ with $k(\Delta f) = \{x\}$ we have $kg = g$, (1) is proved.

By Lemma 3.4, $\{u, v\} \subseteq E(\Delta f)$ is a comparable pair if and only if there exists an $n2r$-map $f' \in \operatorname{End}(X)$ with $f' > \eta_f^{-1}(u), \eta_f^{-1}(v)$ because kg is an r-map for any $k \in S(\mathscr{F}, g, f)$. From Statement 3.11(3) it follows that $\{u, v\}$ is a comparable pair if and only if $\{\nu_f(u), \nu_f(v)\}$ is a comparable pair. $\square$

5. $3r$-maps and blocks

In this section, we define suitable maximal collections consisting of $c2r$-maps or $pt2r$-maps, and investigate their relation and preservation by monoid isomorphisms. To make such collections coherent, we need some additional concepts.

Definition. We say that an idempotent $f \in \operatorname{End}(X)$ is a *$3r$-map* if there are exactly three distinct classes of r-maps $g_i < f$ for $i = 0, 1, 2$, exactly three distinct classes of $t2r$-maps $f_i < f$ for $i = 0, 1, 2$, and $f = \sup\{g_0, g_1, g_2\}$.

Lemma 5.1. *Let $f \in \operatorname{End}(X)$ be a $3r$-map. Let $g_0, g_1, g_2 < f$ be pairwise non-equivalent r-maps, let $f_0, f_1, f_2 < f$ be pairwise non-equivalent $t2r$-maps, and let $g_i, g_{i+1} < f_{i+2}$—with the addition modulo 3. Then exactly one of the following three cases occurs:*

(1) *f_0, f_1, f_2 are $c2r$-maps, and there are distinct $C_0, C_1, C_2 \in \mathbb{C}(X)$ with isomorphic Stone nuclei satisfying*

$$\operatorname{Im}(g_i) \setminus \operatorname{Im}(g_{i+1}) = \operatorname{Im}(g_i) \setminus \operatorname{Im}(g_{i+2}) \subseteq C_i \text{ for } i = 0, 1, 2;$$

(2) *f_0, f_1, f_2 are pt2r-maps, and there is a component C and distinct non-defective $x_0, x_1, x_2 \in \mathrm{Mid}(C)$ such that $\{x_0, x_1, x_2\} \subseteq E(x_0)$ is an antichain and*

$$\mathrm{Im}(g_i) \setminus \mathrm{Im}(g_{i+1}) = \mathrm{Im}(g_i) \setminus \mathrm{Im}(g_{i+2}) = \{x_i\} \text{ for } i = 0, 1, 2;$$

(3) *exactly one $2r$-map, say f_0, from $\{f_0, f_1, f_2\}$ is a pt2r-map, and the other two are c2r-maps, and there are distinct components $C_0, C_1 \in \mathbb{C}(X)$ with isomorphic Stone nuclei and distinct incomparable non-defective $x_1, x_2 \in \mathrm{Mid}(C_1)$ with $x_2 \in E(x_1)$ such that*

$$\mathrm{Im}(g_0) \setminus \mathrm{Im}(g_1) = \mathrm{Im}(g_0) \setminus \mathrm{Im}(g_2) \subseteq C_0,$$

$$\mathrm{Im}(g_1) \setminus \mathrm{Im}(g_0), \mathrm{Im}(g_2) \setminus \mathrm{Im}(g_0) \subseteq C_1,$$

$$\mathrm{Im}(g_i) \setminus \mathrm{Im}(g_{3-i}) = \{x_i\} \text{ for } i = 1, 2.$$

P r o o f. First, by Lemma 3.1, $\mathrm{Im}(f_i) = \mathrm{Im}(g_{i+1}) \cup \mathrm{Im}(g_{i+2})$ and hence

(c1) $$\Delta f_i = (\Delta f_{i+1} \setminus \Delta f_{i+2}) \cup (\Delta f_{i+2} \setminus \Delta f_{i+1}) \text{ for } i = 0, 1, 2.$$

Furthermore, $\Delta f_i \cap \Delta f_j \neq \emptyset$ for distinct $i, j = 0, 1, 2$, for otherwise there would exist four distinct Stone kernels contained in $\sup\{f_i, f_j\} \leqslant f$, by Lemma 4.1.

Suppose that f_0, f_1, f_2 are *c*2r-maps. Since $\Delta f_0 \cap \Delta f_1 \neq \emptyset$, by Lemma 3.2 and (c1) there exist three distinct components C_0, C_1, C_2 with isomorphic Stone nuclei satisfying $\Delta f_i \subseteq C_{i+1} \cup C_{i+2}$ for $i = 0, 1, 2$. This describes the case under (1).

Suppose that f_0, f_1, f_2 are *pt*2r-maps. Since $\Delta f_0 \cap \Delta f_1 \neq \emptyset$, by Lemma 3.2 and (c1) there exist three distinct points x_0, x_1, x_2 such that $\{x_0, x_1, x_2\} \subseteq E(x_0)$ is an antichain and $\Delta f_i = \{x_{i+1}, x_{i+2}\}$ for $i = 0, 1, 2$, and this describes the case under (2).

Suppose that f_0 is a *pt*2r-map and f_1 is a *c*2r-map. Since $\Delta f_0 \cap \Delta f_1 \neq \emptyset$, from Lemma 3.2 and (c1) it follows that f_2 is a *c*2r-map and the description given in (3) occurs. □

Definition. A $3r$-map $f \in \mathrm{End}(X)$ is called

*ct*3*r-map* if it satisfies the condition (1) in Lemma 5.1;
*pt*3*r-map* if it satisfies the condition (2) in Lemma 5.1;
*m*3*r-map* if it satisfies the condition (3) in Lemma 5.1.

We say that a $3r$-map is a *t*3*r-map* if it is either a *ct*3r-map or a *pt*3r-map.

Lemma 5.2. *Let $f_0 \in \mathrm{End}(X)$ be a $t2r$-map and let $g < f_0$ be an r-map. Then for every component $C \in \mathbb{C}(X)$ with $\mathrm{Nuc}(C) \cong \mathrm{Nuc}(K(\Delta f_0 \cap \mathrm{Im}(g)))$ and $C \cap \mathrm{Im}(f_0) = \emptyset$ and for every dp-subspace $N \subseteq C$ isomorphic to $\mathrm{Nuc}(C)$ there exists a $3r$-map $f \in \mathrm{End}(X)$ with $\mathrm{Im}(f) = \mathrm{Im}(f_0) \cup N$.*

Consequently, if $g \in \mathrm{End}(X)$ is an r-map and $C_1, C_2 \in \mathbb{C}(X)$ are distinct components such that $\mathrm{Nuc}(C_1) \cong \mathrm{Nuc}(C_2)$ and $C_1 \cap \mathrm{Im}(g) = \emptyset = C_2 \cap \mathrm{Im}(g)$, then for any dp-subspaces $N_i \subseteq C_i$ isomorphic to $\mathrm{Nuc}(C_i)$ for $i = 1, 2$ there exists a $ct3r$-map $f \in \mathrm{End}(X)$ with $\mathrm{Im}(f) = \mathrm{Im}(g) \cup N_1 \cup N_2$.

If $f \in \mathrm{End}(X)$ is a $3r$-map and $g_0, g_1, g_2 < f$ are non-equivalent r-maps and $f_0, f_1, f_2 < f$ are non-equivalent $t2r$-maps, then $\mathrm{Im}(f) = \bigcup\{\mathrm{Im}(g_i) \mid i = 0, 1, 2\} = \mathrm{Im}(f_j) \cup \mathrm{Im}(f_{j+1})$ for each $j = 0, 1, 2$.

P r o o f. To obtain the first statement, we apply Lemma 1.8 to f_0, the component C and its dp-subspace N. The second statement follows from Corollary 3.3 and the first statement of this Lemma.

If $f \in \mathrm{End}(X)$ is a $ct3r$-map or an $m3r$-map, then the third statement follows from the first statement of this Lemma and Lemma 5.1. It remains to consider a $pt3r$-map $f \in \mathrm{End}(X)$. Since $f = \sup\{g_0, g_1, g_2\}$ and $\mathrm{Im}(f)$ contains exactly three distinct Stone kernels, namely $\mathrm{Im}(g_0)$, $\mathrm{Im}(g_1)$, and $\mathrm{Im}(g_2)$, we conclude from Statement 2.1(6) and 2.1(9) that $\mathrm{Im}(f) \setminus \mathrm{Def}(X) = \mathrm{Im}(g_0) \cup \mathrm{Im}(g_1) \cup \mathrm{Im}(g_2)$. By Statement 2.1(1), there exists an r-map $g' \in \mathrm{End}(\mathrm{Im}(f))$. Define a mapping f' by setting, for $u \in X$,

$$f'(u) = \begin{cases} f(u) & \text{if } f(u) \notin \mathrm{Def}(X), \\ g'(f(u)) & \text{if } f(u) \in \mathrm{Def}(X). \end{cases}$$

Clearly $\mathrm{Im}(f') = \mathrm{Im}(g_0) \cup \mathrm{Im}(g_1) \cup \mathrm{Im}(g_2)$. Since f and g' are idempotent dp-maps, we conclude that $f' \in \mathrm{End}(X)$ is idempotent. From $f = \sup\{g_0, g_1, g_2\}$ it then follows that $\mathrm{Im}(f) = \mathrm{Im}(f')$. □

Lemma 5.3. *Let f be a $3r$-map and let $g_0, g_1, g_2 < f$ be non-equivalent r-maps. Then*

(1) *f is a $t3r$-map if and only if for some $g \in [g_0]$ there exists an $h \in \mathrm{End}(X)$ such that $hg \in [g_1]$, $h^2 g \in [g_2]$ and $h^3 g = g$;*
(2) *if f is an $m3r$-map then $f_0 = \sup\{g_1, g_2\}$ is a $pt2r$-map if and only if there exists an $h \in \mathrm{End}(X)$ such that for some $g \in [g_1]$ we have $hg \in [g_2]$, $h^2 g = g$ and $hg_0 = g_0$.*

P r o o f. Let $f > g_0, g_1, g_2$ be a $t3r$-map. Then, by Statement 2.1(4), the three r-maps g_i can be chosen so that $g_i g_j = g_i$ for $i, j = 0, 1, 2$. Write $\mathrm{Im}(g_i) \setminus \mathrm{Im}(g_{i+1}) =$

$\mathrm{Im}(g_i) \setminus \mathrm{Im}(g_{i+2}) = M_i$ for $i = 0, 1, 2$, and $E = f^{-1}(M_0 \cup M_1 \cup M_2)$, and set

$$h(t) = \begin{cases} f(t) & \text{for } t \notin E, \\ g_{i+1} f(t) & \text{for } t \in f^{-1}(M_i),\ i = 0, 1, 2. \end{cases}$$

The image of $f \in \mathrm{End}(X)$ is finite, and hence h is a *dp*-map. Clearly $hg_i = g_{i+1}$ for $i = 0, 1, 2$.

To prove the converse in (1), assume that f is not a $t3r$-map. Then, by Lemma 5.1, there are components C_0, C_1 with isomorphic Stone nuclei so that $(\mathrm{Im}(g_0) \setminus \mathrm{Im}(g_j)) \cap C_0 \neq \emptyset$ and $(\mathrm{Im}(g_j) \setminus \mathrm{Im}(g_0)) \cap C_1 \neq \emptyset$ for $j = 1, 2$. If for some $g \in [g_0]$ and $h \in \mathrm{End}(X)$ we have $hg \in [g_1]$, $h^2 g \in [g_2]$ and $h^3 g = g$, then $h(\mathrm{Im}(g_i)) = \mathrm{Im}(g_{i+1})$ for $i = 0, 1, 2$ and hence $h(C_1) \subseteq C_1$ and simultaneously $h(C_1) \subseteq C_0$—a contradiction. This completes the proof of (1).

To prove (2), let $f > g_0, g_1, g_2$ be non-equivalent r-maps such that $f_0 = \sup\{g_1, g_2\}$ is a $pt2r$-map and $g_i g_{3-i} = g_i$ for $i = 1, 2$. Denote $\Delta f_0 = \{x_1, x_2\}$, and set

$$h(t) = \begin{cases} f(t) & \text{for } t \notin f^{-1}\{x_1, x_2\}, \\ x_1 & \text{for } t \in f^{-1}\{x_2\}, \\ x_2 & \text{for } t \in f^{-1}\{x_1\}. \end{cases}$$

Then $h \in \mathrm{End}(X)$ because $\mathrm{Im}(f)$ is finite, $x_2 \in E(x_1)$, and $\{x_1, x_2\}$ is an antichain, while $hg_2 = g_1$ and $hg_1 = g_2$ follow from $g_2 g_1 = g_2$ and $g_1 g_2 = g_1$. Clearly $hg_0 = g_0$.

Conversely, if there exists an $h \in \mathrm{End}(X)$ with $hg_0 = g_0$, $hg \in [g_2]$ and $h^2 g = g$ for some $g \in [g_1]$, then $h(C) \subseteq C$ for any component C intersecting $\mathrm{Im}(g_0)$. If g_0 and g_1 are close, then g_0 and g_2 are close because $h(\mathrm{Im}(g_1)) = \mathrm{Im}(g_2)$, and this contradicts Lemma 5.1(3). Thus there exists a component C which intersects $\mathrm{Im}(g_1)$ but not $\mathrm{Im}(g_0)$. From Lemma 5.1(3) and $h(\mathrm{Im}(g_i)) = \mathrm{Im}(g_{3-i})$ for $i = 1, 2$ it follows that g_1 and g_2 are close. But then $f_0 = \sup\{g_1, g_2\}$ is a $pt2r$-map, by Lemma 3.2. □

The observation below now follows directly from the respective definitions.

Lemma 5.4. *For $X, Y \in \mathbb{AR}$ let $\psi\colon \mathrm{End}(X) \longrightarrow \mathrm{End}(Y)$ be an R-isomorphism. Then, for every $f \in \mathrm{End}(X)$,*

(1) *f is a $3r$-map if and only if $\psi(f)$ is a $3r$-map;*
(2) *f is a $t3r$-map if and only if $\psi(f)$ is a $t3r$-map;*
(3) *f is an $m3r$-map if and only if $\psi(f)$ is an $m3r$-map;*
(4) *if f is an $m3r$-map and $g_1, g_2 < f$ are non-equivalent r-maps, then $f_0 = \sup\{g_1, g_2\}$ is a $pt2r$-map if and only if $\psi(f_0)$ is a $pt2r$-map.*

If ψ is also a C-isomorphism, then

(5) *f is a $ct3r$-map if and only if $\psi(f)$ is a $ct3r$-map;*
(6) *f is a $pt3r$-map if and only if $\psi(f)$ is a $pt3r$-map.*

Definition. A set $\mathbb{G}$ of equivalence classes of r-maps is called a *block* if it is maximal with respect to these two properties:

(d1) for every pair $[g_0] \neq [g_1]$ of classes from $\mathbb{G}$, there is a $t2r$-map $f > g_0, g_1$,

(d2) for every triple $\{[g_i] \mid i = 0, 1, 2\}$ of distinct members of $\mathbb{G}$, there is a $t3r$-map $k > g_0, g_1, g_2$.

Lemma 5.5. *Let $\mathbb{G}$ be a block. Then*

(1) $\mathrm{Im}(g_0) \setminus \mathrm{Im}(g_1) = \mathrm{Im}(g_0) \setminus \mathrm{Im}(g_2)$ *and* $\mathrm{Im}(g_0) \cap \mathrm{Im}(g_1) = \mathrm{Im}(g_0) \cap \mathrm{Im}(g_2) = \mathrm{Im}(g_1) \cap \mathrm{Im}(g_2)$ *whenever* $[g_0], [g_1], [g_2] \in \mathbb{G}$ *are pairwise distinct;*

(2) $\mathrm{Im}(g_0) \setminus \mathrm{Im}(g_1) \cong \mathrm{Im}(g_2) \setminus \mathrm{Im}(g_3)$ *for any quadruple* $[g_0], [g_1], [g_2], [g_3] \in \mathbb{G}$ *with* $[g_0] \neq [g_1]$ *and* $[g_2] \neq [g_3]$.

P r o o f. The first statement follows from the fact that $\mathrm{Im}(g_0) \setminus \mathrm{Im}(g_1) = \mathrm{Im}(g_0) \setminus \mathrm{Im}(g_2)$ and $\mathrm{Im}(g_0) \cap \mathrm{Im}(g_1) = \mathrm{Im}(g_0) \cap \mathrm{Im}(g_2) = \mathrm{Im}(g_1) \cap \mathrm{Im}(g_2)$ for any $t3r$-map f and pairwise non-equivalent r-maps $g_0, g_1, g_2 < f$, see Lemma 5.1. The second statement follows from the first because $\mathrm{Im}(g_0) \setminus \mathrm{Im}(g_1) \cong \mathrm{Im}(g_1) \setminus \mathrm{Im}(g_0)$ for any $t2r$-map $f > g_0, g_1$ and pairwise non-equivalent r-maps g_0, g_1 – see Lemma 3.2. $\square$

Lemma 5.5 implies that the mapping β below is correctly defined.

Notation. For any block $\mathbb{G}$ and any $[g] \in \mathbb{G}$ define $\beta(\mathbb{G}, [g]) = \mathrm{Im}(g) \setminus \mathrm{Im}(g')$ for any $[g'] \in \mathbb{G} \setminus \{[g]\}$.

Lemma 5.6. *If $\mathbb{G}$ is a block, then exactly one of these two cases occurs:*

(1) *there is a Stone nucleus N such that for any $[g] \in \mathbb{G}$, the dp-subspace $\beta(\mathbb{G}, [g]) \subseteq X$ is isomorphic to $N \cong \mathrm{Nuc}(K(\beta(\mathbb{G}, [g])))$ and the mapping $\beta' \colon \mathbb{G} \to \mathbb{C}_N(X)$ given by $\beta'([g]) = K(\beta(\mathbb{G}, [g]))$ for all $[g] \in \mathbb{G}$ is a bijection of $\mathbb{G}$ onto $\mathbb{C}_N(X)$;*

(2) *there is a non-defective $x \in \mathrm{Mid}(X)$ such that the mapping $\beta(\mathbb{G}, -)$ maps $\mathbb{G}$ injectively into $E(x)$ and $\{\beta(\mathbb{G}, [g]) \mid [g] \in \mathbb{G}\}$ is an antichain.*

Suppose that N is a Stone nucleus with $|\mathbb{C}_N(X)| > 1$. For every $C \in \mathbb{C}_N(X)$, let $N_C \subseteq C$ be an arbitrarily selected dp-subspace isomorphic to $\mathrm{Nuc}(C) \cong N$. Then for any Stone kernel S of X and for every $C \in \mathbb{C}_N(X)$, there is an r-map $g_C \in \mathrm{End}(X)$ with $\mathrm{Im}(g_C) = (S \setminus (\bigcup\{D \mid D \in \mathbb{C}_N(X)\})) \cup N_C$. The collection $\mathbb{G} = \{g_C \mid C \in \mathbb{C}_N(X)\}$ of these r-maps is a block, and $\beta(\mathbb{G}, [g_C]) = N_C$ for all $C \in \mathbb{C}_N(X)$.

P r o o f. If (2) fails to hold then Lemmas 5.1 and 5.5 imply that there exists a Stone nucleus N such that $\beta(\mathbb{G}, [g])$ is a *dp*-subspace isomorphic to $N \cong \mathrm{Nuc}(K(\beta(\mathbb{G}, [g])))$ for any $[g] \in \mathbb{G}$, and $K(\beta(\mathbb{G}, [g_0])) \neq K(\beta(\mathbb{G}, [g_1]))$ whenever

$[g_0], [g_1] \in \mathbb{G}$ are distinct. Corollary 3.3, Lemma 5.2 and the maximality of a block imply that (1) holds.

Since the subspace $S_C = (S \setminus (\bigcup\{D \mid D \in \mathbb{C}_N(X)\})) \cup N_C$ is a Stone kernel for any $C \in \mathbb{C}_N(X)$, from Statement 2.1(1) it follows that there exists an r-map $g_C \in \mathrm{End}(X)$ with $\mathrm{Im}(g_C) = S_C$. By Corollary 3.3 and Lemma 5.2, the collection $\mathbb{G}$ satisfies (d1) and (d2) from the definition of a block, and Lemmas 3.2 and 5.1 imply the maximality of $\mathbb{G}$. Thus $\mathbb{G}$ is a block. The remainder is clear. $\square$

Definition. Any block $\mathbb{G}$ satisfying statement (1) in Lemma 5.6 is called a *component block*. We say that a component block $\mathbb{G}$ *corresponds to* a Stone nucleus N if $\beta(\mathbb{G}, [g]) \cong N$ for some $[g] \in \mathbb{G}$. If $\beta(\mathbb{G}, [g])$ is a point in $\mathrm{Mid}(X)$ for some $[g] \in \mathbb{G}$, we call $\mathbb{G}$ a *point block*.

Lemma 5.7. *Let $\mathbb{G}_0$, $\mathbb{G}_1$ be blocks such that $[g] \in \mathbb{G}_0 \cap \mathbb{G}_1$. Then the conditions* (1), (2) *and* (3) *below are mutually equivalent, and the same is true also for the conditions* (4), (5), *and* (6).

(1) *For $i = 0, 1$, there exist classes $[g_i] \in \mathbb{G}_i \setminus \{[g]\}$ such that t2r-maps $f_i > g, g_i$ are independent over g,*

(2) *for $i = 0, 1$ and arbitrary classes $[g_i] \in \mathbb{G}_i \setminus \{[g]\}$, any two t2r-maps $f_i > g, g_i$ are independent over g,*

(3) $(\mathrm{Im}(g) \setminus \mathrm{Im}(g_0)) \cap (\mathrm{Im}(g) \setminus \mathrm{Im}(g_1)) = \emptyset$ *for any $[g_i] \in \mathbb{G}_i \setminus \{[g]\}$ with $i = 0, 1$.*

(4) *For $i = 0, 1$, there exist $[g_i] \in \mathbb{G}_i$ with an m3r-map $k > g, g_0, g_1$ such that $k_0 = \sup\{g, g_1\}$ is a pt2r-map;*

(5) *for $i = 0, 1$ and arbitrary $[g_i] \in \mathbb{G}_i \setminus \{[g]\}$, there exist an m3r-map $k > g, g_0, g_1$ such that $k_0 = \sup\{g, g_1\}$ is a pt2r-map;*

(6) *for any $[g_i] \in \mathbb{G}_i \setminus \{[g]\}$ with $i = 0, 1$, $(\mathrm{Im}(g) \setminus \mathrm{Im}(g_1))$ is a singleton which is a non-defective point in the Stone nucleus $(\mathrm{Im}(g) \setminus \mathrm{Im}(g_0))$.*

Proof. According to Lemma 4.1, (1) $\Longrightarrow$ (3) and, by Lemmas 5.5 and 4.1, (3) $\Longrightarrow$ (2). The implication (2) $\Longrightarrow$ (1) is clear.

From Lemmas 5.1, 5.2, 5.3(2) and 5.5 we obtain (4) $\Longrightarrow$ (6) $\Longrightarrow$ (5). The implication (5) $\Longrightarrow$ (4) is obvious. $\square$

Definition. Blocks $\mathbb{G}_0$ and $\mathbb{G}_1$ with $[g] \in \mathbb{G}_0 \cap \mathbb{G}_1$ are called

independent over g if $(\mathrm{Im}(g) \setminus \mathrm{Im}(g_0)) \cap (\mathrm{Im}(g) \setminus \mathrm{Im}(g_1)) = \emptyset$ for any $[g_i] \in \mathbb{G}_i \setminus \{[g]\}$ with $i = 0, 1$,

mixed over g if $(\mathrm{Im}(g) \setminus \mathrm{Im}(g_0)) \cap (\mathrm{Im}(g) \setminus \mathrm{Im}(g_1)) \neq \emptyset$ and $\mathrm{Im}(g) \setminus \mathrm{Im}(g_0) \neq \mathrm{Im}(g) \setminus \mathrm{Im}(g_1)$ for any $[g_i] \in \mathbb{G}_i \setminus \{[g]\}$ with $i = 0, 1$,

similar over g if $\mathrm{Im}(g) \setminus \mathrm{Im}(g_0) = \mathrm{Im}(g) \setminus \mathrm{Im}(g_1)$ for any $[g_i] \in \mathbb{G}_i \setminus \{[g]\}$ with $i = 0, 1$.

Thus each of the first three conditions of Lemma 5.7 characterizes independent blocks, and each of its last three conditions characterizes blocks which are mixed.

Definition. Let g be an r-map, and let $\mathbb{T}$ be a collection of blocks $\mathbb{G}$ such that $[g] \in \mathbb{G}$. If $\mathbb{T}$ has the following three properties:

(e1) any two distinct blocks $\mathbb{G}_0, \mathbb{G}_1 \in \mathbb{T}$ are independent over g,

(e2) if $\mathbb{G} \in \mathbb{T}$ and if $\mathbb{G}_1 \ni [g]$ is a block such that $\mathbb{G}$ and $\mathbb{G}_1$ are mixed, then $\mathbb{G}$ is a component block,

(e3) $\mathbb{T}$ is a maximal collection satisfying (e1) and (e2),

then we say that $\mathbb{T}$ is a *representing collection* over g.

Lemma 5.8. *If $\mathbb{T}$ is a representing collection over an r-map $g \in \mathrm{End}(X)$ then, for every Stone nucleus N with $|\mathbb{C}_N(X)| > 1$, there exists a block $\mathbb{G} \in \mathbb{T}$ corresponding to N.*

Any collection $\mathbb{T}'$ of component blocks $\mathbb{G} \ni [g]$ independent over an r-map $g \in \mathrm{End}(X)$ can be extended to a representing collection $\mathbb{T}$ over g.

Any representing collection $\mathbb{T}$ over an r-map $g \in \mathrm{End}(X)$ is finite—in fact, $|\mathbb{T}| \leqslant |\mathbb{C}(S)| + |\mathrm{Im}(g) \setminus \mathrm{Ext}(X)|$, where S is a Stone kernel of X.

If $\mathbb{T}$ is a representing collection over an r-map $g \in \mathrm{End}(X)$ and if a block $\mathbb{G}'$ is similar to some block $\mathbb{G} \in \mathbb{T}$ over g, then $\mathbb{T}' = (\mathbb{T} \setminus \{\mathbb{G}\}) \cup \{\mathbb{G}'\}$ is also a representing collection over g.

P r o o f. Let $\mathbb{T}$ be a representing collection over g, and let N be a Stone nucleus with $|\mathbb{C}_N(X)| > 1$. By the second statement of Lemma 5.6, there exists a block $\mathbb{G}_N \ni [g]$ corresponding to N because g is an r-map. Since $\mathbb{T}$ is a representing collection, it must contain a block $\mathbb{G}_0$ such that $\mathbb{G}_N$ and $\mathbb{G}_0$ are not independent. Since $\mathbb{G}_N$ is a component block we conclude, by Lemma 5.7 and (e2) in the definition of a representing collection, that $\mathbb{G}_N$ and $\mathbb{G}_0$ cannot be mixed. Thus they are similar, and hence $\mathbb{G}_0$ corresponds to N.

For a given r-map g, let $\mathbb{T}'$ be a collection of independent component blocks containing $[g]$. Then $\mathbb{T}'$ satisfies (e1) and (e2) in the definition of a representing collection. Consider the set $\mathscr{H}$ of all collections $\mathbb{T} \supseteq \mathbb{T}'$ of blocks containing $[g]$ that satisfy (e1) and (e2) from the definition of a representing collection. By Statement 2.1(1), $\mathrm{Im}(g)$ is finite and hence, by Lemma 5.7, all inclusion-ordered chains in $\mathscr{H}$ are finite, so that $\mathscr{H}$ has a maximal element $\mathbb{T}$ containing $\mathbb{T}'$. Any such $\mathbb{T}$ is a representing collection over g.

The third statement follows from the fact that blocks in $\mathbb{T}$ are independent over g.

The fourth statement follows from the definition. □

The claim below concerning component blocks follows from the definition of similarity and Lemma 3.2.

Corollary 5.9. *Let $\mathbb{G}_0$ and $\mathbb{G}_1$ be blocks with $[g] \in \mathbb{G}_0 \cap \mathbb{G}_1$.*

(1) *If $\mathbb{G}_0$ and $\mathbb{G}_1$ are similar then $\mathbb{G}_0$ is a component block if and only if $\mathbb{G}_1$ is a component block.*

(2) *Let $\mathbb{G}$ be a component block and $[g_0] \in \mathbb{G}$. For any $[g] \in \mathbb{G} \setminus \{[g_0]\}$, let $h_g \in \mathrm{End}(X)$ be an r-map such that $\mathrm{Im}(h_g) \cap \mathrm{Im}(g_0) = \mathrm{Im}(g) \cap \mathrm{Im}(g_0)$, and $\mathrm{Im}(h_g) \cap C \neq \emptyset$ exactly when $\mathrm{Im}(g) \cap C \neq \emptyset$ for any component $C \in \mathbb{C}(X)$. Set $h_{g_0} = h_g$. Then $\mathbb{G}' = \{[h_g] \mid [g] \in \mathbb{G}\}$ is a component block similar to $\mathbb{G}$ over g.*

Following is a summary of preservation properties of R-isomorphisms.

Statement 5.10. *Let $X, Y \in \mathbb{AR}$, and let $\psi\colon \mathrm{End}(X) \longrightarrow \mathrm{End}(Y)$ be an R-isomorphism. Then a collection $\mathbb{G}$ of equivalence classes of r-maps is a block in X if and only if its image $\psi(\mathbb{G}) = \{[\psi(g)] \mid [g] \in \mathbb{G}\}$ is a block in Y.*

If $\mathbb{G}_0$ and $\mathbb{G}_1$ are blocks in X with $\{[g]\} = \mathbb{G}_0 \cap \mathbb{G}_1$ then

(1) *$\mathbb{G}_0$ and $\mathbb{G}_1$ are independent in X if and only if $\psi(\mathbb{G}_0)$ and $\psi(\mathbb{G}_1)$ are independent in Y;*

(2) *$\mathbb{G}_0$ and $\mathbb{G}_1$ are mixed in X if and only if $\psi(\mathbb{G}_0)$ and $\psi(\mathbb{G}_1)$ are mixed in Y;*

(3) *$\mathbb{G}_0$ and $\mathbb{G}_1$ are similar in X if and only if $\psi(\mathbb{G}_0)$ and $\psi(\mathbb{G}_1)$ are similar in Y;*

(4) *if $\mathbb{G}_0$ and $\mathbb{G}_1$ are mixed in X, then $\mathbb{G}_0$ is a component block if and only if $\psi(\mathbb{G}_0)$ is a component block.*

If $\mathbb{T}$ is a collection of blocks in X then $\mathbb{T}$ is a representing collection over g in X if and only if $\psi(\mathbb{T}) = \{\psi(\mathbb{G}) \mid \mathbb{G} \in \mathbb{T}\}$ is a representing collection over $\psi(g)$ in Y.

6. Equivalences

In this section we build a decreasing sequence of nine equivalences and employ it to show that in the Main Theorem (3) implies (2).

Definition. Any finitely generated variety $\mathbf{V}$ of almost regular distributive double p-algebras with $P(\mathbf{V}) \subseteq \mathbb{AR}$ will be called an $\mathbb{AR}$-*variety*.

Notation. To any $\mathbb{AR}$-variety $\mathbf{V}$ we assign the following cardinals:

$n_1(\mathbf{V})$, the number of non-isomorphic Stone kernels in $P(\mathbf{V})$;

$n_2(\mathbf{V}) = \max\{|S| \mid S \in P(\mathbf{V}) \text{ is a Stone kernel}\}$;

$n_3(\mathbf{V}) = \max\{|\mathbb{C}(S)| \mid S \in P(\mathbf{V}) \text{ is a Stone kernel}\}$;

$n_4(\mathbf{V}) = \max\{|S \setminus \mathrm{Ext}(S)| \mid S \in P(\mathbf{V}) \text{ is a Stone kernel}\}$;

$n_5(\mathbf{V}) = \max\{|\{g \mid g \in \mathrm{End}(X),\ g \leqslant f\}| \mid f \in \mathrm{End}(X) \text{ is a } br\text{-map, };\ X \in P(\mathbf{V})\}$

$n_7(\mathbf{V}) = \max\{|Aut(\mathrm{End}(S))| \mid S \in P(\mathbf{V}) \text{ is a Stone kernel}\};$
$n_8(\mathbf{V}) = \max\{|\{(x,y) \mid x < y,\ x,y \in \mathrm{Ext}(S)\}| \mid S \in P(\mathbf{V}) \text{ is a Stone kernel}\}.$

Observe that $n_3(\mathbf{V})$ is also the number of pairwise non-isomorphic Stone nuclei which belong to $\mathbf{V}$.

We need the definition below to specify $n_6(\mathbf{V})$.

Definition and notation. Let $S(\mathbf{V}) \subseteq P(\mathbf{V})$ be a set of non-isomorphic Stone nuclei such that for every Stone nucleus $N \in P(\mathbf{V})$ there is an $N_1 \in S(\mathbf{V})$ isomorphic to N. For any Stone nucleus N, select once and for all an isomorphism i_N of N onto a member of $S(\mathbf{V})$.

We need to consider *dp*-spaces such that

(b) $\mathrm{Def}(X) = \emptyset$, and X is a connected space containing exactly two distinct elements x, y with $\mathrm{Ext}(x) = \mathrm{Ext}(y)$.

Clearly, these are the *dp*-spaces which are the union of exactly two intersecting nuclei.

Let $S_1(\mathbf{V}) \subseteq P(\mathbf{V})$ be a set of non-isomorphic *dp*-spaces satisfying (b), and such that for every $X \in P(\mathbf{V})$ with the property (b) there exists an $X_1 \in S_1(\mathbf{V})$ isomorphic to X. For any *dp*-space satisfying (b), select once and for all an isomorphism j_X of X onto a member of $S_1(\mathbf{V})$.

Next, let $\mathscr{H}(\mathbf{V})$ consist of all *dp*-maps $k\colon X \to Y$ with $X \in S(\mathbf{V})$ and $Y \in S_1(\mathbf{V})$ such that $\mathrm{Im}(k)$ contains the two distinct elements $x, y \in Y$ with $\mathrm{Ext}(x) = \mathrm{Ext}(y)$. From (b) it follows that X is a non-singleton nucleus.

Set $n_6(\mathbf{V}) = |\mathscr{H}(\mathbf{V})|$.

Lemma 6.1. *For any $\mathbb{AR}$-variety $\mathbf{V}$, the cardinals $n_1(\mathbf{V})$, $n_2(\mathbf{V})$, $n_3(\mathbf{V})$, $n_4(\mathbf{V})$, $n_5(\mathbf{V})$, $n_6(\mathbf{V})$, $n_7(\mathbf{V})$, and $n_8(\mathbf{V})$ are finite.*

Proof. The finiteness of $n_1(\mathbf{V})$ was shown in [10], the finiteness of $n_2(\mathbf{V})$, $n_3(\mathbf{V})$, $n_4(\mathbf{V})$, $n_7(\mathbf{V})$, and $n_8(\mathbf{V})$ follows from Statement 2.1(1) and from the finiteness of $n_1(\mathbf{V})$. Statements 2.6(2), 2.6(3), 2.1(2) and Lemma 2.5 imply that $n_5(\mathbf{V})$ is finite. The finiteness of $n_6(\mathbf{V})$ follows from the fact that $n_3(\mathbf{V})$ is finite and from Statement 2.1(1). $\square$

Let an $\mathbb{AR}$-variety $\mathbf{V}$ be given, and let $\mathscr{S} \subseteq P(\mathbf{V})$ be a class of equimorphic *dp*-spaces, that is, let $\mathrm{End}(X) \cong \mathrm{End}(Y)$ for all $X, Y \in \mathscr{S}$. For $X, Y, Z \in \mathscr{S}$, select isomorphisms $\psi_{XY}\colon \mathrm{End}(X) \longrightarrow \mathrm{End}(Y)$ so that $\psi_{XZ} = \psi_{YZ} \circ \psi_{XY}$ and $\psi_{XY} \circ \psi_{YX} = \psi_{YY} = id_{\mathrm{End}(Y)}$.

We now intend to define a family of equivalences $\sim_i$ with $i = 1, 2, \ldots, 9$ on $\mathscr{S}$ in such a way that $\sim_{i+1}$ will be finer than $\sim_i$ for every i, each $\sim_i$ will have only finitely many classes, and $Y \sim_9 Z$ will imply that the *dp*-spaces Y, Z are isomorphic.

For $X, Y \in \mathscr{S}$, the *first equivalence* $\sim_1$ will be defined by the requirement that

$X \sim_1 Y$ if and only if the Stone kernels of X and Y are isomorphic.

The lemma below is a consequence of Statement 2.1(2).

Lemma 6.2. *The equivalence $\sim_1$ has at most $n_1(\mathbf{V})$ classes.*

In any class $\mathscr{S}_1$ of $\sim_1$ choose a *dp*-space $X \in \mathscr{S}_1$ and a *br*-map $b_X \in \operatorname{End}(X)$ arbitrarily. The existence of b_X follows from Statement 2.6(1). For any $Y \in \mathscr{S}_1$ set $b_Y = \psi_{XY}(b_X)$. Then b_Y is a *br*-map, by Statement 2.6(4) and, by Statement 2.6(3), for any $Y \in \mathscr{S}_1$ there exists a unique r-map $f_Y \in \operatorname{End}(X)$ with $f_Y \leqslant b_Y$. We now define the *second equivalence* $\sim_2$ on $\mathscr{S}$ by the requirement that

$Y \sim_2 Z$ if and only if $Y \sim_1 Z$ and $\psi_{YX}(f_Y) = \psi_{ZX}(f_Z)$.

The claim below now follows from Statement 2.1(3).

Lemma 6.3. *If the equivalence $\sim_1$ has s_1 classes then the equivalence $\sim_2$ has at most $s_1 n_5(\mathbf{V})$ classes. Furthermore, if $Y \sim_2 Z$ then ψ_{YZ} is an R-isomorphism.*

Next, in any class $\mathscr{S}_2$ of $\sim_2$ choose a *dp*-space $X \in \mathscr{S}_2$ and an r-map $r_X \in \operatorname{End}(X)$ such that $e(r_X) \geqslant n(f)$ for every r-map $f \in \operatorname{End}(X)$. By Lemma 4.6, such an r_X exists and is nice. For any $Y \in \mathscr{S}_2$ set $r_Y = \psi_{XY}(r_X)$. By Statements 4.9(2), 3.11(4) and Lemma 4.6, the map r_Y is a nice r-map and $e(r_Y) \geqslant n(f)$ for every r-map $f \in \operatorname{End}(Y)$.

For any $Y \in \mathscr{S}_2$, there exists an isomorphism $\varphi'_Y\colon \operatorname{Im}(r_X) \longrightarrow \operatorname{Im}(r_Y)$. For any $f \in \operatorname{End}(\operatorname{Im}(r_X))$, write $\psi'_Y(f) = \varphi'_Y f (\varphi'_Y)^{-1}$. Then $\psi'_Y\colon \operatorname{End}(\operatorname{Im}(r_X)) \longrightarrow \operatorname{End}(\operatorname{Im}(r_Y))$ is a monoid isomorphism and $\varphi'_Y f = \psi'_Y(f)\varphi'_Y$ for every $f \in \operatorname{End}(\operatorname{Im}(r_X))$.

By Lemma P.5(1), for any $Y \in \mathscr{S}$, the map $\xi_Y\colon \operatorname{End}(\operatorname{Im}(r_Y)) \to r_Y \operatorname{End}(Y) r_Y$ given by $\xi_Y(f) = f r_Y$ is an isomorphism whose inverse ξ_Y^{-1} is given by $\xi_Y^{-1}(h) = h \restriction \operatorname{Im}(r_Y)$ for every $h \in r_Y \operatorname{End}(Y) r_Y$. Therefore

$$\varphi'_Y r_X f r_X(x) = \psi'_Y(\xi_X^{-1}(r_X f r_X))\varphi'_Y r_X(x) = \psi'_Y(r_X f \restriction \operatorname{Im}(r_X))\varphi'_Y r_X(x)$$

for all $f \in \operatorname{End}(X)$ and $x \in X$. Also, the domain-range restriction of ψ_{YZ} maps $r_Y \operatorname{End}(Y) r_Y$ bijectively onto $r_Z \operatorname{End}(Z) r_Z$ because $\psi_{YZ}(r_Y) = r_Z$.

We now define the *third equivalence* $\sim_3$ on $\mathscr{S}$ by setting

$Y \sim_3 Z$ if and only if $Y \sim_2 Z$ and $\xi_X^{-1}\psi_{YX}\xi_Y\psi'_Y = \xi_X^{-1}\psi_{ZX}\xi_Z\psi'_Z$.

For $Y \sim_3 Z$ write $\varphi_{YZ} = \varphi'_Z(\varphi'_Y)^{-1}$.

Lemma 6.4. *If the equivalence $\sim_2$ has s_2 classes, then the equivalence $\sim_3$ has at most $s_2 n_7(\mathbf{V})$ classes. Furthermore, if $Y \sim_3 Z$, then $\varphi_{YZ}\colon \mathrm{Im}(r_Y) \longrightarrow \mathrm{Im}(r_Z)$ is a dp-isomorphism such that, for $Y \sim_3 Z \sim_3 U$ and any $f \in \mathrm{End}(Y)$,*

$$\varphi_{YZ} r_Y f r_Y = \psi_{YZ}(r_Y f)\varphi_{YZ} r_Y = r_Z \psi_{YZ}(f)\varphi_{ZY} r_Y,$$

$$\varphi_{ZU}\varphi_{YZ} = \varphi_{YU}, \text{ and}$$

$$\varphi_{UY}\varphi_{YU} = \varphi_{YY} \text{ is the identity map on } \mathrm{Im}(r_Y).$$

Proof. From Lemma P.5(1) it follows that the composite $\xi_Y^{-1}\psi_{YX}\xi_Y\psi'_Y$ is an automorphism of $\mathrm{End}(\mathrm{Im}(r_X))$ for every $Y \in \mathscr{S}_2$. Thus if $\sim_2$ has s_2 equivalence classes, then $\sim_3$ has at most $s_2 n_7(\mathbf{V})$ equivalence classes.

If $Y \sim_3 Z$ then $\xi_X^{-1}\psi_{YX}\xi_Y\psi'_Y = \xi_X^{-1}\psi_{ZX}\xi_Z\psi'_Z$ implies $\xi_Z^{-1}\psi_{YZ}\xi_Y = \psi'_Z(\psi'_Y)^{-1}$. Thus for any $f \in \mathrm{End}(Y)$ we obtain

$$\begin{aligned}
\varphi_{YZ} r_Y f r_Y &= \varphi'_Z(\varphi'_Y)^{-1} r_Y f r_Y = \varphi'_Z(\psi'_Y(r_Y f r_Y \restriction \mathrm{Im}(r_Y))(\varphi'_Y)^{-1} r_Y \\
&= \psi'_Z(\psi'_Y)^{-1}(r_Y f r_Y \restriction \mathrm{Im}(r_Y))\varphi'_Z(\varphi'_Y)^{-1} r_Y \\
&= \xi_Z^{-1}\psi_{YZ}\xi_Y(r_Y f r_Y \restriction \mathrm{Im}(r_Y))\varphi_{YZ} r_Y \\
&= \xi_Z^{-1}\psi_{YZ}(r_Y f r_Y)\varphi_{YZ} r_Y = r_Z\psi_{YZ}(f)\varphi_{YZ} r_Y
\end{aligned}$$

because $\xi_Y(r_Y f r_Y \restriction \mathrm{Im}(r_Y)) = r_Y f r_Y$, $\psi_{YZ}(r_Y) = r_Z$, and $\mathrm{Im}(r_Z) = \mathrm{Im}(\varphi_{YZ})$. The remaining equalities follow by a straightforward calculation. □

In any class $\mathscr{S}_3$ of $\sim_3$ choose a *dp*-space $X \in \mathscr{S}_3$. By Lemma 5.8, there exists a representing collection $\mathbb{T}_X$ over r_X in X. Select one such collection and, for any $Y \in \mathscr{S}_3$, set $\mathbb{T}_Y = \psi_{XY}(\mathbb{T}_X)$. Then, by Statement 5.10, $\mathbb{T}_Y$ is a representing collection over r_Y in Y.

For any $Y \in \mathscr{S}_3$ and every $\mathbb{G} \in \mathbb{T}_X$, set $\gamma_Y(\mathbb{G}) = \varphi_{YX}(\beta(\psi_{XY}(\mathbb{G}), [r_Y]))$, where β is the map defined just before Lemma 5.6.

Then $\gamma_Y(\mathbb{G}) \in W$, where $W = \{C \cap \mathrm{Im}(r_X) \mid C \in \mathbb{C}(X)\} \cup (\mathrm{Im}(r_X) \setminus \mathrm{Ext}(X))$.

We now define the *fourth equivalence* $\sim_4$ on $\mathscr{S}$ by requiring that

$$Y \sim_4 Z \text{ if and only if } Y \sim_3 Z \text{ and } \gamma_Y = \gamma_Z.$$

By Lemma 4.1 and Statement 5.10, the mapping γ_Y is one-to-one and, by (1) and (2) in Lemma 5.6, members of its domain can be naturally identified with elements of W. Therefore $\{\gamma_Y \mid Y \sim_4 X\}$ is a collection of partial permutations of W with the same domain.

Lemma 6.5. *If $\sim_3$ has s_3 euivalence classes then $\sim_4$ has at most $s_3(n_3(\mathbf{V}) + n_4(\mathbf{V}))!$ equivalence classes.*

If $Y \sim_4 Z$ then ψ_{YZ} is a C-isomorphism such that $\mathbb{G} \in \mathbb{T}_Y$ is a component block corresponding to a Stone nucleus N if and only if $\psi_{YZ}(\mathbb{G}) \in \mathbb{T}_Z$ is a component block corresponding to N.

Proof. The first claim follows from the observation just above the statement of this Lemma.

Let $Y \sim_4 Z$. The definition of $\sim_4$ implies that a block $\mathbb{G} \in \mathbb{T}_Y$ is a point block if and only if $\varphi_{YZ}(\beta(\mathbb{G}, [r_Y])) = \beta(\psi_{YZ}(\mathbb{G}), [r_Z])$ is a non-extremal point, while $\mathbb{G} \in \mathbb{T}_Y$ corresponds to a Stone nucleus N if and only if the Stone nucleus of the component $\varphi_{YZ}(\beta(\mathbb{G}, [r_Y])) = \beta(\psi_{YZ}(\mathbb{G}), [r_Z])$ is isomorphic to N. Furthermore, for any Stone nucleus N with $|\mathbb{C}_N(Y)| > 1$, any representing collection $\mathbb{T}_Y$ contains a block $\mathbb{G}$ corresponding to N, by the first claim of Lemma 5.8. For any $c2r$-map $f > g, r_Y$ with $[g] \in \mathbb{G} \setminus \{[r_Y]\}$, the map $\psi_{YZ}(f) > \psi_{YZ}(g), r_Z$ is a $2r$-map of Z. Since $[\psi_{YZ}(g)] \in \psi_{YZ}(\mathbb{G})$ and $\psi_{YZ}(\mathbb{G})$ is a component block corresponding to N, the map $\psi_{YZ}(f)$ is a $c2r$-map such that $\Delta\psi_{YZ}(f)$ is a disjoint union of two Stone nuclei isomorphic to N. By Statement 3.12, ψ_{YZ} is a C-isomorphism. □

Let $\mathscr{S}_4$ be an equivalence class of $\sim_4$ and let $Y, Z \in \mathscr{S}_4$.

For every component $C \in \mathbb{C}(Y)$ with $C \cap \mathrm{Im}(r_Y) = \emptyset$, there exists a component block $\mathbb{G} \in \mathbb{T}_Y$ and $[g_C] \in \mathbb{G}$ with $\mathrm{Im}(g_C) \cap C \neq \emptyset$ and $\mathrm{Im}(g_C) \setminus C \subseteq \mathrm{Im}(r_Y)$. By Lemma 6.5, the Stone nucleus of $K(\mathrm{Im}(\psi_{YZ}(g_C)) \setminus \mathrm{Im}(r_Z))$ is isomorphic to $\mathrm{Nuc}(C)$. Therefore the mapping $\varepsilon_{YZ} \colon \mathbb{C}(Y) \longrightarrow \mathbb{C}(Z)$ given by

$$\varepsilon_{YZ}(C) = \begin{cases} K(\mathrm{Im}(\psi_{YZ}(g_C)) \setminus \mathrm{Im}(r_Z)) & \text{if } \mathrm{Im}(r_Y) \cap C = \emptyset, \\ K(\varphi_{YZ}(C \cap \mathrm{Im}(r_Y))) & \text{if } \mathrm{Im}(r_Y) \cap C \neq \emptyset \end{cases}$$

is well defined, and $\mathrm{Nuc}(\varepsilon_{YZ}(C)) \cong \mathrm{Nuc}(C)$ for every $C \in \mathbb{C}(Y)$. For any $Y, Z, U \in \mathscr{S}_4$, equalities $\varepsilon_{ZU}\varepsilon_{YZ} = \varepsilon_{YU}$ and $\varepsilon_{ZY}\varepsilon_{YZ} = id_{\mathbb{C}(Y)}$ follow from the choice of isomorphisms ψ_{YZ} and Lemma 6.4.

For any class $\mathscr{S}_4$ of the equivalence $\sim_4$ choose an $X \in \mathscr{S}_4$. By Lemmas 1.8 and 4.6, for every component $C \in \mathbb{C}(X)$ with $C \cap \mathrm{Im}(r_X) = \emptyset$ there is an r-map g_C satisfying

$$\text{(gC)} \qquad \mathrm{Im}(g_C) \cap C \neq \emptyset \text{ and } \mathrm{Im}(g_C) \setminus C \subseteq \mathrm{Im}(r_X),$$

such that $g_C(\mathrm{Im}(r_X)) = \mathrm{Im}(g_C)$, and $e(g_C) \geqslant n(g)$ for every r-map $g \in \mathrm{End}(X)$ satisfying (gC). For every Stone nucleus N with $|\mathbb{C}_N(X)| > 1$ set

$$\mathbb{G}_N = \{[r_X]\} \cup \{[g_C] \mid C \in \mathbb{C}_N(X) \text{ and } \mathrm{Im}(r_X) \cap C = \emptyset\}.$$

By Lemma 5.6 and Corollary 5.9(2), $\mathbb{G}_N$ is a component block similar to some block $\mathbb{G} \in \mathbb{T}_X$. When we replace the block $\mathbb{G}$ by $\mathbb{G}_N$ for every Stone nucleus N with $|\mathbb{C}_N(X)| > 1$, then, by Lemma 5.8, we obtain a new representing collection $\mathbb{T}_X$. For any $Y \in \mathscr{S}_4$ and for $C' \in \mathbb{C}(Y)$ with $C' \cap \mathrm{Im}(r_Y) = \emptyset$ define $g_{C'} = \psi_{XY}(g_C)$ where $\varepsilon_{YX}(C') = C$. The correctness of this definition follows from the definition of ε_{YX}. By Lemma 4.6 and Statement 4.9(2), all r-maps g_C and $g_{C'} = \psi_{XY}(g_C)$ are nice. By Statement 5.10, $\mathbb{T}_Y = \psi_{XY}(\mathbb{T}_X)$ is a representing collection for every $Y \in \mathscr{S}_4$.

By Statement 2.1(4), for any component $C \in \mathbb{C}(X)$ with $C \cap \mathrm{Im}(r_X) = \emptyset$ there exist r-maps $q_C \in [r_X]$ with $q_C g_C = q_C$ and $g_C q_C = g_C$. For any $C \in \mathbb{C}(X)$ intersecting $\mathrm{Im}(r_X)$, we set $g_C = q_C = r_X$. For $Y \in \mathscr{S}_4$ and $C \in \mathbb{C}(Y)$, set $q_C = \psi_{XY}(q_{C'})$ where $C' = \varepsilon_{YX}(C)$. For $Y, Z \in \mathscr{S}_4$ we now define a mapping $\sigma_{YZ}\colon \mathrm{Ext}(Y) \to \mathrm{Ext}(Z)$ by setting

$$\sigma_{YZ}(x) = \psi_{YZ}(g_C)\varphi_{YZ} q_C(x) \text{ if } x \in \mathrm{Ext}(Y) \cap C \text{ and } C \in \mathbb{C}(Y).$$

Proposition 6.6. *If $Y \sim_4 Z \sim_4 U$, then*

(1) $\sigma_{YZ}\colon \mathrm{Ext}(Y) \to \mathrm{Ext}(Z)$ *is an order preserving bijection with the dp-property;*
(2) $\sigma_{YZ}(y) = \varphi_{YZ}(y)$ *for every* $y \in \mathrm{Ext}(Y) \cap \mathrm{Im}(r_Y)$*;*
(3) $\sigma_{YU} = \sigma_{ZU}\sigma_{YZ}$ *and* $\sigma_{UY}\sigma_{YU}$ *is the identity of* $\mathrm{Ext}(Y)$*;*
(4) $\sigma_{YZ} f(y) = \psi_{YZ}(f)\sigma_{YZ}(y)$ *for every* $y \in \mathrm{Ext}(Y)$ *and for every* $f \in \mathrm{End}(Y)$ *which is an r-map or a $2r$-map, or which satisfies* $f \lesssim r_Y$.

Proof. It is clear that σ_{YZ} satisfies (1), (2) and (3).

In the six steps below, we prove that the equality

$$\text{(e)} \qquad \sigma_{YZ} f(y) = \psi_{YZ}(f)\sigma_{YZ}(y) \text{ for all } y \in \mathrm{Ext}(Y)$$

holds for any $f \in \mathrm{End}(Y)$ which is an r-map or a $2r$-map or satisfies $f \lesssim r_Y$.

Step 1. If $f \lesssim r_Y$ then $\psi_{YZ}(f) \lesssim \psi_{YZ}(r_Y) = r_Z$, and by Lemma 6.4, for any $y \in \mathrm{Ext}(C)$, $C \in \mathbb{C}(Y)$ we have

$$\begin{aligned}\psi_{YZ}(f)\sigma_{YZ}(y) &= \psi_{YZ}(f)\psi_{YZ}(g_C)\varphi_{YZ} q_C(y) = \psi_{YZ}(r_Y f g_C)\varphi_{YZ} q_C(y)\\ &= \varphi_{YZ} r_Y f g_C q_C(y) = \varphi_{YZ} f(y),\end{aligned}$$

so that (e) holds for such f, by (2).

Step 2. Assume that $f \in [g_C]$ for some $C \in \mathbb{C}(Y)$. If $y \in \mathrm{Ext}(C')$ with $f(y) \in C$, then, by Lemma 6.4

$$\begin{aligned}\sigma_{YZ} f(y) &= \psi_{YZ}(g_C)\varphi_{YZ} q_C f g_{C'} q_{C'}(y) = \psi_{YZ}(g_C q_C f g_{C'})\varphi_{YZ} q_{C'}(y)\\ &= \psi_{YZ}(f)\psi_{YZ}(g_{C'})\varphi_{YZ} q_{C'}(y) = \psi_{YZ}(f)\sigma_{YZ}(y).\end{aligned}$$

Next suppose that $y \in \mathrm{Ext}(C')$ and $f(y) \notin C$. Then $f(y) \in \mathrm{Im}(r_Y) \setminus q_C(C)$. Denote $\varepsilon_{YZ}(C) = D$. Then $\psi_{YZ}(q_C) = q_D$ and thus $\sigma_{YZ}(f(y)) \in \mathrm{Im}(r_Z) \setminus \varphi_{YZ}(q_C(C)) = \mathrm{Im}(r_Z) \setminus q_D(D)$. Now, by Lemma 6.4,

$$\begin{aligned}\sigma_{YZ} f(y) &= \varphi_{YZ} q_C f g_{C'} q_{C'}(y) = \psi_{YZ}(q_C f g_{C'})\varphi_{YZ} q_{C'}(y)\\ &= \psi_{YZ}(q_C)\psi_{YZ}(f)\sigma_{YZ}(y) = q_D \psi_{YZ}(f)\sigma_{YZ}(y).\end{aligned}$$

Denote $z = \psi_{YZ}(f)\sigma_{YZ}(y)$. If $q_D(z) \neq z$ then $\psi_{YZ}(f) \in [\psi_{YZ}(g_C)] = [g_D]$ implies that $z \in \mathrm{Im}(\psi_{YZ}(f)) \setminus \mathrm{Im}(r_Z) = \mathrm{Im}(\psi_{YZ}(g_C)) \setminus \mathrm{Im}(r_Z) \subseteq D$. But then $\sigma_{YZ}(f(y)) \in q_D(D)$—a contradiction. Hence $q_D(z) = z$, and (e) holds again.

Step 3. Assume that f is an r-map such that $fr = f$ for some $r \in [r_Y]$, and that $\mathrm{Im}(f) \subseteq \bigcup\{\mathrm{Im}(g_C) \mid C \in \mathbb{C}_{(2)}(Y)\}$. Then there is a smallest set $\mathscr{A} \subseteq \mathbb{C}_{(2)}(Y)$ with $f \leqslant \sup\{g_C \mid C \in \mathscr{A}\}$ and, clearly, the set $\mathscr{A}$ is finite. Since $fr = f$, from Statement 2.1.(4) it follows that for every $C \in \mathscr{A}$ there exists a $g'_C \in [g_C]$ with $fg'_C = f$. Therefore $\psi_{YZ}(f)\psi_{YZ}(g'_C) = \psi_{YZ}(f)$ for all $C \in \mathscr{A}$, $\psi_{YZ}(f) \leqslant \sup\{\psi_{YZ}(g_C) \mid C \in \mathscr{A}\}$ and $\psi_{YZ}(f) \nleqslant \sup\{\psi_{YZ}(g_C) \mid C \in \mathscr{A}'\}$ for any proper subset $\mathscr{A}'$ of $\mathscr{A}$.

Observe that for any $U \in \mathscr{S}$ and an arbitrary r-map $g \in \mathrm{End}(U)$, if $g \leqslant \sup\{g_C \mid C \in \mathscr{B}\}$ for some finite $\mathscr{B} \subseteq \mathbb{C}(U)$ and $g \nleqslant \sup\{g_C \mid C \in \mathscr{B}'\}$ for every proper subset $\mathscr{B}'$ of $\mathscr{B}$, then $\mathrm{Im}(g) = (\bigcup\{\mathrm{Im}(g_C) \setminus \mathrm{Im}(r_U) \mid C \in \mathscr{B}\}) \cup (\bigcap\{\mathrm{Im}(g_C) \mid C \in \mathscr{B}\})$. Therefore $\mathrm{Im}(f) = (\bigcup\{\mathrm{Im}(g'_C) \setminus \mathrm{Im}(r_Y) \mid C \in \mathscr{A}\}) \cup (\bigcap\{\mathrm{Im}(g'_C) \mid C \in \mathscr{A}\})$, and $\mathrm{Im}(\psi_{YZ}(f)) = (\bigcup\{\mathrm{Im}(\psi_{YZ}(g'_C)) \setminus \mathrm{Im}(r_Z) \mid C \in \mathscr{A}\}) \cup (\bigcap\{\mathrm{Im}(\psi_{YZ}(g'_C)) \mid C \in \mathscr{A}\})$, by the choice of $\mathscr{A}$. Since r, f and g'_C are r-maps, from $fr = f$ and $fg'_C = f$ it follows that the kernels of f, r and g'_C coincide. Hence if $y \in \mathrm{Im}(r) \setminus Im(g'_C)$ then $f(y) = g'_C(y)$ and $f(y) = r(y) = y$ for all $y \in \bigcap\{\mathrm{Im}(g'_C) \mid C \in \mathscr{A}\}$. For the same reason, the kernels $\psi_{YZ}(f)$, $\psi_{YZ}(r)$ and $\psi_{YZ}(g'_C)$ coincide, and if $z \in \mathrm{Im}(r_Z) \setminus \mathrm{Im}(\psi_{YZ}(g'_C))$ then $\psi_{YZ}(f)(z) = \psi_{YZ}(g'_C)(z)$ and $\psi_{YZ}(f)(z) = \psi_{YZ}(r)(z) = z$ for all $z \in \bigcap\{\mathrm{Im}(\psi_{YZ}(g'_C)) \mid C \in \mathscr{A}\}$.

Next, let $y \in \mathrm{Ext}(Y)$ be such that $f(y) \in C$ and $C \in \mathscr{A}$. Then $r(y) \in \mathrm{Im}(r) \setminus \mathrm{Im}(g'_C)$. Since $\mathrm{Nuc}(C) \cong \mathrm{Nuc}(\varepsilon_{YZ}(C))$, we get $\varphi_{YZ}(\mathrm{Im}(r) \setminus \mathrm{Im}(g'_C)) = \mathrm{Im}(r_Z) \setminus \mathrm{Im}(\psi_{YZ}(g'_C))$ and thus $\varphi_{YZ}(r(y)) \in \mathrm{Im}(r_Z) \setminus \mathrm{Im}(\psi_{YZ}(g'_C))$. From (2) and Steps 1 and 2, $\sigma_{YZ}(r(y)) = \psi_{YZ}(r)(\sigma_{YZ}(y)) \in \mathrm{Im}(r_Z) \setminus \mathrm{Im}(\psi_{YZ}(g'_C))$. Hence

$$\begin{aligned}\sigma_{YZ} f(y) &= \sigma_{YZ} g'_C r(y) = \psi_{YZ}(g'_C)\psi_{YZ}(r)\sigma_{YZ}(y) = \psi_{YZ}(f)\psi_{YZ}(r)\sigma_{YZ}(y)\\ &= \psi_{YZ}(f)\sigma_{YZ}(y).\end{aligned}$$

Let $y \in \mathrm{Ext}(Y)$ with $f(y) \in \bigcap\{\mathrm{Im}(g'_C) \mid C \in \mathscr{A}\}$. Then $f(y) = r(y)$, and $\mathrm{Nuc}(K(r(y)))$ is not isomorphic to $\mathrm{Nuc}(C)$ for any $C \in \mathscr{A}$. Hence

$$\mathrm{Nuc}(K(\varphi_{YZ}(r(y)))) \ncong \mathrm{Nuc}(\varepsilon_{YZ}(C))$$

for any $C \in \mathscr{A}$, and thus, by (2) and Step 1,

$$\sigma_{YZ}(r(y)) = \psi_{YZ}(r)\sigma_{YZ}(y) \in \bigcap\{\operatorname{Im}(\psi_{YZ}(g'_C)) \mid C \in \mathscr{A}\}.$$

Therefore

$$\sigma_{YZ}f(y) = \sigma_{YZ}r(y) = \psi_{YZ}(r)\sigma_{YZ}(y) = \psi_{YZ}(f)\psi_{YZ}(r)\sigma_{YZ}(y) = \psi_{YZ}(f)\sigma_{YZ}(y).$$

Altogether, $\psi_{YZ}(f)\sigma_{YZ}(y) = \sigma_{YZ}f(y)$ for any $y \in \operatorname{Ext}(Y)$.

Step 4. Assume that f is an r-map such that $fr = f$ for some $r \in [r_Y]$. Then there exists an r-map $f' \in \operatorname{End}(Y)$ close to f, satisfying $ff' = f$, $f'f = f'$, and such that $\operatorname{Im}(f') \subseteq \cup\{\operatorname{Im}(g_C) \mid C \in \mathscr{A}\}$ or $f' \in [r_Y]$.

Then $\psi_{YZ}(f)$ and $\psi_{YZ}(f')$ are close, by Statement 4.9(6). From $\psi_{YZ}(f)\psi_{YZ}(f') = \psi_{YZ}(f)$ we get $\psi_{YZ}(f) \restriction \operatorname{Ext}(Z) = \psi_{YZ}(f') \restriction \operatorname{Ext}(Z)$. Since $f \restriction \operatorname{Ext}(Y) = f' \restriction \operatorname{Ext}(Y)$, using Steps 1 and 3 we conclude that f satisfies (e).

Step 5. Let $f \in \operatorname{End}(Y)$ be any r-map. Then, by Statement 2.1(4), there exist $g_0 \in [r_Y]$ and $g_1 \in [f]$ such that $g_1g_0 = g_1$ and $g_0g_1 = g_0$. Then $g_0f \lesssim r_Y$ and $g_1g_0f = f$, and then (e) holds because for any $y \in \operatorname{Ext}(Y)$ we have

$$\begin{aligned}\sigma_{YZ}f(y) &= \sigma_{YZ}g_1g_0f(y) = \psi_{YZ}(g_1)\sigma_{YZ}g_0f(y) = \psi_{YZ}(g_1)\psi_{YZ}(g_0f)\sigma_{YZ}(y)\\ &= \psi_{YZ}(f)\sigma_{YZ}(y)\end{aligned}$$

from Steps 4 and 1.

Step 6. Let f be a $2r$-map.

If f is a $p2r$-map, then there is an r-map $f' \in \operatorname{End}(Y)$ such that $f'f = f' = ff'$. But then $\psi_{YZ}(f) \restriction \operatorname{Ext}(Z) = \psi_{YZ}(f') \restriction \operatorname{Ext}(Z)$, and because $f \restriction \operatorname{Ext}(Y) = f' \restriction \operatorname{Ext}(Y)$, we conclude that (e) holds again.

Suppose that f is a $c2r$-map. Let $y \in \operatorname{Ext}(Y)$. There is a Stone kernel S of Y intersecting $K(y)$ such that all its other components intersect the image of f, and such that S intersects $K(f(y))$ whenever $\operatorname{Nuc}(K(f(y))) \not\cong \operatorname{Nuc}(K(y))$. By Statement 2.1(1), there is an r-map g_1 with $\operatorname{Im}(g_1) = S$. Clearly, $fg_1 \leqslant g_2$ for some r-map g_2 and, by Statement 2.1(4), we may assume that g_2 is one-to-one on $\operatorname{Im}(r_Y)$. Hence there exists an $h \in \operatorname{End}(Y)$ such that $fg_1 = g_2h$ and $h \lesssim r_Y$. But then, from Steps 1 and 5,

$$\begin{aligned}\sigma_{YZ}f(y) &= \sigma_{YZ}fg_1(y) = \sigma_{YZ}g_2h(y) = \psi_{YZ}(g_2)\sigma_{YZ}h(y)\\ &= \psi_{YZ}(g_2)\psi_{YZ}(h)\sigma_{YZ}(y) = \psi_{YZ}(f)\psi_{YZ}(g_1)\sigma_{YZ}(y)\\ &= \psi_{YZ}(f)\sigma_{YZ}g_1(y) = \psi_{YZ}(f)\sigma_{YZ}(y)\end{aligned}$$

because $g_1(y) = y$. Therefore (e) holds also for any $2r$-map. □

Let $\mathscr{S}_4$ be a class of $\sim_4$. For the *dp*-space $X \in \mathscr{S}_4$ already selected, choose a proper collection $\mathscr{F}_X$ over r_X. This is possible, by Lemma 4.5, because r_X is nice. For every $Y \in \mathscr{S}_4$ denote $\mathscr{F}_Y = \psi_{XY}(\mathscr{F}_X)$. Then $\mathscr{F}_Y$ is a proper collection over r_Y, by Statement 4.9(4).

Further, for every $C \in \mathbb{C}(X)$ with $\mathrm{Im}(r_X) \cap C = \emptyset$, the r-map g_C is nice, and hence there exists a proper collection $\mathscr{F}'$ over g_C. For every $f \in \mathscr{F}'$ with $\Delta f \cap C = \emptyset$ there exists some $g_f \in \mathscr{F}_X$ with $\Delta f \cap \Delta g_f \neq \emptyset$, and hence also a $2r$-map $h_f \in \mathrm{End}(X)$ with $h_f > g_C$ and $\Delta h_f = \Delta g_f$. Define $\mathscr{F}_C = \{f \in \mathscr{F}' \mid \Delta f \cap C \neq \emptyset\} \cup \{h_f \mid f \in \mathscr{F}',\ \Delta f \cap C = \emptyset\}$. Since $\mathscr{F}_X$ is a proper collection, we obtain, by Lemmas 3.7 and 4.5, that $\mathscr{F}_C$ is a proper collection over g_C.

Let $Y \in \mathscr{S}_4$ and $D \in \mathbb{C}(Y)$ be such that $\mathrm{Im}(r_Y) \cap D = \emptyset$, and let $C = \varepsilon_{YX}(D)$. Then $\mathrm{Im}(r_X) \cap C = \emptyset$, and we set $\mathscr{F}_D = \psi_{XY}(\mathscr{F}_C)$. By Statement 4.9(4), $\mathscr{F}_D$ is a proper collection over $g_D = \psi_{XY}(g_C)$. For any $C \in \mathbb{C}_{(2)}(X)$ for which $C \cap \mathrm{Im}(r_X) \neq \emptyset$, we have $g_C = r_X$ and we set $\mathscr{F}_C = \mathscr{F}_X$. For any $Y \in \mathscr{S}_4$, if $D \in \mathbb{C}_{(2)}(Y)$ is such that $D \cap \mathrm{Im}(r_Y) \neq \emptyset$ then $g_D = r_Y$ and $\varepsilon_{YX}(D) \cap \mathrm{Im}(r_X) \neq \emptyset$, so that $\mathscr{F}_D = \mathscr{F}_Y$. Thus for any $Y, Z \in \mathscr{S}_4$ and any $D \in \mathbb{C}_{(2)}(Y)$, we now have a proper collection $\mathscr{F}_D$ over g_D such that if $f \in \mathscr{F}_D$ and $\Delta f \cap \mathrm{Im}(r_Y) \neq \emptyset$ then $\Delta f = \Delta g$ for some $g \in \mathscr{F}_Y$, and if $\varepsilon_{Y,Z}(D) = D'$ then $\psi_{YZ}(\mathscr{F}_D) = \mathscr{F}_{D'}$.

We also note that, for distinct components $D_1, D_2 \in \mathbb{C}_{(2)}(Y)$ disjoint with $\mathrm{Im}(r_Y)$, the proper collections $\mathscr{F}_{D_1}$, $\mathscr{F}_{D_2}$ are disjoint.

For any $Y \in \mathscr{S}_4$ denote

$$Nd(Y) = \mathrm{Ext}(Y) \cup \{x \in \mathrm{Mid}(Y) \setminus \mathrm{Def}(Y) \mid E(x) \neq \{x\}\}.$$

For any $Y, Z \in \mathscr{S}_4$, we now intend to define an extension $\tau_{YZ}\colon Nd(Y) \to Nd(Z)$ of the mapping σ_{YZ} defined earlier. We set

$$\tau_{YZ}(x) = \begin{cases} \sigma_{YZ}(x) & \text{for } x \in \mathrm{Ext}(Y), \\ \nu_f(x) & \text{for } x \in E(\Delta f) \subseteq C,\ f \in \mathscr{F}_C,\ C \in \mathbb{C}_{(2)}(Y), \end{cases}$$

where ν_f was defined in Statement 4.11.

Lemma 6.7. *For any $Y, Z \in \mathscr{S}_4$, the map τ_{YZ} has the following properties:*

(1) *τ_{YZ} maps $E(x)$ bijectively onto $E(\tau_{YZ}(x))$ for every $x \in Nd(Y) \cap \mathrm{Mid}(Y)$;*
(2) *τ_{YZ} maps $K(\mathrm{Im}(r_Y)) \cap Nd(Y)$ bijectively onto $K(\mathrm{Im}(r_Z)) \cap Nd(Z)$;*
(3) *τ_{YZ} maps $C \cap Nd(Y)$ bijectively onto $\varepsilon_{YZ}(C) \cap Nd(Z)$ for every $C \in \mathbb{C}(Y)$ with $\mathrm{Im}(r_Y) \cap C = \emptyset$;*
(4) *τ_{YZ} is a bijection;*
(5) *if $y, z \in \mathrm{Mid}(Y) \cap Nd(Y)$, then $\{y, z\}$ is a comparable pair in Y if and only if $\{\tau_{YZ}(y), \tau_{YZ}(z)\}$ is a comparable pair in Z;*

(6) *if* $y \in \mathrm{Im}(r_Y) \cap Nd(Y)$ *then* $\tau_{YZ}(y) \in \mathrm{Im}(\varphi_{YZ}) = \mathrm{Im}(r_Z)$*;*

(7) *if also* $U \in \mathscr{S}_4$ *then* $\tau_{YU} = \tau_{ZU}\tau_{YZ}$*, and* $\tau_{UY}\tau_{YU}$ *is the identity mapping on* $Nd(Y)$*.*

P r o o f. From Statement 4.11 and Lemma 6.5 we immediately obtain (1).

Since σ_{YZ} is a bijection and $\psi_{YZ}(\mathscr{F}_Y) = \mathscr{F}_Z$, (2) follows from the definition of τ_{YZ} and (1).

We turn to (3). If C is a component of Y disjoint with $\mathrm{Im}(r_Y)$, then $\mathrm{Im}(g_C) \cap C \neq \emptyset$. For any $f \in \mathscr{F}_C$, either $\Delta f \subseteq C$ or else there is an $f' \in \mathscr{F}_Y$ with $\Delta f' = \Delta f$. Since $\mathscr{F}_Y$ and $\mathscr{F}_C$ are proper, from Lemma 4.7 it follows that $\Delta f' = \Delta f$ if and only if $\Delta\psi_{YZ}(f') = \Delta\psi_{YZ}(f)$. Therefore $\Delta f \subseteq C$ if and only if $\Delta\psi_{YZ}(f) \subseteq \varepsilon_{YZ}(C)$, and (3) follows from (1) because σ_{YZ} and ψ_{YZ} are bijective.

Claim (4) follows from (2) and (3).

Claims (5) and (6) are the respective consequences of Statements 4.11(2) and 4.11(1).

Finally, $\tau_{YU} = \tau_{ZU}\tau_{YZ}$ follows from $\sigma_{YU} = \sigma_{ZU}\sigma_{YZ}$ and $\psi_{YU} = \psi_{ZU}\psi_{YZ}$, and the reason for $\tau_{UY}\tau_{YU} = id_{Nd(Y)}$ is similar. □

For any $Y \in \mathscr{S}$ and $y \in Y$, set $\kappa_Y(y) = q_{K(y)}(y)$. Thus $\kappa_Y\colon Y \to \mathrm{Im}(r_Y)$.

Lemma 6.8. *The mapping* κ_Y *has the following properties:*

(1) *if* $x, y \in Nd(Y)$ *and* $K(x) = K(y)$*, then* $\kappa_Y(x) = \kappa_Y(y)$ *if and only if* $y \in E(x)$*;*

(2) κ_Y *has the dp-property;*

(3) *if* $C_0, C_1 \in \mathbb{C}_N(Y)$ *for a Stone nucleus* N*, and if* $f_i \in \mathscr{F}_{C_i}$ *for* $i = 0, 1$*, then* $\kappa_Y(\Delta f_0) = \kappa_Y(\Delta f_1)$ *if and only if for some* $j = 0, 1$ *there exists an* $h \in \mathrm{End}(Y)$ *such that* $h(\mathrm{Im}(f_j)) = \mathrm{Im}(f_{1-j})$ *and* $q_{C_{1-j}}hq_{C_j} = q_{C_j}$*;*

(4) *if* $Y \sim_4 Z$ *and* $x_0, x_1 \in Nd(Y)$*, then* $\kappa_Y(x_0) = \kappa_Y(x_1)$ *if and only if* $\kappa_Z\tau_{YZ}(x_0) = \kappa_Z\tau_{YZ}(x_1)$*.*

P r o o f. Claims (1) and (2) follow because $q_{K(y)}$ is an r-map for every $y \in Y$.

To prove (3), let $C_0, C_1 \in \mathbb{C}_N(Y)$ and $f_i \in \mathscr{F}_{C_i}$ for $i = 0, 1$. Then $\kappa_Y(\Delta f_i)$ is a singleton because $\Delta f_i \subseteq E(x_i)$ for $x_i \in \Delta f_i \cap \mathrm{Im}(g_{C_i})$.

Assume that $\kappa_Y(\Delta f_0) = \kappa_Y(\Delta f_1)$. Then for $h_i = g_{C_{1-i}}q_{C_i} \in \mathrm{End}(Y)$ we have $h_i(x_i) = x_{1-i}$, $h_i(\mathrm{Im}(g_{C_i})) = \mathrm{Im}(g_{C_{1-i}})$, and $q_{C_{1-i}}h_iq_{C_i} = q_{C_i}$ for $i = 0, 1$. Obviously, for some $j = 0, 1$, f_j is a *pt*2r-map or f_{1-j} is an $n2r$-map. Hence there is an $h \in \mathrm{End}(Y)$ such that $h(\mathrm{Im}(f_j)) = \mathrm{Im}(f_{1-j})$, by Statement 3.9(1).

Conversely, assume that for some $j = 0, 1$ there exists an $h \in \mathrm{End}(Y)$ with $h(\mathrm{Im}(f_j)) = \mathrm{Im}(f_{1-j})$ and $q_{C_{1-j}}hq_{C_j} = q_{C_j}$. Then $h(\Delta f_j) = \Delta f_{1-j}$ and $q_{C_j}(\Delta f_j) =$

$q_{C_{1-j}}(\Delta f_{1-j})$ because f_j and f_{1-j} are $p2r$-maps. Whence $\kappa_Y(\Delta f_0) = \kappa_Y(\Delta f_1)$ and (3) is proved.

Let $x_0, x_1 \in Nd(Y)$. If $x_0 \in \mathrm{Ext}(Y)$ then $\kappa_Y(x_0) = \kappa_Y(x_1)$ implies that $x_1 \in \mathrm{Ext}(Y)$ and $q_{K(x_0)}(x_0) = q_{K(x_1)}(x_1)$. Since τ_{YZ} extends σ_{YZ}, from Proposition 6.6 we obtain $q_{K(\tau_{YZ}(x_0))}(\tau_{YZ}(x_0)) = q_{K(\tau_{YZ}(x_1))}(\tau_{YZ}(x_1))$, and hence $\kappa_Z(\tau_{YZ}(x_0)) = \kappa_Z(\tau_{YZ}(x_1))$.

If $x_i \in \mathrm{Im}(g_{K(x_i)}) \setminus \mathrm{Ext}(Y)$, then there exists a unique $f_i \in \mathscr{F}_{K(x_i)}$ with $x_i \in \Delta f_i$ for $i = 0, 1$. In this case, if $\kappa_Y(x_0) = \kappa_Y(x_1)$ then from Statement 3.9(3) and from (3) we conclude that $\kappa_Z(\tau_{YZ}(x_0)) = \kappa_Z(\tau_{YZ}(x_1))$.

For $i = 0, 1$, there is a unique $z_i \in \mathrm{Im}(g_{K(x_i)})$ with $z_i \in E(x_i)$. By Lemma 6.7(1), we have $\tau_{YZ}(z_i) \in E(\tau_{YZ}(x_i))$ and, by (1), $\kappa_Y(x_i) = \kappa_Y(z_i)$ and $\kappa_Z(\tau_{YZ}(x_i)) = \kappa_Z(\tau_{YZ}(z_i))$. Thus, by the previous paragraph, $\kappa_Y(x_0) = \kappa_Y(x_1)$ implies

$$\kappa_Z(\tau_{YZ}(z_0)) = \kappa_Z(\tau_{YZ}(z_1)),$$

and $\kappa_Z(\tau_{YZ}(x_0)) = \kappa_Z(\tau_{YZ}(x_1))$ follows.

If $\kappa_Z(\tau_{YZ}(x_0)) = \kappa_Z(\tau_{YZ}(x_1))$ then, using what was shown above, we obtain

$$\kappa_Y(x_0) = \kappa_Y(\tau_{ZY}(\tau_{YZ}(x_0))) = \kappa_Y(\tau_{ZY}(\tau_{YZ}(x_1))) = \kappa_Y(x_1),$$

which completes the proof of (4). □

For any $Y \in \mathscr{S}_4$ we now intend to define a partial mapping ϱ_Y from $\mathrm{Im}(r_X)$ into itself as follows: for any $x \in \mathrm{Im}(r_X)$ for which there exists a $y \in Nd(Y)$ such that $x = \varphi_{YX}\kappa_Y(y)$, and only for these elements x, we set $\varrho_Y(x) = \kappa_X(\tau_{YX}(y))$.

Lemma 6.9. *Let $Y, Z \in \mathscr{S}_4$. Then the partial mapping ϱ_Y is correctly defined and has the following properties:*

(1) *ϱ_Y is one-to-one;*
(2) *$\varrho_Y(x)$ is defined and $\varrho_Y(x) = x$ for every $x \in \mathrm{Ext}(\mathrm{Im}(r_X))$;*
(3) *if $\varrho_Y = \varrho_Z$, then τ_{YZ} has the dp-property and $\varphi_{YZ}\kappa_Y = \kappa_Z\tau_{YZ}$.*

P r o o f. From Lemma 6.8(4) it follows that ϱ_{XY} is a correctly defined injection and, by Proposition 6.6(4), $\varrho_Y(x) = x$ for any $x \in \mathrm{Ext}(\mathrm{Im}(r_X))$. Thus it remains to prove (3).

Assume that $\varrho_Y = \varrho_Z$. Let $y \in Nd(Y)$. Then $x = \varphi_{YX}\kappa_Y(y) \in \mathrm{Im}(r_X)$ and hence $\varrho_Z(x) = \varrho_Y(x) = \kappa_X(\tau_{YX}(y))$. Write $z = \tau_{YZ}(y)$. Then

$$\kappa_X(\tau_{ZX}(z)) = \kappa_X(\tau_{ZX}(\tau_{YZ}(y))) = \kappa_X(\tau_{YX}(y)) = \varrho_Z(x)$$

and, by (1), $\kappa_Z(z) = \varphi_{XZ}(x) = \varphi_{YZ}(\varphi_{XY}(x)) = \varphi_{YZ}\kappa_Y(y)$. Thus $\varphi_{YZ}\kappa_Y = \kappa_Z\tau_{YZ}$. From this and Lemma 6.7(2) it follows that τ_{YZ} maps $C \cap Nd(Y)$ bijectively onto $\varepsilon_{YZ}(C) \cap Nd(Z)$ for every component C of Y intersecting $\mathrm{Im}(r_Y)$, while this is true for all other components because of Lemma 6.7(3). Using $\varphi_{YZ}\kappa_Y = \kappa_Z\tau_{YZ}$, the fact that φ_{YZ}, κ_Y and κ_Z have the *dp*-property (see Lemmas 6.4 and 6.8(2)), together with Lemma 6.8(1), we conclude that τ_{YZ} has the *dp*-property. This proves (3). $\square$

Define the *fifth equivalence* $\sim_5$ on $\mathscr{S}$ by

$$Y \sim_5 Z \text{ if and only if } Y \sim_4 Z \text{ and } \varrho_Y = \varrho_Z.$$

The claim below holds because $\mathrm{Im}(\varrho_Y) = \kappa_X(Nd(X))$ for every $Y \in \mathscr{S}_4$.

Lemma 6.10. *If the equivalence $\sim_4$ has s_4 classes then $\sim_5$ has at most $s_4 n_4(\mathbf{V})!$ classes. If $Y \sim_5 Z$, then τ_{YZ} has the dp-property.*

Let $Y \sim_5 Z$. Recall that, for any $x \in Y \setminus \mathrm{Def}(Y)$, we have $x \notin Nd(Y)$ if and only if $x \in \mathrm{Mid}(Y)$ and $E(x) = \{x\}$. Consider such an x, and denote $C = K(x)$. Then $\omega'_{YZ}(x) = \psi_{YZ}(g_C)\varphi_{YZ}q_C(x) \in (\mathrm{Mid}(Z) \setminus \mathrm{Def}(Z))$, and $y = \omega'_{YZ}(x) \notin Nd(Z)$ because τ_{ZY} has the *dp*-property and maps $Nd(Z)$ onto $Nd(Y)$. Therefore $E(y) = \{y\}$.

This enables us to extend τ_{YZ} to a mapping $\omega_{YZ}\colon Y \setminus \mathrm{Def}(Y) \to Z \setminus \mathrm{Def}(Z)$ by

$$\omega_{YZ}(x) = \begin{cases} \tau_{YZ}(x) & \text{for } x \in Nd(Y), \\ \omega'_{YZ}(x) & \text{for } x \in Y \setminus (Nd(Y) \cup \mathrm{Def}(Y)). \end{cases}$$

Lemma 6.11. *If $Y \sim_5 Z \sim_5 U$, then*

(1) *ω_{YZ} is a bijection and has the dp-property;*
(2) *$\omega_{YZ}(x) = \tau_{YZ}(x)$ for any $x \in Nd(Y)$ and $\omega_{YZ}(x) = \varphi_{YZ}(x)$ for any $x \in \mathrm{Im}(r_Y)$;*
(3) *$\omega_{ZU}\omega_{YZ} = \omega_{YU}$, and $\omega_{UY}\omega_{YU}$ is the identity on $Y \setminus \mathrm{Def}(Y)$;*
(4) *$\omega_{YZ}f = \psi_{YZ}(f)\omega_{YZ}$ whenever $f \in \mathrm{End}(Y)$ is an r-map or a $c2r$-map.*

P r o o f. Claims (1), (3) and the first statement in (2) follow easily. The second statement in (2) follows from Proposition 6.6(2) and Lemma 6.7(6) because ω_{YZ} has the *dp*-property, by (1).

It remains to prove (4).

We know that $\sigma_{YZ}f = \psi_{YZ}(f)\sigma_{YZ}$ on $\mathrm{Ext}(Y)$ for every f which is an r-map or a $c2r$-map. In order to prove (4), we only need to show that $\omega_{YZ}(\mathrm{Im}(f) \cap \mathrm{Mid}(Y)) = \mathrm{Im}(\psi_{YZ}(f)) \cap \mathrm{Mid}(Z)$. Since the image of a $c2r$-map is the union of images of r-maps below it, see Lemma 3.1, we may assume that f is an r-map.

Let $x \in \mathrm{Im}(f) \cap \mathrm{Mid}(Y)$. If $E(x) = \{x\}$, then $E(\omega_{YZ}(x)) = \{\omega_{YZ}(x)\}$ and, because the bijection ω_{YZ} extends σ_{YZ} and has the *dp*-property, $\omega_{YZ}(x) \in \mathrm{Im}(\psi_{YZ}(f)) \cap \mathrm{Mid}(Z)$.

If $E(x) \neq \{x\}$ then there exists an $f' \in \mathscr{F}_{K(x)}$ with $\Delta f' \subseteq E(x)$ and either f' is an $n2r$-map or $E(x)$ is an antichain. Since $\mathscr{F}_{K(x)}$ is proper, by Theorem 4.10 and Statement 4.11, for every $z \in E(x)$ there exists a $k_z \in S(\mathscr{F}_{K(x)}, g_{K(x)}, f')$ such that $k_z g_{K(x)}$ is an r-map for which $z \in \mathrm{Im}(k_z g_{K(x)})$ and $\omega_{YZ}(z) = \nu_{f'}(z) \in \mathrm{Im}(\psi_{YZ}(k_z g_{K(x)}))$. Since f and $k_z g_{K(x)}$ are r-maps such that $\mathrm{Im}(f)$ and $\mathrm{Im}(k_z g_{K(x)})$ intersect $E(x)$ we conclude, by Proposition 6.6(1) and 6.6(4), that $\mathrm{Im}(\psi_{YZ}(f))$ and $\mathrm{Im}(\psi_{YZ}(k_z g_{K(x)}))$ intersect $E(\Delta\psi_{YZ}(f'))$. Furthermore, from the hypothesis on f' and from Statements 3.11(3) and 4.11(2) it follows that either $\psi_{YZ}(f')$ is an $n2r$-map or $E(\Delta\psi_{YZ}(f'))$ is an antichain. Therefore, by Statement 3.13, $\mathrm{Im}(f) \cap E(\Delta f') = \mathrm{Im}(k_z g_{K(x)}) \cap E(\Delta f')$ if and only if

$$\mathrm{Im}(\psi_{YZ}(f)) \cap E(\Delta\psi_{YZ}(f')) = \mathrm{Im}(\psi_{YZ}(k_z g_{K(x)})) \cap E(\Delta\psi_{YZ}(f')).$$

Hence $f(x) = k_z g_{K(x)}(x)$ if and only if

$$\psi_{YZ}(f)(\omega_{YZ}(x)) = \psi_{YZ}(k_z g_{K(x)})(\omega_{YZ}(x)) = k_{\omega_{YZ}(z)} g_{K(\omega_{YZ}(x))}(\omega_{YZ}(x)).$$

Therefore $x = z$ if and only if $\psi_{YZ}(f)(\omega_{YZ}(x)) = \omega_{YZ}(z)$. Thus $\omega_{YZ}(x) \in \mathrm{Im}(\psi_{YZ}(f))$. This proves (4). $\square$

For any class $\mathscr{S}_5$ of the fifth equivalence, choose an $X \in \mathscr{S}_5$. Let $\mathscr{G}_0(X)$ be a collection of equivalence classes of $p2r$-maps of X such that

(v1) $\mathrm{Im}(f_0) \ncong \mathrm{Im}(f_1)$ whenever $[f_0], [f_1] \in \mathscr{G}_0(X)$ are distinct,
(v2) for every $p2r$-map $f \in \mathrm{End}(X)$ there is an $f' \in \mathscr{G}_0(X)$ with $\mathrm{Im}(f) \cong \mathrm{Im}(f')$,
(v3) if f is a $p2r$-map, $[f'] \in \mathscr{G}_0(X)$ and $\mathrm{Im}(f) \cong \mathrm{Im}(f')$, then

$$|\{C \in \mathbb{C}(X) \mid C \cap \mathrm{Im}(r_X) \cap \mathrm{Im}(f) \neq \emptyset\}| \geqslant |\{C \in \mathbb{C}(X) \mid C \cap \mathrm{Im}(r_X) \cap \mathrm{Im}(f') \neq \emptyset\}|.$$

The existence and the finiteness of $\mathscr{G}_0(X)$ follow from Lemma 3.1 and Statement 2.1(1).

Let $[g] \in \mathscr{G}_0(X)$ and let $C, C' \in \mathbb{C}(X)$ be components such that $M = C \cap \mathrm{Im}(r_X) \neq \emptyset = C \cap \mathrm{Im}(g)$, and $C' \cap \mathrm{Im}(g) = M'$ satisfies (b) from the definition of $n_6(\mathbf{V}) = |\mathscr{H}(\mathbf{V})|$ at the beginning of this section. Suppose that there is a $k\colon N \to N' \in \mathscr{H}(\mathbf{V})$ with $M \cong N$ and $M' \cong N'$. By Lemma 1.8, there exists a map $\langle gk \rangle \in [g]$ such that $\langle gk \rangle \restriction C = j_{M'}^{-1} k i_M r_X \restriction C$ and $\mathrm{Nuc}(K(\langle gk \rangle(D))) \cong \mathrm{Nuc}(D)$ for every $D \in \mathbb{C}(X) \setminus \{C\}$ with $D \cap \mathrm{Im}(r_X) \neq \emptyset$. Select one such $\langle gk \rangle \in [g]$ for each $k \in \mathscr{H}(\mathbf{V})$ and $[g] \in \mathscr{G}_0(X)$, and let $\mathscr{G}_1(X)$ denote the collection of all these $\langle gk \rangle \in \mathrm{End}(X)$.

Lemma 6.12. *For any $Y \sim_5 X \sim_5 Z$,*

(1) *if, for a $p2r$-map $f \in \operatorname{End}(Y)$ and $y \in Y$, there exists a $c2r$-map or an r-map $g \in \operatorname{End}(Y)$ for which $y \in \operatorname{Im}(g)$ and fg is an r-map, then $\omega_{YZ} f(y) = \psi_{YZ}(f)\omega_{YZ}(y)$;*

(2) *$\omega_{YZ}(\operatorname{Im}(f)) = \operatorname{Im}(\psi_{YZ}(f))$ for any $p2r$-map $f \in \operatorname{End}(Y)$;*

(3) *if for a $p2r$-map $f \in \operatorname{End}(Y)$ and $y \in Y$ there exists a dp-subspace $M \ni y$ isomorphic to $\operatorname{Nuc}(K(y))$ and such that $\Delta f \subseteq f(M)$, then there exist r-maps $g_0, g_1 \in \operatorname{End}(Y)$ and $p2r$-maps $f_0, f_1 \in \operatorname{End}(Y)$ such that $g_1 \in [r_Y]$, $f_0 \in \psi_{XY}(\mathscr{G}_1(X))$, $\operatorname{Im}(g_0) \cap K(y) = M$, $f_1(\operatorname{Im}(f_0)) = \operatorname{Im}(f)$, and $fg_0 = f_1 f_0 g_1$;*

(4) *if $\omega_{YX}\psi_{XY}(f)\omega_{XY} \restriction \operatorname{Im}(r_X) = \omega_{ZX}\psi_{XZ}(f)\omega_{XZ} \restriction \operatorname{Im}(r_X)$ for every $f \in \mathscr{G}_1(X)$, then $\omega_{YZ} g = \psi_{YZ}(g)\omega_{YZ}$ for every $p2r$-map $g \in \operatorname{End}(Y)$;*

(5) *$|\mathscr{G}_1(X)| \leqslant n_6(\mathbf{V})$.*

P r o o f. To prove (1) we apply Lemma 6.11(4). Then

$$\omega_{YZ} f(x) = \omega_{YZ} fg(x) = \psi_{YZ}(fg)\omega_{YZ}(x) = \psi_{YZ}(f)\omega_{YZ} g(x) = \psi_{YZ}(f)\omega_{YZ}(x),$$

and (1) is proved.

Let $f \in \operatorname{End}(X)$ be a $p2r$-map. Then there are non-equivalent r-maps $g_0, g_1 < f$ such that $\operatorname{Im}(f) = \operatorname{Im}(g_0) \cup \operatorname{Im}(g_1)$ and $\operatorname{Im}(\psi_{YZ}(f)) = \operatorname{Im}(\psi_{YZ}(g_0)) \cup \operatorname{Im}(\psi_{YZ}(g_1))$— see Lemma 3.1 and Statements 3.11(3) and 3.12(2). By Lemma 6.11(4), $\omega_{YZ}(\operatorname{Im}(g_i)) = \operatorname{Im}(\psi_{YZ}(g_i))$ for $i = 0, 1$ and the proof of (2) is complete.

To prove (3) first assume that $Y = X$. By (v1), there exists an $[f'] \in \mathscr{G}_0(X)$ such that $\operatorname{Im}(f') \cong \operatorname{Im}(f)$. Denote $M' = \operatorname{Im}(f) \cap K(\Delta f)$, then M' satisfies (b). Since M is a Stone nucleus of $K(y)$ such that $\Delta f \subseteq f(M)$, we conclude that the map $k = j_{M'} f i_M^{-1}$ belongs to $\mathscr{H}(\mathbf{V})$.

Set $f_0 = \langle f'k \rangle \in \mathscr{G}_1(X)$ and $N' = K(\Delta f') \cap \operatorname{Im}(f')$. Since $\operatorname{Im}(f_0) = \operatorname{Im}(f') \cong \operatorname{Im}(f)$, from Statement 3.9(1) and Lemma 1.8 we obtain an $f_1 \in \operatorname{End}(X)$ such that $f_1 j_{N'}^{-1} = j_{M'}^{-1}$, $f_1(\operatorname{Im}(f_0)) = \operatorname{Im}(f)$ and $f_1(z) = z$ for any $z \in \operatorname{Ext}(\operatorname{Im}(f))$. Let $C_r, C_f \in \mathbb{C}(X)$ have their Stone nuclei isomorphic to M and let $C_r \cap \operatorname{Im}(r_X) \neq \emptyset \neq C_f \cap \operatorname{Im}(f)$. Then the set $M \cup (\operatorname{Im}(g') \setminus C_f)$ is a Stone kernel of X, and, by Statement 2.1(4), there is an r-map $g_1 \in [r_X]$ such that $g_1(C_f) \subseteq C_r$, $i_{g_1(M)} g_1 \restriction M = i_M$, $g_1(\operatorname{Im}(f) \setminus C_f) = \operatorname{Im}(r_X) \setminus C_r$ and $f_1 f_0 g_1(x) = x$ for any $x \in \operatorname{Im}(f) \setminus C_f$. The last property holds because $f_1(z) = z$ for any $z \in \operatorname{Ext}(\operatorname{Im}(f))$. Since $f \restriction M = j_{M'}^{-1} k i_M \restriction M = f_1 j_{N'}^{-1} k i_{g_1(M)} g_1 \restriction M = f_1 f_0 g_1 \restriction M$ and because f is an idempotent we conclude that $f_1 f_0 g_1 \restriction (M \cup (\operatorname{Im}(f) \setminus C_f)) = f \restriction (M \cup (\operatorname{Im}(f) \setminus C_f))$. By Statement 2.1(1) and 2.1(4), there exists an r-map g_0 with $g_1 g_0 = g_1$ and $\operatorname{Im}(g_0) = M \cup (\operatorname{Im}(f) \setminus C_f)$. Then for any $x \in X$, $f_1 f_0 g_1(x) = f_1 f_0 g_1 g_0(x) = f g_0(x)$ because $g_0(x) \in M \cup (\operatorname{Im}(f) \setminus C_f)$. Therefore (3) holds for $Y = X$.

If $Y \sim_5 X$ and $f \in \mathrm{End}(Y)$ satisfies the hypothesis, then $\psi_{YX}(f)$ is a $p2r$-map because ψ_{YX} is a C-isomorphism, by Lemma 6.5. Let $h \in \mathrm{End}(Y)$ be any r-map such that $M \subseteq \mathrm{Im}(h) \subseteq \mathrm{Im}(f) \cup M$. Then, by Lemma 6.11, $\omega_{YX}h = \psi_{YX}(h)\omega_{YX}$ and hence $\omega_{YX}(M)$ is a nucleus. From (1) we obtain that $\omega_{YX}f(x) = \psi_{YX}(f)\omega_{YX}(x)$ for every $x \in \mathrm{Im}(h) \setminus M$. If $g < f$ is any r-map then $gfh \neq fh$, and hence $\Delta\psi_{YX}(f) \subseteq \psi_{YX}(f)(\mathrm{Im}(\psi_{YX}(h)))$. Therefore $\Delta\psi_{YX}(f) \subseteq \psi_{YX}(f)(\omega_{YX}(M))$, and the hypothesis of (3) is satisfied by $\psi_{YX}(f)$. From the first part of the proof it then follows that (3) holds.

We turn to (4). Let $f \in \mathrm{End}(Y)$ be a $p2r$-map and $y \in Y$. If there is a $g \in \mathrm{End}(Y)$ that is either an r-map or a $c2r$-map for which fg is an r-map, then (1) implies that $\omega_{YZ}f(y) = \psi_{YZ}(f)\omega_{YZ}(y)$. If there is no such g, then there is a Stone nucleus $N \ni y$ isomorphic to $\mathrm{Nuc}(K(y))$ such that $\Delta f \subseteq f(N)$. By (3), there exist r-maps $g_0, g_1 \in \mathrm{End}(Y)$ and $p2r$-maps $f_0, f_1 \in \mathrm{End}(Y)$ such that $N \subseteq \mathrm{Im}(g_0)$, $g_1 \in [r_Y]$, $f_1(\mathrm{Im}(f_0)) = \mathrm{Im}(f)$, $f_0 \in \psi_{XY}(\mathscr{G}_1(X))$, and $fg_0 = f_1f_0g_1$. From $f_1(\mathrm{Im}(f_0)) = \mathrm{Im}(f)$ and (1) it follows that $\omega_{YZ}f_1(u) = \psi_{YZ}(f_1)\omega_{YZ}(u)$ for every $u \in \mathrm{Im}(f_0)$. The hypothesis $\omega_{YZ}\psi_{XY}(f_0) \restriction \mathrm{Im}(r_Y) = \psi_{XZ}(f_0)\omega_{YZ} \restriction \mathrm{Im}(r_Y)$ for $f_0 \in \mathscr{G}_1(X)$ and Lemma 6.11(4) imply that $\omega_{YZ}f_1f_0g_1(y) = \psi_{YZ}(f_1f_0g_1)\omega_{YZ}(y)$ because $\mathrm{Im}(r_Y) = \mathrm{Im}(g_1)$. But then

$$\begin{aligned}\omega_{YZ}f(y) &= \omega_{YZ}fg_0(y) = \omega_{YZ}f_1f_0g_1(y) = \psi_{YZ}(f_1f_0g_1)\omega_{YZ}(y)\\ &= \psi_{YZ}(fg_0)\omega_{YZ}(y) = \psi_{YZ}(f)\omega_{YZ}g_0(y) = \psi_{YZ}(f)\omega_{YZ}(y),\end{aligned}$$

and (4) is proved.

The definition of $\mathscr{G}_1(X)$ implies (5) immediately. □

We now define the *sixth equivalence* $\sim_6$ on $\mathscr{S}$ as follows.

$Y \sim_6 Z$ if and only if $Y \sim_5 Z$ and
$\omega_{YX}\psi_{XY}(f)\omega_{XY} \restriction \mathrm{Im}(r_X) = \omega_{ZX}\psi_{XZ}(f)\omega_{XZ} \restriction \mathrm{Im}(r_X)$ for every $f \in \mathscr{G}_1(X)$.

Lemma 6.13. *If the equivalence $\sim_5$ has s_5 classes, then $\sim_6$ has at most $s_5 2^{n_6(\mathbf{V})}$ classes.*

If $Y \sim_6 Z$, then $\psi_{YZ}(f)\omega_{YZ} = \omega_{YZ}f$ for every f that is an r-map or a $2r$-map.

Proof. By Lemma 6.12(2), $\mathrm{Im}(\omega_{YX}\psi_{XY}(f)\omega_{XY}) = \mathrm{Im}(f)$ for all $f \in \mathscr{G}_1(X)$ and $Y \sim_5 X$. Let $k\colon \mathrm{Im}(f) \to \mathrm{Im}(f)$ denote the non-identity involution with $k(z) = z$ for all $z \in \mathrm{Im}(f) \setminus \Delta f$. By Lemma 6.11(1), ω_{XY} and ω_{YX} have the dp-property, and hence either $\omega_{YX}\psi_{XY}(f)\omega_{XY} = f$ or $\omega_{YX}\psi_{XY}(f)\omega_{XY} = kf$. But then Lemma 6.12(5) implies the first claim and Lemma 6.12(4) implies the second. □

Lemma 6.14. *If $Y \sim_6 Z$ and $y \in \operatorname{Im}(\kappa_Y) \cap Mid(Y)$, then ω_{YZ} is either an order isomorphism or an order anti-isomorphism of $\kappa_Y^{-1}\{y\}$ onto $\kappa_Z^{-1}\{\varphi_{YZ}(y)\}$.*

Proof. The statement follows from Lemma P.6, (1) and (3) of Statement 3.9, and Lemma 6.7(5). $\square$

Now we define the *seventh equivalence* $\sim_7$. We set

$$Y \sim_7 Z \text{ if and only if } Y \sim_6 Z \text{ and } \omega_{YZ} \text{ is order preserving.}$$

The relation $\sim_7$ is an equivalence because of Lemmas 6.11(3) and 6.14. From Lemma 6.14 and from $\operatorname{Im}(\kappa_Y) \cap \operatorname{Mid}(Y) \subseteq \operatorname{Im}(r_Y) \cap \operatorname{Mid}(Y)$, we get the claim below.

Lemma 6.15. *If the equivalence $\sim_6$ has s_6 classes then the equivalence $\sim_7$ has at most $s_6 2^{n_4(\mathbf{V})}$ classes.*

If $Y \sim_7 Z$, then ω_{YZ} has the dp-property, preserves order, and satisfies $\psi_{YZ}(f)\omega_{YZ} = \omega_{YZ}f$ for every $f \in \operatorname{End}(Y)$ that is an r-map or a $2r$-map.

For any $Y, Z \in \mathscr{S}$ with $Y \sim_7 Z$, we now define

$$\lambda_{YZ}(y) = \begin{cases} \omega_{YZ}(y) & \text{for every } y \in Y \setminus \operatorname{Def}(Y), \\ d(\psi_{YZ}(f)) & \text{if } f \in \operatorname{End}(Y) \text{ is a } dr\text{-map and } d(f) = y \in \operatorname{Def}(Y). \end{cases}$$

Lemma 6.16. *Let $\mathscr{S}_7$ be a class of the seventh equivalence. Then, for any $Y, Z \in \mathscr{S}_7$, the map $\lambda_{YZ}\colon Y \to Z$ is a correctly defined bijection that extends ω_{YZ} in such a way that*

(1) $\lambda_{YZ}(K(\operatorname{Im}(r_Y))) = K(\operatorname{Im}(r_Z))$;
(2) $\lambda_{YZ}(C) = \varepsilon_{YZ}(C)$ *for every* $C \in \mathbb{C}(Y)$ *with* $\operatorname{Im}(r_Y) \cap C = \emptyset$;
(3) *if also $U \in \mathscr{S}_7$ then $\lambda_{YU} = \lambda_{ZU}\lambda_{YZ}$, and $\lambda_{UY}\lambda_{YU}$ is the identity mapping of Y.*

Proof. By Statement 2.3(1), for any $x \in \operatorname{Def}(X)$ there exists a dr-map f with $d(f) = x$. Then, by Statement 2.3(6a), $\psi_{YZ}(f)$ is a dr-map, and Statement 2.3(6b) ensures the correctness of the definition of λ_{YZ}. Moreover, from Statements 2.3(6a), 2.3(6b), 2.3(1) and Lemma 6.11(1) it follows that the map λ_{YZ} is a bijection of Y onto Z, and that λ_{YZ} extends ω_{YZ}.

Since λ_{YZ} extends ω_{YZ}, from $\psi_{YZ}(r_Y) = r_Z$ and Statement 4.9(6) it follows that λ_{YZ} maps $K(\operatorname{Im}(r_Y))$ onto $K(\operatorname{Im}(r_Z))$.

If $\operatorname{Im}(r_Y) \cap K(x) = \emptyset$ for $x \in \operatorname{Def}(Y)$, then by Statement 2.1(6) and Lemma 2.2 there is a dr-map f such that $d(f) = x$ and $\operatorname{Im}(f) \setminus K(x) \subseteq \operatorname{Im}(r_Y)$. From Statements 2.3(6b) and 4.9(6) it then follows that $\lambda_{YZ}(x) = d(\psi_{XY}(f)) \in \varepsilon_{YZ}(K(x))$.

Finally, $\lambda_{YU} = \lambda_{ZU}\lambda_{YZ}$ follows from $\psi_{YU} = \psi_{ZU}\psi_{YZ}$ and $\omega_{YU} = \omega_{ZU}\omega_{YZ}$, and $\lambda_{UY}\lambda_{YU} = id_Y$ because $\psi_{UY}\psi_{YU}$ and $\omega_{UY}\omega_{YU}$ are identity maps. $\square$

For any $Y \in \mathscr{S}$ and every $x \in \mathrm{Def}(Y)$ we now define a subset $\zeta_Y(x)$ of $\mathrm{Im}(r_Y)$ by

$$\zeta_Y(x) = \{\kappa_Y(y) \mid x, y \text{ are comparable in } Y\}.$$

Lemma 6.17. *Let $x_0, x_1 \in \mathrm{Def}(Y)$. Then*

(1) *$\zeta_Y(x_0) = \zeta_Y(x_1)$ if and only if there exist dr-maps f_0, f_1 such that $f_i f_{1-i} = f_i$, $d(f_i) = x_i$ and $q_{K(x_i)} = q_{K(x_{1-i})} f_{1-i}$ for $i = 0, 1$;*

(2) *for any $Z \in \mathscr{S}$ with $Y \sim_7 Z$ we have $\zeta_Y(x_0) = \zeta_Y(x_1)$ if and only if $\zeta_Z \lambda_{YZ}(x_0) = \zeta_Z \lambda_{YZ}(x_1)$.*

P r o o f. If $\zeta_Y(x_0) = \zeta_Y(x_1)$, then, using Statements 2.1(4) and 2.3(7), we obtain the required dr-maps f_i for $i = 0, 1$.

Conversely, if $f_i f_{1-i} = f_i$ then $f_i(x_0) = f_i(x_1) = x_i$ for $i = 0, 1$. From $q_{K(x_i)} = q_{K(x_{1-i})} f_{i-1}$ it then follows that $\zeta_Y(x_0) = \zeta(x_1)$.

To prove (2), it suffices to note that $\psi_{YZ}(f_i)$ is a dr-map exactly when f_i is a dr-map, that $d(\psi_{YZ}(f_i)) = \lambda_{YZ}(x_i)$ for $i = 0, 1$, and then apply (1). $\square$

For any class $\mathscr{S}_7$ of $\sim_7$ choose an $X \in \mathscr{S}_7$. For every $Y \in \mathscr{S}_7$ we intend to define a mapping μ_Y from $\mathrm{Im}(\zeta_X)$ into the set of all subsets of $\mathrm{Im}(r_X)$ as follows: for any $A \in \mathrm{Im}(\zeta_X)$ we set $\mu_Y(A) = \varphi_{YX} \zeta_Y \lambda_{XY}(x)$, where $x \in \mathrm{Def}(X)$ and $\zeta_X(x) = A$.

Lemma 6.18. *If $Y, Z \in \mathscr{S}_7$ then:*

(1) *μ_Y is a correctly defined one-to-one mapping;*

(2) *if $\mu_Y = \mu_Z$, then λ_{YZ} has the dp-property and, for any $u \in \mathrm{Def}(Y)$ and any $v \in Y \setminus \mathrm{Def}(Y)$,*

(a) *$u < v$ exactly when $\lambda_{YZ}(u) < \lambda_{YZ}(v)$;*

(b) *$v < u$ exactly when $\lambda_{YZ}(v) < \lambda_{YZ}(u)$;*

and, for any two min-defective or max-defective $u, v \in \mathrm{Def}(Y)$,

(c) *$u \leqslant v$ exactly when $\lambda_{YZ}(u) \leqslant \lambda_{YZ}(v)$.*

P r o o f. (1) follows from Lemma 6.17(2).

For (2), let $y \in \mathrm{Def}(Y)$. Then there exists an $x \in \mathrm{Def}(X)$ with $\lambda_{XY}(x) = y$, and hence

$$\mu_Y(\zeta_X(x)) = \varphi_{YX}(\zeta_Y(\lambda_{XY}(x))) = \varphi_{YX}(\zeta_Y(y)).$$

Thus, from $\mu_Y = \mu_Z$ we get

$$\begin{aligned}\varphi_{ZX}(\zeta_Z(\lambda_{YZ}(y))) &= \varphi_{ZX}(\zeta_Z(\lambda_{XZ}(x))) = \mu_Z(\zeta_X(x)) \\ &= \mu_Y(\zeta_X(x)) = \varphi_{YX}(\zeta_Y(y)) = \varphi_{ZX}(\varphi_{YZ}(\zeta_Y(y))),\end{aligned}$$

and hence $\zeta_Z(\lambda_{YZ}(y)) = \varphi_{YZ}(\zeta_Y(y)) = \lambda_{YZ}(\zeta_Y(y))$ because $\lambda_{YZ} \restriction \mathrm{Im}(r_Y) = \varphi_{YZ}$ is one-to-one. Furthermore, $\mathrm{Ext}(y) = \mathrm{Ext}(K(y)) \cap \kappa_Y^{-1}(\zeta_Y(y))$. By Lemma 6.16(1) and 6.16(2), and from the definition of λ_{YZ} we obtain $\lambda_{YZ}(\mathrm{Ext}(K(y))) = \mathrm{Ext}(K(\lambda_{YZ}(y)))$. Since λ_{YZ} extends τ_{YZ}, and from Lemmas 6.9(3) and 6.11(2), it follows that

$$\lambda_{YZ}(\kappa_Y^{-1}(\zeta_Y(y)) \cap \mathrm{Ext}(Y)) = \kappa_Z^{-1}(\lambda_{YZ}(\zeta_Y(y))) \cap \mathrm{Ext}(Z).$$

Therefore

$$\begin{aligned}\lambda_{YZ}(\mathrm{Ext}(y)) &= \mathrm{Ext}(K(\lambda_{YZ}(y))) \cap \kappa_Z^{-1}(\lambda_{YZ}(\zeta_Y(y))) \\ &= \mathrm{Ext}(K(\lambda_{YZ}(y))) \cap \kappa_Z^{-1}(\zeta_Z(\lambda_{YZ}(y))) = \mathrm{Ext}(\lambda_{YZ}(y)),\end{aligned}$$

and hence λ_{XY} has the *dp*-property.

Assume that $u \in \mathrm{Def}(Y)$ and $v \in \mathrm{Mid}(K(u)) \setminus \mathrm{Def}(Y)$. Then $((u] \cup [u)) \cap E(v) \neq \emptyset$ if and only if $\kappa_Y(v) \in \zeta_Y(u)$. Since λ_{YZ} has the *dp*-property, we have $\lambda_{YZ}\kappa_Y(v) = \kappa_Z\lambda_{YZ}(v)$, and hence $\kappa_Z(\lambda_{YZ}(v)) \in \zeta_Z(\lambda_{YZ}(u))$ because $\zeta_Z\lambda_{YZ}(u) = \lambda_{YZ}\zeta_Y(u)$. Using Statement 2.3(8), we conclude that $u < v$ exactly when $\lambda_{YZ}(u) < \lambda_{YZ}(v)$, and $v < u$ exactly when $\lambda_{YZ}(v) < \lambda_{YZ}(u)$.

The claim in (2c) follows from (1) and (2) of Statement 2.3. □

Now we define the *eighth equivalence* $\sim_8$ by

$$Y \sim_8 Z \text{ if and only if } Y \sim_7 Z \text{ and } \mu_Y = \mu_Z.$$

The claim below easily follows.

Lemma 6.19. *If the equivalence $\sim_7$ has s_7 classes then the equivalence $\sim_8$ has at most $s_7(2^{n_2(\mathbf{V})})!$ classes.*

If $Y \sim_8 Z$ then λ_{YZ} has the dp-property, preserves the order on $E(y)$ for any $y \in Y$ which is not doubly defective, and preserves the order between the defective and the non-defective elements.

Lemma 6.20. *If $Y \sim_8 Z$, then λ_{YZ} is an order isomorphism or an order anti-isomorphism of the set of all doubly defective elements of Y onto the set of all doubly defective elements of Z.*

Proof. The statement follows from Lemma 3.8, (2) and (3) of Statement 3.9, and Lemma P.6. □

Lemma 6.21. *If $Y \sim_8 Z$, then*

$$\lambda_{YZ} f = \psi_{YZ}(f)\lambda_{YZ}$$

for every $f \in \operatorname{End}(Y)$ which is an r-map, or a dr-map, or a $2r$-map.

Proof. If $f \in \operatorname{End}(Y)$ is an r-map or $2r$-map then by Lemmas 6.13 and 6.18(2), $\lambda_{YZ} f = \psi_{YZ}(f)\lambda_{YZ}$. Thus assume that $f \in \operatorname{End}(Y)$ is a dr-map. By the above, for an r-map $r(f) < f$ we have $\lambda_{YZ} r(f) = \psi_{YZ}(r(f))\lambda_{YZ}$ and therefore it suffices to show that $\psi_{YZ}(f)(\lambda_{YZ}(x)) = d(\psi_{YZ}(f))$ exactly when $f(x) = d(f)$. Since $f(x) = d(f)$ implies that $x \in \operatorname{Def}(X)$, Statement 2.3(6b) and 2.3(11) completes the proof. $\square$

Lemma 6.22. *If $Y \sim_8 Z$ then λ_{YZ} is continuous.*

Proof. By Corollary 3.6 and Lemma P.2, the set

$$\{f^{-1}\{z\} \mid z \in \operatorname{Im}(f),\ f \in \operatorname{End}(Z) \text{ is an } r\text{-map or a } dr\text{-map or a } 2r\text{-map}\}$$

is a subbase of the topology on Z. Further, an endomorphism f of Z is an r-map (or a $2r$-map, or a dr-map) if and only if $\psi_{ZY}(f)$ is an r-map (or a $2r$-map, or a dr-map, respectively). By Lemma 6.21, $\lambda_{ZY} f = \psi_{ZY}(f)\lambda_{ZY}$ for any $f \in \operatorname{End}(Z)$ which is an r-map or a dr-map or a $2r$-map. Since λ_{YZ} is a bijection and $\lambda_{ZY} = \lambda_{YZ}^{-1}$, for any such f and each $z \in \operatorname{Im}(f)$ we have $\lambda_{YZ}^{-1}(f^{-1}\{z\}) = \psi_{ZY}(f)^{-1}(\lambda_{YZ}^{-1}\{z\}) = \psi_{ZY}(f)^{-1}(\lambda_{ZY}\{z\})$. Thus $\lambda_{YZ}^{-1}(f^{-1}\{z\})$ is clopen in Y, and hence λ_{YZ} is continuous by Lemma P.3. $\square$

In a dp-space $Y \in \mathbb{DC}$, let $u \in \operatorname{Im}(r_Y) \cap \operatorname{Min}(Y)$ and $v \in \operatorname{Im}(r_Y) \cap \operatorname{Max}(Y)$ be such that $u < v$ and $\operatorname{Ext}(K(u)) \neq \{u, v\}$. It is clear that comparable $x \in \kappa_Y^{-1}\{u\} \cap \operatorname{Mid}(Y) = B_u$ and $y \in \kappa_Y^{-1}\{v\} \cap \operatorname{Mid}(Y) = T_v$ must satisfy $x < y$. If there are no such comparable pairs, or if $x < y$ for all $x \in B_u$, $y \in T_v$ with $y \in K(x)$, we say that the pair $\{u, v\}$ is *degenerate.*

Lemma 6.23. *Let $Y, Z \in \mathbb{DC} \cap \mathscr{S}$ and $Y \sim_8 Z$. Then, for any $u \in \operatorname{Im}(r_Y) \cap \operatorname{Min}(Y)$ and $v \in \operatorname{Im}(r_Y) \cap \operatorname{Max}(Y)$ with $u < v$, one of the following two possibilities occurs:*

(1) *the pairs $\{u, v\}$ and $\{\lambda_{YZ}(u), \lambda_{YZ}(v)\}$ are non-degenerate, in which case, for any $x \in \kappa_Y^{-1}\{u\} \cap \operatorname{Mid}(Y)$ and $y \in \kappa_Y^{-1}\{v\} \cap \operatorname{Mid}(Y)$ we have $x < y$ exactly when $\lambda_{YZ}(x) < \lambda_{YZ}(y)$;*
(2) *the pairs $\{u, v\}$ and $\{\lambda_{YZ}(u), \lambda_{YZ}(v)\}$ are degenerate.*

P r o o f. Since $Y, Z \in \mathbb{DC}$, the hypothesis of Lemma 3.10 is (vacuously) satisfied. If $\{u, v\}$ is non-degenerate and $x \in B_u$, $y \in T_v$ then, by Lemmas 3.10 and 6.19, we have $x < y$ if and only if $\lambda_{YZ}(x) < \lambda_{YZ}(y)$, and $\{\lambda_{YZ}(u), \lambda_{YZ}(v)\}$ is a non-degenerate pair because λ_{YZ} has the *dp*-property. $\square$

Finally, we define the *ninth equivalence* $\sim_9$ by

$$Y \sim_9 Z \text{ if and only if } Y \sim_8 Z \text{ and } \lambda_{YZ} \text{ is an order isomorphism.}$$

For $Y \in \mathbb{DC}$, $u \in \mathrm{Im}(r_Y) \cap \mathrm{Min}(Y)$ and $v \in \mathrm{Im}(r_Y) \cap \mathrm{Max}(Y)$, a comparable pair $\{u, v\}$ that is not a component of $\mathrm{Im}(r_Y)$ falls into one of the two cases described by Lemma 6.23. Under (1) of Lemma 6.23, the order on $\kappa_Y^{-1}\{u, v\}$ is fully determined, while under (2), there are two possible orders on this set. If $\{u, v\}$ is a component of $\mathrm{Im}(r_Y)$ then, according to Lemma 6.20, there are two possible orders on $\kappa_Y^{-1}\{u, v\}$. The lemma below now easily follows.

Lemma 6.24. *If $\mathscr{S} \subseteq \mathbb{DC}$ and the equivalence $\sim_8$ has s_8 classes, then the equivalence $\sim_9$ has at most $s_8 2^{n_8(\mathbf{V})}$ classes. If $Y \sim_9 Z$, then $\lambda_{YZ}\colon Y \longrightarrow Z$ is a dp-isomorphism.*

Theorem 6.25. *If $P(\mathbf{V}) \subseteq \mathbb{DC}$, then $\mathbf{V}$ is n-determined for some finite n.*

P r o o f. By Lemmas 6.2, 6.3, 6.4, 6.5, 6.10, 6.13, 6.15, 6.19 and 6.24, there exists a finite cardinal

$$m \leqslant n_1(\mathbf{V})n_5(\mathbf{V})n_7(\mathbf{V})((n_3(\mathbf{V}) + n_4(\mathbf{V}))!)(n_4(\mathbf{V})!)(2^{n_2(\mathbf{V})}!)2^{n_4(\mathbf{V})+n_6(\mathbf{V})+n_8(\mathbf{V})}$$

such that the equivalence $\sim_9$ has at most m classes. $\square$

7. Conclusion

This section completes the proof of Main Theorem, and shows why the set

$$\{n(\mathbf{V}) \mid \mathbf{V} \subseteq \mathbf{R} \text{ is finitely generated }\}$$

has no finite upper bound.

We begin with a proof of the latter claim.

For any integer $n > 0$, let A_n be the *dp*-space on the set $\{0, 1, \ldots, 2n + 1\}$ whose order is given by $2i < 2i + 1 > 2i + 2$ for $i = 0, 1, \ldots, n - 1$ and $2n < 2n + 1$.

Lemma 7.1. *For any $n > 0$, the algebra $D(A_n)$ dual to the dp-space A_n is rigid and regular.*

P r o o f. Since $\mathrm{Mid}(A_n) = \emptyset$, the algebra $D(A_n)$ is regular, and $\mathrm{End}(A_n) = Aut(A_n)$ because A_n is connected. If $a, b \in A_n$ then $|\,\mathrm{Max}(a)| = 1$ for $a = 0$ alone, and $|\,\mathrm{Min}(b)| = 1$ only for $b = 2n + 1$. Since the unique order path connecting a to b passes through all elements of A_n, the identity is the only endomorphism of A_n. $\square$

For every positive integer n, let $\mathbf{V}_n$ be the variety of *dp*-algebras generated by the duals $D(A_i)$ of all A_i with $i \leqslant n$.

Corollary 7.2. *The finitely generated variety $\mathbf{V}_n \subseteq \mathbf{R}$ contains at least $n + 2$ non-isomorphic equimorphic algebras.*

We still need to show that (1) of Main Theorem implies (3).

Remark. For any $X \in \mathbb{FG}$ and any $x \in X$, the *dp*-subspace $Q_x = \{x\} \cup \mathrm{Ext}(K(x))$ of X is the Priestley dual of a subdirectly irreducible algebra, and the dual Q of any finite subdirectly irreducible algebra satisfies $|Q \setminus \mathrm{Ext}(Q)| \leqslant 1$, see [4]. According to [11], for $X, X' \in \mathbb{FG}$, the algebras $D(X)$ and $D(X')$ generate the same variety if and only if, up to *dp*-isomorphisms, the sets $\{Q_x \mid x \in X\}$ and $\{Q_{x'} \mid x' \in X'\}$ coincide.

Let $\mathbf{V}$ be an $\mathbb{AR}$-variety, and let $P(\mathbf{V}) \nsubseteq \mathbb{DC}$. Then, by [11] and Remark 1.9, there exists a finite order connected *dp*-space $X \in P(\mathbf{V})$ such that $\mathrm{Mid}(X) = \{x, y, z\}$, where x is min-defective, y is max-defective, z is non-defective, and $x < z < y$. Let Y denote the finite *dp*-space on the set $X \setminus \{z\}$ whose order is obtained from the order of X by the removal of comparability $x < y$. Then $Y \in P(\mathbf{V})$.

Let $\mathbb{P}_2$ denote the category whose objects are all triples (D, a, b), where D is a Priestley space in which $a \in \mathrm{Min}(D)$ and $b \in \mathrm{Max}(D)$ are incomparable elements, and whose morphisms $f\colon (D, a, b) \to (D', a', b')$ are all continuous, order preserving mappings for which $f(a) = a'$ and $f(b) = b'$. By [6], the category dual to $\mathbb{P}_2$ is universal.

We now define a functor $\mathscr{L}\colon \mathbb{P}_2 \longrightarrow P(\mathbf{V})$ as follows. For any object $(D, a, b) \in \mathbb{P}_2$ we set $\mathscr{L}(D, a, b) = (D \cup Y, \leqslant, \tau)$, where D and Y are disjoint. The order $\leqslant$ of $\mathscr{L}(D, a, b)$ is the joint extension of the respective orders on Y and D in which $u < d < v$ whenever $d \in D$, $u \in \mathrm{Min}(z)$ and $v \in \mathrm{Max}(z)$ in X, and $x \leqslant b$, $a \leqslant y$. The topology τ of $\mathscr{L}(D, a, b)$ is the extension of the topology on D by the discrete topology on Y. For any morphism $f\colon (D, a, b) \longrightarrow (D', a', b')$, we define $\mathscr{L}(f)$ to be the extension of f by the identity map of Y. Routine calculations show that $\mathscr{L}(D, a, b)$ is a *dp*-space and $\mathscr{L}(f)$ is a *dp*-map, and from the remark above it follows that $\mathscr{L}(D, a, b) \in P(\mathbf{V})$.

Lemma 7.3. *$\mathscr{L}\colon \mathbb{P}_2 \longrightarrow P(\mathbf{V})$ is a functor and if $f\colon \mathscr{L}(D,a,b) \longrightarrow \mathscr{L}(D',a',b')$ is a dp-map satisfying $f(x) = x$ and $f(y) = y$, then $f(D) \subseteq D'$ and the restriction of f to D is a $\mathbb{P}_2$-morphism from (D,a,b) to $(D',a',b') \subset \mathscr{L}(D',a',b')$.*

P r o o f. A verification that $\mathscr{L}$ is a functor is straightforward.

If $f\colon \mathscr{L}(D,a,b) \longrightarrow \mathscr{L}(D',a',b')$ is a *dp*-map with $f(x) = x$ and $f(y) = y$ then $x = f(x) \leqslant f(b)$ and $f(a) \leqslant f(y) = y$ because f preserves order, and $f(a) \in E(f(b))$ because f has the *dp*-property. Hence $f(a), f(b) \in D'$ and thus $f(a) = a'$ and $f(b) = b'$, by the definition of the order on $Y \cup D'$. The map f has the *dp*-property, and hence $f(D) \subseteq D'$. □

Let $\mathscr{E} = \{(D_i, a_i, b_i) \mid i \in I\}$ be any family of objects from $\mathbb{P}_2$. For simplicity's sake, let $D_i \cap Y = \emptyset$ and $D_i \cap D_j = \emptyset$ whenever $i, j \in I$ are distinct. Let $(Z', \leqslant, \tau)$ be the disjoint union of all $\mathscr{L}(D_i, a_i, b_i)$ with $i \in I$, let $\leqslant$ be the union of the individual orders of $\mathscr{L}(D_i, a_i, b_i)$, and let the topology τ be the union of topologies on $\mathscr{L}(D_i, a_i, b_i)$. For any finite I we set $\mathscr{K}(\mathscr{E}) = (Z', \leqslant, \tau)$. If I is infinite then $\mathscr{K}(\mathscr{E}) = (Z' \cup \{z'\}, \leqslant, \sigma)$ where $z' \notin Z'$, the order $\leqslant$ extends the order of Z' in such a way that z' is incomparable to any member of Z', and σ is the one-point compactification of τ by $\{z'\}$. Set $Z = Z'$ when I is finite, and $Z = Z' \cup \{z'\}$ when I is infinite, so that Z is the underlying set of $\mathscr{K}(\mathscr{E})$ in either case. To simplify the notation, all elements of $Y \subset \mathscr{L}(D_i, a_i, b_i)$ will also carry the index i.

Lemma 7.4. *If $\mathscr{E}$ is a family of $\mathbb{P}_2$-objects, then $\mathscr{K}(\mathscr{E}) \in P(\mathbf{V})$. If $f \in \mathrm{End}(\mathscr{K}(\mathscr{E}))$ satisfies $f(x_i) = x_j$ and $f(y_i) = y_j$ then $f(D_i) \subseteq D_j$ and the domain-range restriction of f to D_i and D_j is a $\mathbb{P}_2$-morphism from (D_i, a_i, b_i) to (D_j, a_j, b_j).*

P r o o f. Since $\mathscr{L}(D,a,b) \in P(\mathbf{V})$ for every $(D,a,b) \in \mathscr{E}$ and because $\mathscr{K}(\mathscr{E})$ is a disjoint union of all $\mathscr{L}(D,a,b)$ with $(D,a,b) \in \mathscr{E}$ for any finite $\mathscr{E}$, and $\mathscr{K}(\mathscr{E})$ is the one-point compactification of a disjoint union of all $\mathscr{L}(D,a,b)$ with $(D,a,b) \in \mathscr{E}$ for any infinite $\mathscr{E}$, the remark concerning subdirectly irreducibles implies that $\mathscr{K}(\mathscr{E}) \in P(\mathbf{V})$.

If $f(x_i) = x_j$ and $f(y_i) = y_j$ then the domain-range restriction of f to $\mathscr{L}(D_i, a_i, b_i)$ and $\mathscr{L}(D_j, a_j, b_j)$ is a *dp*-morphism from $\mathscr{L}(D_i, a_i, b_i)$ to $\mathscr{L}(D_j, a_j, b_j)$. This follows from Lemma 7.3 because these subspaces are closed order components of $\mathscr{K}(\mathscr{E})$. □

A family $\mathscr{E} = \{(D_i, a_i, b_i) \mid i \in I\}$ of $\mathbb{P}_2$-objects is *mutually rigid* when for all $i, j \in I$, if $f\colon (D_i, a_i, b_i) \longrightarrow (D_j, a_j, b_j)$ is a $\mathbb{P}_2$-morphism, then $j = i$ and f is the identity map on D_i. Since $\mathbb{P}_2$ is dually universal, arbitrarily large mutually rigid families $\mathscr{E} \subseteq \mathbb{P}_2$ exist.

For any $I' \subseteq I$, let $\mathscr{K}(\mathscr{E}, I')$ be the *dp*-space obtained from $\mathscr{K}(\mathscr{E})$ by setting $x_i < y_i$ for every $i \in I'$. Thus x_i and y_i are comparable in $\mathscr{K}(\mathscr{E}, I')$ exactly when $i \in I'$, and the remainder is unchanged from $\mathscr{K}(\mathscr{E})$.

Lemma 7.5. *If $\mathscr{E}$ is a mutually rigid family of objects in $\mathbb{P}_2$ then $\mathscr{K}(\mathscr{E}, I') \in P(\mathbf{V})$ and $\operatorname{End}(\mathscr{K}(\mathscr{E}, I')) = \operatorname{End}(\mathscr{K}(\mathscr{E}))$ for any $I' \subseteq I$.*

Proof. The remark on subdirectly irreducibles shows that $\mathscr{K}(\mathscr{E}, I') \in P(\mathbf{V})$.

First, note that the topologies of $\mathscr{K}(\mathscr{E})$ and $\mathscr{K}(\mathscr{E}, I')$ coincide on Z and that, for any $v \in Z$, $\operatorname{Ext}(v)$ in $\mathscr{K}(\mathscr{E})$ is the same as $\operatorname{Ext}(v)$ in $\mathscr{K}(\mathscr{E}, I')$, and $v < w$ in $\mathscr{K}(\mathscr{E})$ just when $v < w$ in $\mathscr{K}(\mathscr{E}, I')$ for any $v, w \in Z$ such that $\{v, w\} \notin \{\{x_i, y_i\} \mid i \in I\}$.

Let $f \in \operatorname{End}(\mathscr{K}(\mathscr{E})) \cup \operatorname{End}(\mathscr{K}(\mathscr{E}, I'))$. Then $f^{-1}(x_i) \subseteq \{x_j \mid j \in I\}$ and $f^{-1}(y_i) \subseteq \{y_j \mid j \in I\}$ for any $i \in I$. Denote $\{t\} = \operatorname{Min}(x)$, $\{u\} = \operatorname{Max}(y)$ in X. Then $\{t_i\} = \operatorname{Min}(x_i)$, $\{u_i\} = \operatorname{Max}(y_i)$, $x_i < u_i$ and $t_i < y_i$ for all $i \in I$ in both $\mathscr{K}(\mathscr{E})$ and $\mathscr{K}(\mathscr{E}, I')$.

Next we note that for any $i \in I$, if $f(x_i) \notin \{x_j \mid j \in I\}$, then $f(x_i) = f(t_i)$, and if $f(y_i) \notin \{y_j \mid j \in I\}$ then $f(y_i) = f(u_i)$ because f has the *dp*-property. Hence if $f(x_i) \notin \{x_j \mid j \in I\}$ or $f(y_i) \notin \{y_j \mid j \in I\}$ then $f(x_i) \leqslant f(y_i)$ in both $\mathscr{K}(\mathscr{E})$ and $\mathscr{K}(\mathscr{E}, I')$. Since $\mathscr{E}$ is a mutually rigid family, we conclude from Lemma 7.4 that if $f(x_i) = x_j$ and $f(y_i) = y_k$ then $i = j = k$. Thus f is continuous, has the *dp*-property and preserves the order in both $\mathscr{K}(\mathscr{E})$ and $\mathscr{K}(\mathscr{E}, I')$.

Whence $\operatorname{End}(\mathscr{K}(\mathscr{E})) = \operatorname{End}(\mathscr{K}(\mathscr{E}, I'))$. □

Theorem 7.6. *If $\mathbf{V}$ is an $\mathbb{AR}$-variety of dp-algebras and $P(A) \notin \mathbb{DC}$ for some $A \in \mathbf{V}$, then $\mathbf{V}$ is not α-determined for any cardinal α.*

Proof. Let α be a cardinal. The category $\mathbb{P}_2$ is dually universal [6], and hence it contains a mutually rigid family $\mathscr{E}$ of cardinality α. Since $\mathscr{K}(\mathscr{E}, I')$ is isomorphic to $\mathscr{K}(\mathscr{E}, I'')$ exactly when $|I'| = |I''|$, Lemma 7.5 implies that $P(\mathbf{V})$ must contain a family of non-isomorphic equimorphic *dp*-spaces of cardinality $|\{\beta \mid \beta < \alpha \text{ is a cardinal}\}|$ for every cardinal α. □

The proof of Main Theorem is now complete.

Remark. Let $\mathbb{L}$ be the class formed by all $X \in \mathbb{AR}$ for which, for any min-defective $x \in X$, max-defective $y \in X$ and non-defective $z \in X$, $[x) \cap E(z) \neq \emptyset \neq (y] \cap E(z)$ implies $[x) \cap (y] \cap E(z) \neq \emptyset$. Arguments presented here can be used to show that $\mathbb{L}$ is $\aleph_1$-determined, that is, every class $\mathscr{C} \subseteq \mathbb{L}$ of equimorphic non-isomorphic *dp*-spaces is countable. This result, of course, does not affect Main Theorem, because $P(\mathbf{V}) \subseteq \mathbb{L}$ implies $P(\mathbf{V}) \subseteq \mathbb{DC}$ for any variety $\mathbf{V}$.

Concluding remarks. Let $\mathbf{V}$ be a finitely generated variety $\mathbf{V}$ of distributive double p-algebras. The result of [10] quoted in the introduction says that $\mathbf{V}$ is universal exactly when it contains a nucleus $C \in \mathbf{V}$ with a three-element order component M of $\operatorname{Mid}(P(C))$ for which the identity map is the only *dp*-endomorphism of $P(C)$ extending the identity map of M.

If, for every nucleus $C \in \mathbf{V}$, the union $M \subseteq \operatorname{Mid}(P(C))$ of all order components having at least three elements fails to have such extension property, then the arguments of [10] imply that $\mathbf{V}$ has arbitrarily large algebras whose endomorphism monoids have a finitely bounded size. Since Theorem 2.4 says that every infinite member of any $\mathbb{AR}$-variety $\mathbf{V}$ has infinitely many endomorphisms, it seems natural to ask about the remaining case, that in which all order components of $\operatorname{Mid}(P(C))$ of any nucleus $C \in \mathbf{V}$ have at most two elements. We conjecture that any such variety $\mathbf{V}$ will contain infinite algebras with finite endomorphism monoids, or, equivalently, that Theorem 2.4 cannot be strengthened. This conjecture is supported, somewhat indirectly, by properties of the construction presented in this section. In fact, we also believe that no finitely generated variety $\mathbf{V} \not\subseteq \mathbb{AR}$ is α-determined for any cardinal α.

Our third conjecture concerns finitely generated varieties of double Heyting algebras. It appears that any such variety will be n-determined for some finite $n = n(\mathbf{V})$. The key question here is whether or not an analogue of Lemma 3.1 holds for double Heyting algebras.

Finally, we note that Theorem 1.5, the principal application of Lemma 1.3, and Theorem 1.7 imply that

> every directly indecomposable homomorphic image D of any algebra $A \in \mathbf{V} \subseteq \mathbb{AR}$ is a subdirect power of (finite) retracts of D.

Is there a reasonably transparent algebraic reason why this is true? And what other familiar finitely generated varieties other than varieties of double Heyting algebras may have this property?

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