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AN ASYMPTOTIC THEOREM FOR A CLASS OF NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract. The neutral differential equation

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}[x(t)+x(t-\tau)]+\sigma F(t,x(g(t)))=0, \tag{1.1}$$

is considered under the following conditions: $n \geqslant 2$, $\tau > 0$, $\sigma = \pm 1$, $F(t,u)$ is nonnegative on $[t_0,\infty)\times(0,\infty)$ and is nondecreasing in $u \in (0,\infty)$, and $\lim g(t) = \infty$ as $t \to \infty$. It is shown that equation (1.1) has a solution $x(t)$ such that

$$\lim_{t\to\infty}\frac{x(t)}{t^k} \text{ exists and is a positive finite value if and only if} \tag{1.2}$$
$$\int_{t_0}^{\infty} t^{n-k-1}F(t,c[g(t)]^k)\,\mathrm{d}t<\infty \text{ for some } c>0.$$

Here, k is an integer with $0 \leqslant k \leqslant n-1$. To prove the existence of a solution $x(t)$ satisfying (1.2), the Schauder-Tychonoff fixed point theorem is used.

1. Introduction

In this paper we consider nonlinear neutral differential equations of the form

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}[x(t)+x(t-\tau)]+\sigma F(t,x(g(t)))=0, \tag{1.1}$$

where the following conditions are assumed: $n \geqslant 2$; $\tau > 0$ is a positive constant; $\sigma = +1$ or $\sigma = -1$; $F\colon [t_0,\infty)\times(0,\infty)\to\mathbb{R}$ is continuous, $F(t,u)\geqslant 0$ on $[t_0,\infty)\times(0,\infty)$ and $F(t,u)$ is nondecreasing in $u\in(0,\infty)$ for each fixed $t\in[t_0,\infty)$; $g\colon [t_0,\infty)\to\mathbb{R}$ is continuous and $\lim g(t)=\infty$ as $t\to\infty$. These conditions are always assumed throughout the paper. By a solution of (1.1) we mean a function $x(t)$ which is continuous and satisfies (1.1) on $[t_x, \infty)$ for some $t_x \geqslant t_0$. This implies that if $x(t)$ is a solution of (1.1), then $x(t) + x(t-\tau)$ is n-times continuously differentiable on $[t_x, \infty)$, whereas $x(t)$ is not required to be n-times continuously differentiable. Our interest here is the problem of the existence of a solution $x(t)$ of (1.1) satisfying the asymptotic condition

$$\lim_{t\to\infty} \frac{x(t)}{t^k} \quad \text{exists and is a positive finite value.} \tag{1.2}$$

Here, k is an integer with $0 \leqslant k \leqslant n-1$.

Now consider the equation

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}[x(t) + \lambda x(t-\tau)] + \sigma F(t, x(g(t))) = 0, \tag{1.3}$$

where n, τ, σ, F and g are as above, and λ is a real number. If $\lambda = +1$, then (1.3) becomes (1.1). As is easily seen, αt^k ($\alpha \in \mathbb{R}$, $\alpha \neq 0$, $k \in \mathbb{Z}$, $0 \leqslant k \leqslant n-1$) is a nontrivial solution of the unperturbed equation $(\mathrm{d}^n/\mathrm{d}t^n)[x(t) + \lambda x(t-\tau)] = 0$, and so it is natural to expect that, if F is small enough in some sense, equation (1.3) has a solution $x(t)$ satisfying (1.2). For the case $|\lambda| < 1$, the smallness condition on F is characterized by the integral condition

$$\int_{t_0}^{\infty} t^{n-k-1} F(t, c[g(t)]^k)\,\mathrm{d}t < \infty \quad \text{for some } c > 0. \tag{1.4}$$

In fact, it is known ([5, 6, 14, 15]) that equation (1.3) with $|\lambda| < 1$ has a solution $x(t)$ satisfying (1.2) if and only if (1.4) holds. This result is regarded as an extension of the well-known result for the non-neutral case (i.e., the case $\lambda = 0$). However, it has been recently observed that there is a slight difference between the case $\lambda = -1$ and the case $|\lambda| < 1$. For the details, see the papers of Kitamura and Kusano [9], and Y. Naito [16]. In this paper, to complete the theory from the mathematical point of view, we discuss the case $\lambda = +1$. It is shown that, for the case $\lambda = +1$, the same result as the case $|\lambda| < 1$ holds. More precisely, we have the following theorem.

Theorem. *Let k be an integer with $0 \leqslant k \leqslant n-1$. Then equation (1.1) has a solution $x(t)$ satisfying (1.2) if and only if (1.4) holds.*

The proof of "only if" part of Theorem is given in Section 2. The proof of "if" part of Theorem is divided into the two cases $k \neq 0$ and $k = 0$. The cases $k \neq 0$ and $k = 0$ are considered in Sections 2 and 3, respectively. In both cases, we make use of the Schauder-Tychonoff fixed point theorem to prove the existence of a solution $x(t)$ satisfying (1.2).

Recently there have been several papers concerning the oscillatory and asymptotic properties of solutions of neutral differential equations. See, for instance, the papers [1–17]. However, little is known for the neutral differential equations of the form (1.1).

2. Proof of Theorem

In this section we prove the "only if" part and the "if" part for the case $k \neq 0$. The proof of the "if" part for the case $k = 0$ is given in Section 3.

Proof of "only if" part. Let $k \in \{0, 1, \dots, n-1\}$ and suppose that $x(t)$ is an eventually positive solution of (1.1) which satisfies the asymptotic condition (1.2). Put $\ell = \lim_{t\to\infty} x(t)/t^k$. We have $0 < \ell < \infty$. Define the function $y(t)$ by $y(t) = x(t) + x(t-\tau)$. It follows from (1.1) that

$$y^{(n)}(t) = -\sigma F(t, x(g(t))) \tag{2.1}$$

for all large t, and so $y^{(n)}(t)$ is eventually of constant sign. Then we see that $y^{(i)}(t), i = 0, 1, \dots, n-1$, are eventually monotonic and that the limits $\lim_{t\to\infty} y^{(i)}(t), i = 0, 1, \dots, n-1$, exist in the extended real line $\overline{\mathbb{R}}$. Since $\lim_{t\to\infty} y(t)/t^k = 2\ell \in (0, \infty)$, we find that $\lim_{t\to\infty} y^{(i)}(t) = 0$ for $i = k+1, \dots, n-1$, $\lim_{t\to\infty} y^{(k)}(t) = 2\ell k!$ and $\lim_{t\to\infty} y^{(i)}(t) = \infty$ for $i = 0, 1, \dots, k-1$. Then, integrating (2.1), we get

$$\begin{aligned}
y^{(i)}(t) &= (-1)^{n-i-1}\sigma \int_t^\infty \frac{(s-t)^{n-i-1}}{(n-i-1)!} F(s, x(g(s)))\,\mathrm{d}s, \ \ i = k+1, \dots, n-1,\\
y^{(k)}(t) &= 2\ell k! + (-1)^{n-k-1}\sigma \int_t^\infty \frac{(s-t)^{n-k-1}}{(n-k-1)!} F(s, x(g(s)))\,\mathrm{d}s,
\end{aligned}$$

and

$$\begin{aligned}
y^{(i)}(t) = &\sum_{j=0}^{k-i-1} y^{(i+j)}(T)\frac{(t-T)^j}{j!} + 2\ell k!\frac{(t-T)^{k-i}}{(k-i)!}\\
&+ (-1)^{n-k-1}\sigma \int_T^t \frac{(t-s)^{k-i-1}}{(k-i-1)} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} F(r, x(g(r)))\,\mathrm{d}r\,\mathrm{d}s,\\
&i = 0, 1, \dots, k-1,
\end{aligned}$$

for $t \geqslant T$, where $T(\geqslant t_0)$ is taken sufficiently large. As an immediate consequence we have

$$\int_T^\infty \frac{(s-T)^{n-k-1}}{(n-k-1)!} F(s, x(g(s)))\,\mathrm{d}s < \infty.$$

Then, in view of $\lim_{t\to\infty} x(g(t))/[g(t)]^k = \ell \in (0,\infty)$, we conclude that (1.4) holds.

In the proof of "if" part of Theorem for the case $k \neq 0$, the following $\Phi[\varphi]$ plays an important role. Let $\tau > 0$ and $t_1 \leqslant t_2$. Then, for each $\varphi \in C[t_1,\infty)$ with $\varphi(t) = 0 (t_1 \leqslant t \leqslant t_2)$, we define the function $\Phi[\varphi]$ on the interval $[t_1,\infty)$ by

$$(2.2)\quad \Phi[\varphi](t) = \begin{cases} 0, & t_1 \leqslant t \leqslant t_2, \\ \sum_{j=0}^{m} (-1)^j \varphi(t - j\tau), & t_2 + m\tau < t \leqslant t_2 + (m+1)\tau \ (m = 0,1,\ldots). \end{cases}$$

It is easily seen that $\Phi[\varphi] \in C[t_1,\infty)$ and

$$(2.3)\qquad \Phi[\varphi](t) + \Phi[\varphi](t-\tau) = \varphi(t),\ t \geqslant t_2 + \tau.$$

Proof of "if" part ($k \neq 0$). Let $k \in \{1,2,\ldots,n-1\}$ and suppose that (1.4) is satisfied. We can take a number $t_2 (\geqslant t_0)$ such that

$$(2.4)\qquad \inf\{\min\{t, g(t)\}\colon\ t \geqslant t_2\} \geqslant \max\{t_0, 0\}$$

and

$$(2.5)\qquad \int_{t_2}^{\infty} s^{n-k-1} F(s, c[g(s)]^k)\,\mathrm{d}s \leqslant \frac{k!(n-k-1)!}{2} c.$$

Put $t_1 = \inf\{\min\{t, g(t)\} : t \geqslant t_2\}$. Then it is clear that $0 \leqslant t_1 \leqslant t_2$ and $g(t) \geqslant t_1$ for $t \geqslant t_2$.

We regard the set $C[t_1,\infty)$ as a Fréchet space equipped with the topology of uniform convergence on every compact subinterval of $[t_1,\infty)$, and consider the subset X of $C[t_1,\infty)$ defined by

$$X = \{x \in C[t_1,\infty)\colon\ \tfrac{1}{2}ct^k \leqslant x(t) \leqslant ct^k,\ t \geqslant t_1\}.$$

Then, for each $x \in X$, we assign the function $I[x]$ on $[t_1,\infty)$ as follows:

$$(2.6)\qquad I[x](t) = \begin{cases} \displaystyle\int_{t_2}^{t} \frac{(t-s)^{k-1}}{(k-1)!} \int_{s}^{\infty} \frac{(r-s)^{n-k-1}}{(n-k-1)!} F(r, x(g(r)))\,\mathrm{d}r\,\mathrm{d}s, & t \geqslant t_2, \\ 0, & t_1 \leqslant t \leqslant t_2. \end{cases}$$

Notice that $I[x]$ is well defined and belongs to $C[t_1,\infty)$.

Let us now put

$$k(c) = \begin{cases} \tfrac{1}{2}c & \text{if } (-1)^{n-k-1}\sigma = +1, \\ c & \text{if } (-1)^{n-k-1}\sigma = -1, \end{cases}$$

and define the mapping $M\colon\ X \to C[t_1,\infty)$ by

$$(Mx)(t) = k(c)t^k + (-1)^{n-k-1}\sigma\Phi[I[x]](t),\ \ t \geqslant t_1. \tag{2.7}$$

We will show that the Schauder-Tychonoff theorem ensures the existence of a fixed element $x = Mx \in X$, and that this x is a solution of (1.1) satisfying the desired asymptotic condition (1.2). To see that the Schauder-Tychonoff fixed point theorem can be applied to the mapping M, it is enough to verify that (a) M maps X into X; (b) M is continuous on X; and (c) $M(X)$ is relatively compact.

(a) *M maps X into X.* Let $x \in X$. We first claim that

$$0 \leqslant \Phi[I[x]](t) \leqslant I[x](t) \tag{2.8}$$

for $t \geqslant t_1$. If $t \in [t_1, t_2]$, then (2.8) is clear. Let $t \in (t_2, \infty)$. There is an $m \in \mathbb{Z}$, $m \geqslant 0$, such that $t \in (t_2 + m\tau, t_2 + (m+1)\tau]$. Then, $\Phi[I[x]](t)$ is given by

$$\Phi[I[x]](t) = \sum_{j=0}^{m} (-1)^j I[x](t - j\tau).$$

If m is even, we can rewrite $\Phi[I[x]](t)$ as

$$\Phi[I[x]](t) = I[x](t) - \sum_{j=1}^{m/2} \{I[x](t-(2j-1)\tau) - I[x](t - 2j\tau)\} \tag{2.9}$$

and

$$\Phi[I[x]](t) = \sum_{j=1}^{m/2} \{I[x](t-(2j-2)\tau) - I[x](t-(2j-1)\tau)\} + I[x](t - m\tau). \tag{2.10}$$

If m is odd, we can rewrite $\Phi[I[x]](t)$ as

$$\Phi[I[x]](t) = \sum_{j=0}^{(m-1)/2} \{I[x](t-2j\tau) - I[x](t-(2j+1)\tau)\} \tag{2.11}$$

and

$$\Phi[I[x]](t) = I[x](t) - \sum_{j=1}^{(m-1)/2} \{I[x](t-(2j-1)\tau) - I[x](t-2j\tau)\} - I[x](t - m\tau). \tag{2.12}$$

Note here that $I[x](t)$ is nonnegative and nondecreasing on $[t_1, \infty)$. Then we see that (2.10) and (2.11) imply $\Phi[I[x]](t) \geqslant 0$, and that (2.9) and (2.12) imply $\Phi[I[x]](t) \leqslant I[x](t)$. Thus (2.8) is satisfied for $t \geqslant t_1$.

From (2.5) and (2.6) it follows that

$$\begin{aligned} I[x](t) &\leqslant \frac{t^k}{k!(n-k-1)!} \int_{t_2}^{\infty} r^{n-k-1} F(r, c[g(r)]^k) \, \mathrm{d}r \\ &\leqslant \frac{c}{2} t^k, \quad t \geqslant t_1. \end{aligned}$$

This inequality combined with (2.8) yields $0 \leqslant \Phi[I[x]](t) \leqslant (c/2)t^k$ for $t \geqslant t_1$. Then it is a matter of simple computation to verify that $(Mx)(t)$, which is given by (2.7), satisfies

$$\frac{c}{2} t^k \leqslant (Mx)(t) \leqslant ct^k, \quad t \geqslant t_1.$$

This proves that M maps X into X.

(b) *M is continuous on X.* Let x, $x_i \in X (i = 1, 2, \ldots)$ and $x_i \to x$ as $i \to \infty$ in the space $C[t_1, \infty)$. This means that $x_i(t) \to x(t)$ as $i \to \infty$ uniformly on any compact subinterval of $[t_1, \infty)$. Since $(Mx_i)(t) = (Mx)(t) = k(c)t^k (i = 1, 2, \ldots)$ on $[t_1, t_2]$, it is trivial that $(Mx_i)(t) \to (Mx)(t) (i \to \infty)$ uniformly on $[t_1, t_2]$. Let us consider the convergence $\{(Mx_i)(t)\}$ on the interval $[t_2 + m\tau, t_2 + (m+1)\tau]$, $m \in \mathbb{Z}$, $m \geqslant 0$. As a routine computation, we can show that $I[x_i](t) \to I[x](t)$ as $i \to \infty$ uniformly on every compact subinterval of $[t_1, \infty)$. Then we see that

$$(Mx_i)(t) = k(c)t^k + (-1)^{n-k-1} \sigma \sum_{j=0}^{m} (-1)^j I[x_i](t - j\tau)$$

converges to

$$(Mx)(t) = k(c)t^k + (-1)^{n-k-1} \sigma \sum_{j=0}^{m} (-1)^j I[x](t - j\tau)$$

as $i \to \infty$ uniformly on $[t_2 + m\tau, t_2 + (m+1)\tau]$. Consequently we conclude that $(Mx_i)(t) \to (Mx)(t)$ as $i \to \infty$ uniformly on $[t_2 + m\tau, t_2 + (m+1)\tau]$. Thus, $(Mx_i)(t) \to (Mx)(t)$ as $i \to \infty$ uniformly on any compact subinterval of $[t_1, \infty)$, and hence $Mx_i \to Mx$ as $i \to \infty$ in $C[t_1, \infty)$.

(c) *$M(X)$ is relatively compact.* By the Arzela-Ascoli theorem, it is sufficient to prove that $M(X)$ is uniformly bounded and equicontinuous at every point $t \in [t_1, \infty)$. The uniform boundedness of $M(X)$ is clear since $(c/2)t^k \leqslant (Mx)(t) \leqslant ct^k$ $(t \geqslant t_1)$ for any $x \in X$. To prove the equicontinuity of $M(X)$ on every compact subinterval of $[t_2, \infty)$, we first consider the case $k > 1$. In this case, $(\mathrm{d}/\mathrm{d}t)I[x](t)$ is nonnegative and nondecreasing on $[t_2,\infty)$. Note that $(\mathrm{d}/\mathrm{d}t)\Phi[I[x]] = \Phi[(\mathrm{d}/\mathrm{d}t)I[x]]$. Then, as in the proof of (2.8), we can verify that

$$0 \leqslant \frac{\mathrm{d}}{\mathrm{d}t}\Phi[I[x]](t) \leqslant \frac{\mathrm{d}}{\mathrm{d}t}I[x](t), \quad t \geqslant t_2. \tag{2.13}$$

Therefore, if $[a,b] \subset [t_2,\infty)$, then

$$|\Phi[I[x]](T_2) - \Phi[I[x]](T_1)| \leqslant \frac{\mathrm{d}}{\mathrm{d}t}I[x](b) \cdot |T_2 - T_1|$$

for all $T_1, T_2 \in [a,b]$. Notice that

$$\frac{\mathrm{d}}{\mathrm{d}t}I[x](b) \leqslant \int_{t_2}^{b} \frac{(b-s)^{k-2}}{(k-2)!} \int_{s}^{\infty} \frac{(r-s)^{n-k-1}}{(n-k-1)!} F(r, c[g(r)]^k)\,\mathrm{d}r\,\mathrm{d}s$$

for any $x \in X$ and that the right-hand side of the above inequality is independent of $x \in X$. Then we find that $M(X)$ is equicontinuous on $[a,b] \subset [t_2,\infty)$.

Next consider the case $k = 1$. In this case, $(\mathrm{d}/\mathrm{d}t)I[x](t)$ is nonnegative and nonincreasing on $[t_2,\infty)$. Let $t \in (t_2 + m\tau, t_2 + (m+1)\tau)$ for some $m = 0,1,2,\ldots$. We estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi[I[x]](t) = \sum_{j=0}^{m} (-1)^j \frac{\mathrm{d}}{\mathrm{d}t}I[x](t - j\tau)$$

by using the expressions which are analogous to (2.9) – (2.12). Then we see that if m is even, then

$$0 \leqslant \frac{\mathrm{d}}{\mathrm{d}t}\Phi[I[x]](t) \leqslant \frac{\mathrm{d}}{\mathrm{d}t}I[x](t - m\tau);$$

and that if m is odd, then

$$-\frac{\mathrm{d}}{\mathrm{d}t}I[x](t - m\tau) \leqslant \frac{\mathrm{d}}{\mathrm{d}t}\Phi[I[x]](t) \leqslant 0.$$

In either case we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\Phi[I[x]](t)\right| \leqslant \frac{\mathrm{d}}{\mathrm{d}t}I[x](t_2). \tag{2.14}$$

It is to be noted that (2.14) is valid for $t \in (t_2 + m\tau, t_2 + (m+1)\tau)$, $m = 0,1,2,\ldots$, and, in general, $(\mathrm{d}/\mathrm{d}t)\Phi[I[x]](t)$ does not exist at $t = t_2 + m\tau$, $m = 0,1,2,\ldots$. Let $[a,b]$ be any compact subinterval of $[t_2,\infty)$, and suppose that $T_1, T_2 \in [a,b]$, $T_1 < T_2$. There are $m_1, m_2 \in \mathbb{Z}$, $0 \leqslant m_1 \leqslant m_2$, such that $T_1 \in [t_2 + m_1\tau, t_2 + (m_1+1)\tau)$ and

$T_2 \in [t_2 + m_2\tau, t_2 + (m_2+1)\tau)$. Then, using (2.14), we have

$$\begin{aligned} |\Phi[I[x]](T_2) - \Phi[I[x]](T_1)| &\leqslant |\Phi[I[x]](T_2) - \Phi[I[x]](t_2 + m_2\tau)| \\ &\quad + \sum_{i=m_1+1}^{m_2-1} |\Phi[I[x]](t_2 + (i+1)\tau) - \Phi[I[x]](t_2 + i\tau)| \\ &\quad + |\Phi[I[x]](t_2 + (m_1+1)\tau) - \Phi[I[x]](T_1)| \\ &\leqslant \frac{\mathrm{d}}{\mathrm{d}t} I[x](t_2) \Big[T_2 - (t_2 + m_2\tau) \\ &\quad + \sum_{i=m_1+1}^{m_2-1} \{(t_2 + (i+1)\tau) - (t_2 + i\tau)\} + (t_2 + (m_1+1)\tau) - T_1 \Big] \\ &= \frac{\mathrm{d}}{\mathrm{d}t} I[x](t_2) |T_2 - T_1|. \end{aligned}$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}t} I[x](t_2) \leqslant \int_{t_2}^{\infty} \frac{(r-t_2)^{n-2}}{(n-2)!} F(r, cg(r))\,\mathrm{d}r$$

for any $x \in X$ and that the right-hand side of the above does not depend on $x \in X$. Then we see that $M(X)$ is equicontinuous on $[a,b] \subset [t_2, \infty)$. In both of the cases $k > 1$ and $k = 1$, the equicontinuity of $M(X)$ on $[t_1, t_2]$ is obvious. Thus we can conclude that $M(X)$ is equicontinuous on every compact subinterval of $[t_1, \infty)$.

From the above observation we can apply the Schauder-Tychonoff fixed point theorem to the mapping $M\colon X \to X$. Let $x \in X$ be a fixed point of M. We have

$$x(t) = k(c)t^k + (-1)^{n-k-1}\sigma\Phi[I[x]](t), \quad t \geqslant t_1. \tag{2.15}$$

Then, using (2.3), we obtain

$$x(t) + x(t-\tau) = k(c)(t^k + (t-\tau)^k) + (-1)^{n-k-1}\sigma I[x](t), \quad t \geqslant t_2 + \tau,$$

from which it follows that $x(t)$ is a solution of (1.1). It is easy to see that $I[x](t)/t^k$ tends to 0 as $t \to \infty$, and hence (2.8) and (2.15) yield

$$\lim_{t\to\infty} \frac{x(t)}{t^k} = k(c).$$

This completes the proof of Theorem for the case $k \neq 0$.

3. Proof of Theorem (continued)

In this section we give the proof of "if" part of Theorem for the case $k = 0$.

Let $\psi \in C[t_1, \infty)$ be nonincreasing on $[t_1, \infty)$ and $\lim \psi(t) = 0$ as $t \to \infty$. Then we define the function $\Psi[\psi]$ by

$$\Psi[\psi](t) = \sum_{j=1}^{\infty} (-1)^j \psi(t + j\tau), \quad t \geqslant t_1 - \tau. \tag{3.1}$$

For each $t \in [t_1 - \tau, \infty)$, the sequence of real numbers $\{\psi(t + j\tau)\}_{j=1}^{\infty}$ is a nonincreasing sequence. Furthermore, the sequence of continuous functions $\{\psi(\cdot + j\tau)\}_{j=1}^{\infty}$ converges to 0 uniformly on the interval $[t_1 - \tau, \infty)$ since

$$\begin{aligned} \sup\{|\psi(t + j\tau)| \colon\ t \geqslant t_1 - \tau\} &= \psi(t_1 - \tau + j\tau) \\ &\to 0 \quad \text{as } j \to \infty. \end{aligned}$$

Therefore, by Dirichlet's test, we see that $\sum\limits_{j=1}^{\infty} (-1)^j \psi(t + j\tau)$ converges uniformly on $[t_1 - \tau, \infty)$, and in particular, $\Psi[\psi]$ is well defined and is a continuous function on $[t_1 - \tau, \infty)$. It is easy to see that

$$\Psi[\psi](t) + \Psi[\psi](t - \tau) = -\psi(t), \quad t \geqslant t_1. \tag{3.2}$$

In the proof of "if" part of Theorem with $k = 0$, $\Psi[\psi]$ plays a crucial role.

Proof of "if" part ($k = 0$). Let $k = 0$ and suppose that (1.4) is satisfied:

$$\int_{t_0}^{\infty} t^{n-1} F(t, c)\,\mathrm{d}t < \infty \quad \text{for some } c > 0. \tag{3.3}$$

We choose a number $t_2 (\geqslant t_0)$ satisfying

$$\inf\{\min\{t, g(t)\} \colon\ t \geqslant t_2\} \geqslant \max\{t_0, 0\} \tag{3.4}$$

and

$$\int_{t_2}^{\infty} t^{n-1} F(t, c)\,\mathrm{d}t \leqslant \frac{c}{4}\Big\{\frac{1}{(n-1)!} + \frac{1}{(n-2)!}\Big\}^{-1}. \tag{3.5}$$

Put $t_1 = \inf\{\min\{t, g(t)\} \colon\ t \geqslant t_2\}$. We have $0 \leqslant t_1 \leqslant t_2$ and $g(t) \geqslant t_1$ for $t \geqslant t_2$.

Define the subset X of the Fréchet space $C[t_1, \infty)$ as follows:

$$X = \Big\{x \in C[t_1, \infty) \colon\ \frac{c}{2} \leqslant x(t) \leqslant c,\ t \geqslant t_1\Big\},$$

where $c > 0$ is a constant in (3.3). Moreover, for $x \in X$, we define the function $I[x]$ on $[t_1, \infty)$ by

$$I[x](t) = \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} F(s, x(g(s)))\,\mathrm{d}s,\ t \geqslant t_2,$$

and

$$\begin{aligned} I[x](t) = &\int_{t_2}^\infty \frac{(s-t_2)^{n-1}}{(n-1)!} F(s, x(g(s)))\,\mathrm{d}s \\ &+ (t_2 - t)\int_{t_2}^\infty \frac{(s-t_2)^{n-2}}{(n-2)!} F(s, x(g(s)))\,\mathrm{d}s,\ t_1 \leqslant t \leqslant t_2. \end{aligned}$$

It is easily seen that, for each $x \in X$, $I[x]$ has the following properties: $I[x] \in C^1[t_1, \infty)$, $I[x](t) \geqslant 0$ and $(\mathrm{d}/\mathrm{d}t)I[x](t) \leqslant 0 (t \geqslant t_1)$, and $\lim\limits_{t\to\infty} I[x](t) = 0$. Notice here that if $x \in X$, then $\Psi[I[x]](t)$ is well defined for $t \geqslant t_1$. Thus we can consider the mapping $M\colon\ X \to C[t_1, \infty)$ which is defined by

$$(Mx)(t) = \frac{3}{4}c + (-1)^{n-1}\sigma\Psi[I[x]](t_1) - (-1)^{n-1}\sigma\Psi[I[x]](t), \quad t \geqslant t_1. \tag{3.6}$$

Making use of the Schauder-Tychonoff theorem, we will show that the mapping M has a fixed point $x \in X$.

(a) *M maps X into X.* Let $x \in X$. We claim that

$$-I[x](t+\tau) \leqslant \sum_{j=1}^m (-1)^j I[x](t+j\tau) \leqslant 0 \tag{3.7}$$

for $t \geqslant t_1, m \in \mathbb{Z}, m \geqslant 1$. If m is even, then we have

$$\sum_{j=1}^m (-1)^j I[x](t+j\tau) = -\sum_{j=1}^{m/2} \{I[x](t+(2j-1)\tau) - I[x](t+2j\tau)\} \tag{3.8}$$

and

$$\begin{aligned} \sum_{j=1}^m (-1)^j I[x](t+j\tau) = &-I[x](t+\tau) \\ &+ \sum_{j=2}^{m/2} \{I[x](t+(2j-2)\tau) - I[x](t+(2j-1)\tau)\} + I[x](t+m\tau). \end{aligned} \tag{3.9}$$

If m is odd, then

$$\begin{aligned} &\sum_{j=1}^m (-1)^j I[x](t+j\tau) \\ &\quad = -\sum_{j=1}^{(m-1)/2} \{I[x](t+(2j-1)\tau) - I[x](t+2j\tau)\} - I[x](t+m\tau) \end{aligned} \tag{3.10}$$

and

$$\sum_{j=1}^{m}(-1)^j I[x](t+j\tau) = -I[x](t+\tau) + \sum_{j=1}^{(m-1)/2}\{I[x](t+2j\tau) - I[x](t+(2j+1)\tau)\}. \tag{3.11}$$

Then, by virtue of the nonnegativity and the nonincreasing property of $I[x]$, we easily see that (3.8) and (3.10) yield the right-hand side inequality of (3.7), and that (3.9) and (3.11) yield the left-hand side inequality of (3.7).

Letting $m \to \infty$ in (3.7), we obtain

$$-I[x](t+\tau) \leqslant \Psi[I[x]](t) \leqslant 0,\ t \geqslant t_1. \tag{3.12}$$

Since (3.5) implies

$$0 \leqslant I[x](t) \leqslant I[x](t_1) \leqslant \int_{t_2}^{\infty}\frac{s^{n-1}}{(n-1)!}F(s,c)\,\mathrm{d}s + t_2\int_{t_2}^{\infty}\frac{s^{n-2}}{(n-2)!}F(s,c)\,\mathrm{d}s \leqslant \frac{c}{4}$$

for $t \geqslant t_1$, it follows from (3.12) that

$$-\frac{c}{4} \leqslant \Psi[I[x]](t) \leqslant 0, \quad t \geqslant t_1.$$

Then we easily see that

$$\frac{c}{2} \leqslant (Mx)(t) \leqslant c, \quad t \geqslant t_1,$$

which implies $M(X) \subset X$.

(b) *M is continuous on X.* Before proving the continuity of M, we show that, for each $x \in X$, $\Psi[I[x]] \in C^1[t_1,\infty)$ and $(\mathrm{d}/\mathrm{d}t)\Psi[I[x]](t)$ can be obtained by termwise differentiation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi[I[x]](t) = \sum_{j=1}^{\infty}(-1)^j\frac{\mathrm{d}}{\mathrm{d}t}I[x](t+j\tau), \quad t \geqslant t_1. \tag{3.13}$$

To see this, it is enough to verify that $\sum\limits_{j=1}^{\infty}(-1)^j(\mathrm{d}/\mathrm{d}t)I[x](t+j\tau)$ converges uniformly on any compact subinterval $[a,b]$ of $[t_1,\infty)$. Let $[a,b] \subset [t_1,\infty)$. There is a $j_0 \in \mathbb{N}$ such that $t+j\tau \geqslant t_2$ for $t \in [a,b]$ and $j \geqslant j_0$. Then, for $t \in [a,b]$ and $j \geqslant j_0$, we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}I[x](t+j\tau)\right| = \int_{t+j\tau}^{\infty}\frac{(s-(t+j\tau))^{n-2}}{(n-2)!}F(s,x(g(s)))\,\mathrm{d}s \leqslant \int_{a+j\tau}^{\infty}\frac{s^{n-2}}{(n-2)!}F(s,c)\,\mathrm{d}s.$$

As is easily seen, condition (3.3) implies

$$\int_{j_0}^{\infty}\Big(\int_{a+u\tau}^{\infty}\frac{s^{n-2}}{(n-2)!}F(s,c)\,\mathrm{d}s\Big)\,\mathrm{d}u<\infty$$

and hence we can apply Cauchy's integral test to obtain

$$\sum_{j=j_0}^{\infty}\int_{a+j\tau}^{\infty}\frac{s^{n-2}}{(n-2)!}F(s,c)\,\mathrm{d}s<+\infty.$$

Then the Weierstrass M-test ensures the uniform (and absolute) convergence of $\sum\limits_{j=j_0}^{\infty}(-1)^j(\mathrm{d}/\mathrm{d}t)I[x](t+j\tau)$ on $[a,b]$. Thus we have (3.13).

Now, to prove the continuity of M, suppose that $x,x_i\in X$ $(i=1,2,\dots)$ and that $\lim\limits_{i\to\infty}x_i(t)=x(t)$ uniformly on any compact subinterval of $[t_1,\infty)$. In view of (3.13), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\{\Psi[I[x_i]](t)-\Psi[I[x]](t)\}=\sum_{j=1}^{\infty}(-1)^j\frac{\mathrm{d}}{\mathrm{d}t}\{I[x_i](t+j\tau)-I[x](t+j\tau)\}$$

for $t\geqslant t_1$. Let $[a,b]$ be an arbitrary compact subinterval of $[t_1,\infty)$. Choose a positive integer $j_0\in\mathbb{N}$ satisfying $t+j\tau\geqslant t_2$ for $t\in[a,b]$ and $j\geqslant j_0$. Then, for $t\in[a,b]$,

$$\begin{aligned}&\Big|\sum_{j=1}^{j_0}(-1)^j\frac{\mathrm{d}}{\mathrm{d}t}\{I[x_i](t+j\tau)-I[x](t+j\tau)\}\Big|\\&\qquad\leqslant j_0\int_{t_2}^{\infty}\frac{(s-t_2)^{n-2}}{(n-2)!}|F(s,x_i(g(s)))-F(s,x(g(s)))|\,\mathrm{d}s\to 0\text{ as }i\to\infty.\end{aligned}$$

Moreover, for $[a,b]$ and $j\geqslant j_0$,

$$\begin{aligned}&\Big|\frac{\mathrm{d}}{\mathrm{d}t}\{I[x_i](t+j\tau)-I[x](t+j\tau)\}\Big|\\&\leqslant\int_{t+j\tau}^{\infty}\frac{(s-(t+j\tau))^{n-2}}{(n-2)!}|F(s,x_i(g(s)))-F(s,x(g(s)))|\,\mathrm{d}s\\&\leqslant\int_{a+j\tau}^{\infty}\frac{s^{n-2}}{(n-2)!}|F(s,x_i(g(s)))-F(s,x(g(s)))|\,\mathrm{d}s,\end{aligned}$$

and consequently, for $t \in [a, b]$,

$$
\begin{aligned}
&\Big| \sum_{j=j_0+1}^{\infty} (-1)^j \frac{\mathrm{d}}{\mathrm{d}t} \{I[x_i](t + j\tau) - I[x](t + j\tau)\} \Big| \\
&\qquad \leqslant \sum_{j=j_0+1}^{\infty} \int_{a+j\tau}^{\infty} \frac{s^{n-2}}{(n-2)!} |F(s, x_i(g(s))) - F(s, x(g(s)))| \, \mathrm{d}s \\
&\qquad \leqslant \int_{j_0}^{\infty} \left(\int_{a+u\tau}^{\infty} \frac{s^{n-2}}{(n-2)!} |F(s, x_i(g(s))) - F(s, x(g(s)))| \, \mathrm{d}s \right) \mathrm{d}u \\
&\qquad = \frac{1}{\tau} \int_{a+j_0\tau}^{\infty} \left(\int_{w}^{\infty} \frac{s^{n-2}}{(n-2)!} |F(s, x_i(g(s))) - F(s, x(g(s)))| \, \mathrm{d}s \right) dw \\
&\qquad \leqslant \frac{1}{\tau} \int_{a+j_0\tau}^{\infty} \frac{s^{n-1}}{(n-2)!} |F(s, x_i(g(s))) - F(s, x(g(s)))| \, \mathrm{d}s \to 0 \text{ as } i \to \infty.
\end{aligned}
$$

From these observation we find that $\{(\mathrm{d}/\mathrm{d}t)\Psi[I[x_i]](t)\}$ converges to $(\mathrm{d}/\mathrm{d}t)\Psi[I[x]](t)$ as $i \to \infty$ uniformly on $[a, b]$, and, therefore, $\{(\mathrm{d}/\mathrm{d}t)(Mx_i)(t)\}$ converges to $(\mathrm{d}/\mathrm{d}t)(Mx)(t)$ as $i \to \infty$ uniformly on any compact subinterval of $[t_1, \infty)$. Then, in view of $(Mx_i)(t_1) = \frac{3}{4}c$ $(i = 1, 2, \ldots)$, we see that $\lim_{i\to\infty} (Mx_i)(t) = (Mx)(t)$ uniformly on any compact subinterval of $[t_1, \infty)$.

(c) $M(X)$ *is relatively compact.* Let $x \in X$ and $[a, b] \subset [t_1, \infty)$. As in the above discussion, we have

$$
\Big| \frac{\mathrm{d}}{\mathrm{d}t} \Psi[I[x]](t) \Big| \leqslant j_0 \int_{t_2}^{\infty} \frac{(s - t_2)^{n-2}}{(n-2)!} F(s, c) \, \mathrm{d}s + \frac{1}{\tau} \int_{a+j_0\tau}^{\infty} \frac{s^{n-1}}{(n-2)!} F(s, c) \, \mathrm{d}s
$$

for $t \in [a, b]$, where $j_0 \in \mathbb{N}$ and $a + j_0\tau \geqslant t_2$. This implies that $M(X)$ is equicontinuous on $[a, b]$. The uniform boundedness of $M(X)$ on $[a, b]$ is evident since $c/2 \leqslant (Mx)(t) \leqslant c$ for $t \geqslant t_1$. Hence, by the Arzela-Ascoli theorem, we find that $M(X)$ is relatively compact.

All the conditions for the Schauder-Tychonoff fixed point theorem are satisfied, and so there is an $x \in X$ such that $x = Mx$, i.e., $x(t) = (Mx)(t)$ for $t \geqslant t_1$. Then, in view of (3.12), we see that $\lim_{t\to\infty} \Psi[I[x]](t) = 0$, and consequently, we find that $\lim x(t) = \lim (Mx)(t) = (3/4)c + (-1)^{n-1}\sigma\Psi[I[x]](t_1)$ as $t \to \infty$. Since $c/2 \leqslant x(t) \leqslant c$ for $t \geqslant t_1$, we have $c/2 \leqslant \lim_{t\to\infty} x(t) \leqslant c$. Applying the formula (3.2), we see that $x(t)$ is a solution of (1.1). This finishes the proof of "if" part of Theorem for the case $k = 0$.

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