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POINTWISE CONVERGENCE FAILS TO BE STRICT

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Abstract. It is known that the ring $B(\mathbb{R})$ of all Baire functions carrying the pointwise convergence yields a sequential completion of the ring $C(\mathbb{R})$ of all continuous functions. We investigate various sequential convergences related to the pointwise convergence and the process of completion of $C(\mathbb{R})$. In particular, we prove that the pointwise convergence fails to be strict and prove the existence of the categorical ring completion of $C(\mathbb{R})$ which differs from $B(\mathbb{R})$.

1.

Consider the set $\mathbb{R}^{\mathbb{R}}$ of all real functions carrying the pointwise (sequential) convergence. If we start with the ring $C(\mathbb{R})$ of all continuous functions, then the ring $B_1(\mathbb{R})$ of all 1-st Baire class functions is the first sequential closure of $C(\mathbb{R})$, the ring $B_\alpha(\mathbb{R})$ of all α-th Baire class functions, $\alpha < \omega_1$, is the α-th sequential closure of $C(\mathbb{R})$, and the ring $B(\mathbb{R})$ of all Baire functions is the smallest subset of $\mathbb{R}^{\mathbb{R}}$ containing $C(\mathbb{R})$ and closed with respect to the pointwise convergence, hence sequentially complete, see [NOV], [LAC].

Let $\mathbb{L}$ be a sequential convergence on $B_1(\mathbb{R})$ such that, for each sequence $\langle f_n \rangle$ of continuous functions, $\langle f_n \rangle$ converges to $f \in B_1(\mathbb{R})$ under $\mathbb{L}$ iff it converges to f pointwise. Then $\mathbb{L}$ is said to be *admissible.* If $\mathbb{L}$ is compatible with the group or ring structure of $B_1(\mathbb{R})$, then each Cauchy sequence of continuous functions converges under $\mathbb{L}$ and we get $B_1(\mathbb{R})$ as a group or ring (sequential) *precompletion* of $C(\mathbb{R})$. Observe that $\mathbb{L}$, for example the pointwise convergence, need not be complete. To get a completion, in such cases we have to iterate the precompletion process. In case of the pointwise convergence the usual sequential completion of $C(\mathbb{R})$ is the ring

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$B(\mathbb{R})$. In the present paper we investigate admissible convergences and alternative ways of (pre)completing $C(\mathbb{R})$.

Strictness is a natural way how to control the convergence of sequences of ideal points in an extension of a convergence space or a precompletion of a sequential group or ring ([FZS], [FKE], [PAU]).

An admissible convergence $\mathbb{L}$ on $B_1(\mathbb{R})$ is said to be *strict* if the following condition is satisfied (see Definition 1.2 in [FZS]):

(s) Let $\langle f_n \rangle$ be a sequence ranging in $B_1(\mathbb{R}) \setminus C(\mathbb{R})$ which converges under $\mathbb{L}$ to $f \in B_1(\mathbb{R})$. Then there are a subsequence $\langle f'_n \rangle$ of $\langle f_n \rangle$ and sequences $\langle g_n^{(k)} \rangle$, $k \in \mathbb{N}$, of continuous functions such that the sequence $\langle g_n^{(k)} \rangle$ pointwise converges to f'_k and each diagonal sequence $\langle g_{d(n)}^{(n)} \rangle$, $d\colon \mathbb{N} \to \mathbb{N}$, pointwise converges to f.

In [FZS] the authors asked whether the pointwise convergence is strict. We prove that the answer is "NO".

Theorem 1.1. *The pointwise convergence on $B_1(\mathbb{R})$ fails to be strict.*

Proof. Let $p_1, p_2, p_3, \ldots$ denote the increasing sequence of all prime numbers. For each $n \in \mathbb{N}$, let $A_n = \{k/p_n;\ k = 1, 2, \ldots, p_n - 1\}$ and let f_n denote the characteristic function of A_n. Let f denote the constant zero function. Then $f_n \in B_1(\mathbb{R}) \setminus C(\mathbb{R})$ for each $n \in \mathbb{N}$ and the sequence $\langle f_n \rangle$ pointwise converges to f. For each $k \in \mathbb{N}$, let $\langle g_n^{(k)} \rangle$ be a sequence of continuous functions which pointwise converges to f_k. We show that there exists a mapping u of $\mathbb{N}$ into $\mathbb{N}$ such that for each mapping v of $\mathbb{N}$ into $\mathbb{N}$, $v(k) > u(k)$ for each $k \in \mathbb{N}$, and for each strictly increasing mapping s of $\mathbb{N}$ into $\mathbb{N}$ the subsequence $\langle g_{v(s(n))}^{(s(n))} \rangle$ of the diagonal sequence $\langle g_{v(n)}^{(n)} \rangle$ does not pointwise converge to f. Clearly, then the pointwise convergence on $B_1(\mathbb{R})$ fails to be strict.

So, since all sets A_k are finite, for each $k \in \mathbb{N}$ choose $u(k) \in \mathbb{N}$ such that $g_n^{(k)}(x) > 1/2$ for each $n > u(k)$ and each $x \in A_k$. Let v be a mapping of $\mathbb{N}$ into $\mathbb{N}$ such that $v(k) > u(k)$ for each $k \in \mathbb{N}$ and let s be a strictly increasing mapping of $\mathbb{N}$ into $\mathbb{N}$. From $g_{v(s(1))}^{(s(1))}(1/p_{s(1)}) > 1/2$ it follows that there exists a closed interval $I_1 \subset (0,1)$ such that $1/p_{s(1)} \in \operatorname{int} I_1$ and $g_{v(s(1))}^{(s(1))}(I_1) > 1/2$. Put $t(1) = 1$. By induction, define a strictly increasing mapping t of $\mathbb{N}$ into $\mathbb{N}$ and a sequence $\langle I_n \rangle$ of closed intervals such that $\operatorname{int} I_n \supset I_{n+1} \neq \emptyset$ and $g_{v(s(t(n)))}^{(s(t(n)))}(I_n) > 1/2$ for all $n \in \mathbb{N}$. Choose $t(2) \in \mathbb{N}$ such that $s(t(2)) \in \{s(2), s(3), \ldots\}$ and $A_{s(t(2))} \cap \operatorname{int} I_1 \neq \emptyset$. Choose a closed interval I_2 such that $\operatorname{int} I_2 \neq \emptyset$, $\operatorname{int} I_1 \supset I_2$ and $g_{v(s(t(2)))}^{(s(t(2)))}(I_2) > 1/2$. Analogously define $t(3)$ and $I_3, \ldots, t(n)$ and I_n, and so on. Now, choose $x_0 \in \bigcap_{i=1}^{\infty} I_i \neq \emptyset$. Since

$g^{(s(t(n)))}_{v(s(t(n)))}(x_0) > 1/2$ for all $n \in \mathbb{N}$, the sequence $\langle g^{(s(n))}_{v(s(n))}\rangle$ does not pointwise converge to f. This completes the proof. □

2.

In this section we prove some simple facts about strict admissible convergences.

Let f and f_n, $n \in \mathbb{N}$, be functions in $\mathbb{R}^{\mathbb{R}}$. We say (cf. [FKE]) that the sequence $\langle f_n\rangle$ and the function f are *linked* if there are sequences $\langle g^{(k)}_n\rangle$, $k \in \mathbb{N}$, in $\mathbb{R}^{\mathbb{R}}$ such that for each $k \in \mathbb{N}$ the sequence $\langle g^{(k)}_n\rangle$ pointwise converges to f_k and each diagonal sequence $\langle g^{(n)}_{d(n)}\rangle$, $d\colon \mathbb{N} \to \mathbb{N}$, pointwise converges to f. Note: condition (s) can be reformulated as "if $\langle f_n\rangle$ converges to f under $\mathbb{L}$ and $f_n \in B_1(\mathbb{R}) \setminus C(\mathbb{R})$ for all $n \in \mathbb{N}$, then there exists a subsequence $\langle f'_n\rangle$ of $\langle f_n\rangle$ which is linked to f via a double sequence, i.e. sequence of sequences, of continuous functions".

Lemma 2.1. *Let $\langle f_n\rangle$ be a sequence of functions linked to a function f. Then $\langle f_n\rangle$ converges pointwise to f.*

P r o o f. Assume that, on the contrary, for some $x \in \mathbb{R}$ the sequence $\langle f_n(x)\rangle$ does not converge to $f(x)$. Then there exists a positive number ε such that $|f_n(x)-f(x)| > \varepsilon$ for infinitely many $n \in \mathbb{N}$. Clearly, this is a contradiction with the assumption that $\langle f_n\rangle$ and f are linked. □

Corollary 2.2. *Every strict convergence on $B_1(\mathbb{R})$ is finer than the pointwise convergence.*

Next, we describe the coarsest and the finest strict convergences on $B_1(\mathbb{R})$ compatible with the ring structures of $B_1(\mathbb{R})$.

Construction 2.3. Denote by $\mathbb{L}_s$ the set of all pairs $(\langle f_n\rangle, f)$ such that $\langle f_n\rangle$ is a sequence of functions of $B_1(\mathbb{R})$, $f \in B_1(\mathbb{R})$ and for each subsequence $\langle f'_n\rangle$ of $\langle f_n\rangle$ there exists its subsequence $\langle f''_n\rangle$ which is linked to f via a double sequence of continuous functions. As a rule, $(\langle f_n\rangle, f) \in \mathbb{L}_s$ means that the sequence $\langle f_n\rangle$ converges to f under $\mathbb{L}_s$.

Claim 2.3.1. $\mathbb{L}_s$ is a strict $\mathcal{L}^*_0$-ring convergence.

P r o o f. It follows easily from Lemma 2.1 that each sequence $\mathbb{L}_s$-converges to at most one limit. The remaining axioms of convergence follow directly from the definition of $\mathbb{L}_s$. Indeed, each constant sequence converges, each subsequence of a convergent sequence converges to the same limit, and $\mathbb{L}_s$ satisfies the Urysohn axiom: if $\langle f_n\rangle$ and f are such that for each subsequence $\langle f'_n\rangle$ of $\langle f_n\rangle$ there exists its subsequence $\langle f''_n\rangle$ such that $(\langle f''_n\rangle, f) \in \mathbb{L}_s$, then $(\langle f_n\rangle, f) \in \mathbb{L}_s$. Further, sums and products of convergent sequences converge to the corresponding sums and products of their limits, hence $\mathbb{L}_s$ is compatible with the ring structure of $B_1(\mathbb{R})$. It follows from Lemma 2.1 that $\mathbb{L}_s$ is admissible and since $\mathbb{L}_s$ is clearly strict, the proof is complete. □

Claim 2.3.2. Let $\mathbb{L}$ be a strict $\mathcal{L}_0^*$-ring convergence on $B_1(\mathbb{R})$. Then $\mathbb{L}_s$ is coarser than $\mathbb{L}$.

Proof. If $\langle f_n\rangle$ converges to f under $\mathbb{L}$, then some subsequence $\langle f'_n\rangle$ of $\langle f_n\rangle$ is linked to f via a double sequence of continuous functions and hence $\langle f'_n\rangle$ converges to f under $\mathbb{L}_s$. Since $\mathbb{L}_s$ satisfies the Urysohn axiom, it follows that $\mathbb{L} \subset \mathbb{L}_s$. □

Construction 2.4. Denote by $\mathcal{N}$ the set of all sequences in $B_1(\mathbb{R})$ of the form $\bigl\langle \sum_{i=1}^{k} (f_{in} - f_i) g_i \bigr\rangle$, where $k \in \mathbb{N}$, $\langle f_{in}\rangle$ is a sequence of continuous functions pointwise converging to $f_i \in B_1(\mathbb{R})$, $i = 1, \ldots, k$. Trivially, $\mathcal{N}$ is closed with respect to subsequences and finite sums. Since

$$\begin{aligned}\langle (f_{1n} - f_1) g_1\rangle \langle (f_{2n} - f_n) g_2\rangle &= \langle (f_{1n} f_{2n} - f_1 f_2) g_1 g_2\rangle \\ &\quad - \langle (f_{1n} - f_1) f_2 g_1 g_2\rangle - \langle (f_{2n} - f_2) f_1 g_1 g_2\rangle,\end{aligned}$$

it follows that $\mathcal{N}$ is closed with respect to finite products, too. By Lemma 2 in [FZE], there exists a unique $\mathcal{L}$-ring convergence under which a sequence $\langle f_n\rangle$ converges to the constant zero function $\mathbb{O}$ iff $\langle f_n\rangle \in \mathcal{N}$. Denote by $\mathbb{L}_r$ its Urysohn modification. Recall that $\langle f_n\rangle$ converges to $\mathbb{O}$ under $\mathbb{L}_r$ iff for each subsequence $\langle f'_n\rangle$ of $\langle f_n\rangle$ there exists its subsequence $\langle f''_n\rangle$ belonging to $\mathcal{N}$.

Claim 2.4.1. $\mathbb{L}_r$ is a strict $\mathcal{L}_0^*$-ring convergence.

Proof. Obviously, $\mathbb{L}_r$ is finer than the pointwise convergence on $B_1(\mathbb{R})$ and hence the limits of $\mathbb{L}_r$-convergent sequences are uniquely determined. Thus $\mathbb{L}_r$ is an $\mathcal{L}_0^*$-ring convergence. If $\langle f_n\rangle$ is a sequence of continuous functions pointwise converging to $f \in B_1(\mathbb{R})$, then $(\langle f_n\rangle, f) \in \mathbb{L}_r$. Consequently, $\mathbb{L}_r$ is admissible. The proof of strictness of $\mathbb{L}_r$ is straightforward. Hint: if $(\langle h_n\rangle, h) \in \mathbb{L}_r$ and $h_n \in B_1(\mathbb{R}) \setminus C(\mathbb{R})$ for all $n \in \mathbb{N}$, then there exists a subsequence $\langle h'_n\rangle$ of $\langle h_n\rangle$ such that $\langle h'_n - h\rangle \in \mathcal{N}$; hence $\langle h'_n\rangle$ is of the form $\bigl\langle \sum_{i=1}^{k} (f_{in} - f_i) g_i + h \bigr\rangle$, where $f_i, g_i, h \in B_1(\mathbb{R})$, and $\langle f_{in}\rangle$ is a sequence of continuous functions pointwise converging to f_i, $i = 1, \ldots, k$; the rest is trivial. □

Claim 2.4.2. Let $\mathbb{L}$ be a strict $\mathcal{L}_0^*$-ring convergence on $B_1(\mathbb{R})$. Then $\mathbb{L}_r \subset \mathbb{L}$.

P r o o f. Since $\mathbb{L}$ is admissible and compatible with the ring structure of $B_1(\mathbb{R})$, $\langle (f_n - f)g\rangle$ converges under $\mathbb{L}$ to $\mathbb{O}$ whenever $\langle f_n\rangle$ is a sequence of continuous functions pointwise converging to $f \in B_1(\mathbb{R})$ and $g \in B_1(\mathbb{R})$. Hence $\mathbb{L}_r \subset \mathbb{L}$.

It is known that each commutative $\mathcal{L}_0^*$-group can have many nonequivalent $\mathcal{L}_0^*$-group completions and its Novák completion ([NOV]) yields its categorical $\mathcal{L}_0^*$-group completion ([FKO]). We show that the Novák $\mathcal{L}_0^*$-group completion of $C(\mathbb{R})$ fails to be an $\mathcal{L}_0^*$-ring completion. □

Example 2.5. Let $\mathbb{L}$ denote the pointwise convergence on $C(\mathbb{R})$. Then the Novák $\mathcal{L}_0^*$-group completion of $C(\mathbb{R})$ has $B_1(\mathbb{R})$ as its underlying group and is equipped with an $\mathcal{L}_0^*$-group convergence $\mathbb{L}_1^*$ defined as follows: $\langle f_n\rangle$ converges to f under $\mathbb{L}_1^*$ iff for each subsequence $\langle f'_n\rangle$ of $\langle f_n\rangle$ there exists its subsequence $\langle f''_n\rangle$ such that $f''_n - f = g_n - g$, $n \in \mathbb{N}$, where $\langle g_n\rangle$ is a sequence of continuous functions pointwise converging to $g \in B_1(\mathbb{R})$. Let h be the characteristic function of the singleton $\{0\}$ and let f_n be the constant function with value $1/n$, $n \in \mathbb{N}$. Then $h \in B_1(\mathbb{R})$ and the sequence of continuous functions $\langle f_n\rangle$ pointwise converges to $\mathbb{O}$, but their product $\langle hf_n\rangle$ fails to converge under $\mathbb{L}_1^*$. Hence $\mathbb{L}_1^*$ fails to be an $\mathcal{L}_0^*$-ring convergence and clearly $\mathbb{L}_1^* \subsetneqq \mathbb{L}_r$.

Note: it is known that an $\mathcal{L}_0^*$-ring need not have an $\mathcal{L}_0^*$-ring completion ([FZE]) and there are known sufficient conditions guaranteeing the existence of the categorical $\mathcal{L}_0^*$-ring completion ([FKO]); $C(\mathbb{R})$ fails to be a field and hence does not satisfy the conditions.

We finish this section by mentioning some problems. First, we do not know whether $\mathbb{L}_s$ or $\mathbb{L}_r$ is complete. Second, if not, then we can ask whether $B_1(\mathbb{R})$ equipped with $\mathbb{L}_s$ or $\mathbb{L}_r$ has an $\mathcal{L}_0^*$-ring completion, at all.

3.

Our final goal is to construct an $\mathcal{L}_0^*$-ring completion of $C(\mathbb{R})$ having a universal extension property or, in categorical terms, an epireflection of $C(\mathbb{R})$ into complete $\mathcal{L}_0^*$-rings. Since $C(\mathbb{R})$ is not a field, we cannot use the construction due to J. Novák. The interested reader is referred to [FKO] for the background information about $\mathcal{L}_0^*$-ring completions and to [HES] about categorical notions. To make the paper more self-contained, we briefly recall some related notions.

Let $\mathbb{K}$ be an $\mathcal{L}_0^*$-convergence on a set $Y \neq \emptyset$. For $A \subset Y$, denote by $\operatorname{cl} A$ the set of all $\mathbb{K}$-limits of sequences ranging in A. Define $0\text{-}\operatorname{cl} A = A$ and, by induction, for each ordinal number $\alpha \leqslant \omega_1$ define $\alpha\text{-}\operatorname{cl} A = \bigcup\limits_{\beta<\alpha} \operatorname{cl}(\beta\text{-}\operatorname{cl} A)$. Then $1\text{-}\operatorname{cl} A = \operatorname{cl} A$ and each α-cl yields a closure operator on Y. Further, ω_1-cl is idempotent and hence a topology; it is the finest of all topologies on Y coarser than cl.

Note: if $\omega_1\text{-cl}\,A = Y$ and f, g are continuous maps of $(Y, \mathbb{K})$ into an $\mathcal{L}_0^*$-space $(Y', \mathbb{K}')$ such that $f(x) = g(x)$ for each $x \in A$, then $f = g$; consequently a morphism with a topologically dense range is an epimorphism in $\mathcal{L}_0^*$-spaces.

Let Y be a ring (commutative, not necessarily possessing a unit element) and let $\mathbb{K}$ be an $\mathcal{L}_0^*$-ring convergence on Y. A sequence $\langle x_n \rangle$ is said to be *Cauchy* if $\langle x'_n - x''_n \rangle$ converges under $\mathbb{K}$ to zero whenever $\langle x'_n \rangle$ and $\langle x''_n \rangle$ are subsequences of $\langle x_n \rangle$. If each Cauchy sequence converges, then we speak of a complete $\mathcal{L}_0^*$-ring. Let X be a subring of Y. Then, for each ordinal number $\alpha \leqslant \omega_1$, the set $\alpha\text{-cl}\,X$ is a subring of Y and if Y is complete, then the subring $\omega_1\text{-cl}\,X$ is complete, too. If Y is complete and $Y = \omega_1\text{-cl}\,X$, then $(Y, \mathbb{K})$ is said to be an $\mathcal{L}_0^*$-ring *completion* of X carrying the restriction of $\mathbb{K}$ to X. Finally, for $\mathcal{L}_0^*$-convergences the coordinatewise convergence on products is the categorical one and the product of complete $\mathcal{L}_0^*$-rings is a complete $\mathcal{L}_0^*$-ring.

Let $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ denote the categories of all $\mathcal{L}_0^*$-rings, all $\mathcal{L}_0^*$-rings having a completion, and all complete $\mathcal{L}_0^*$-rings, respectively. A straightforward application of the usual categorical tricks yields the following

Theorem 3.1. *$\mathcal{C}$ is an epireflective subcategory of $\mathcal{B}$.*

P r o o f. Let X be a ring carrying an $\mathcal{L}_0^*$-ring convergence $\mathbb{L}$ and let $(\overline{X}, \overline{\mathbb{L}})$ be its $\mathcal{L}_0^*$-ring completion. We show that there exists its $\mathcal{L}_0^*$-ring completion $\varrho(X, \mathbb{L}) = (\hat{X}, \hat{\mathbb{L}})$ having the following universal extension property: for each continuous homomorphism f of $(X, \mathbb{L})$ into a complete $\mathcal{L}_0^*$-ring $(Y, \mathbb{K})$ there exists a unique continuous homomorphism $\hat{f}$ of $(\hat{X}, \hat{\mathbb{L}})$ into $(Y, \mathbb{K})$ such that $f(x) = \hat{f}(x)$ for each $x \in X$. Since $\hat{X}$ is the smallest sequentially closed subset containing X, the embedding id: $(X, \mathbb{L}) \to (\hat{X}, \hat{\mathbb{L}})$ is an epimorphism and ϱ yields an epireflector of $\mathcal{B}$ into $\mathcal{C}$. The construction of $(\hat{X}, \hat{\mathbb{L}})$ is divided into two parts. The first has an auxiliary character.

Part 1. There exists a nonempty set $\mathcal{S} = \{f_a \colon (X, \mathbb{L}) \to (X_a, \mathbb{L}_a);\ a \in A\}$ of continuous homomorphisms such that each $(X_a, \mathbb{L}_a)$ is a complete $\mathcal{L}_0^*$-ring and if f is a continuous homomorphism of $(X, \mathbb{L})$ into a complete $\mathcal{L}_0^*$-ring $(Y, \mathbb{K})$, then there exists $a \in A$ and a homeomorphic isomorphism g of $(X_a, \mathbb{L}_a)$ onto a subring $(Y_f, \mathbb{K} \upharpoonright Y_f)$ of $(Y, \mathbb{K})$ such that f is a composition of $g \circ f_a$ and the embedding of $(Y_f, \mathbb{K} \upharpoonright Y_f)$ into $(Y, \mathbb{K})$. Indeed, each f determines a complete $\mathcal{L}_0^*$-subring of $(Y, \mathbb{K})$ the underlying set of which is $Y_f = \omega_1\text{-cl}\,f(X)$. Since $\operatorname{card}(1\text{-cl}\,f(X))$ cannot exceed the cardinality of the set of all countable infinite subsets of $f(X)$ and $\omega_1\text{-cl}\,f(X) = \bigcup\limits_{\beta < \omega_1} \text{cl}(\beta\text{-cl}\,f(X))$, it follows that $\operatorname{card}(Y_f) \leqslant \exp\operatorname{card}(X)$. Hence there is a set $\{(X_b, \mathbb{L}_b);\ b \in B\}$ of complete $\mathcal{L}_0^*$-rings such that $\operatorname{card}(X_b) \leqslant \exp\operatorname{card}(X)$ and each $(Y_f, \mathbb{K} \upharpoonright Y_f)$ is homeomorphic and isomorphic to some $(X_b, \mathbb{L}_b)$, $b \in B$. Note: $\mathcal{S}$ yields a so-called solution set for $(X, \mathbb{L})$ with respect to the inclusion functor of $\mathcal{C}$ into $\mathcal{B}$.

Part 2. The product $\prod_{a\in A} (X_a, \mathbb{L}_a)$ is a complete $\mathcal{L}_0^*$-ring and, via the canonical embedding sending $x \in X$ into $\varphi(x) = (f_a(x), a \in A)$, $(X, \mathbb{L})$ can be view as the corresponding $\mathcal{L}_0^*$-subring of $\prod_{a\in A} (X_a, \mathbb{L}_a)$ (remember, $(X, \mathbb{L})$ is an $\mathcal{L}_0^*$-subring of its completion $(\overline{X}, \overline{\mathbb{L}})$). Denote by $(\hat{X}, \hat{\mathbb{L}})$ the smallest sequentially closed $\mathcal{L}_0^*$-subring of $\prod_{a\in A} (X_a, \mathbb{L}_a)$ containing $(X, \mathbb{L})$. It is easy to see that $(\hat{X}, \hat{\mathbb{L}})$ has the desired properties.

This completes the proof. □

Lemma 3.2. *Let $(Y, \mathbb{K})$ be a complete $\mathcal{L}_0^*$-ring and let X be a subring of Y. Put $\overline{X} = \omega\text{-cl}\, X$ and define $\overline{\mathbb{L}} \subset \overline{X}^{\mathbb{N}} \times \overline{X}$ as follows: $(\langle x_n \rangle, x) \in \overline{\mathbb{L}}$ whenever $(\langle x_n \rangle, x) \in \mathbb{K}$ and there exists a natural number k such that $x_n \in (k\text{-cl}\, X)$ for each $n \in \mathbb{N}$. Then $(\overline{X}, \overline{\mathbb{L}})$ is a complete $\mathcal{L}_0^*$-ring and the identity mapping on $\overline{X}$ is a continuous isomorphism of $(\overline{X}, \overline{\mathbb{L}})$ onto $(\overline{X}, \mathbb{K} \upharpoonright \overline{X})$.*

P r o o f. It is easy to verify that $\overline{\mathbb{L}}$ is an $\mathcal{L}_0^*$-ring convergence on $\overline{X}$ finer than $\mathbb{K} \upharpoonright \overline{X}$. If $\langle x_n \rangle$ is a Cauchy sequence in $(\overline{X}, \overline{\mathbb{L}})$, then there exists $k \in \mathbb{N}$ such that $x \in (k\text{-cl}\, X)$ for each $n \in \mathbb{N}$. Thus $(\overline{X}, \overline{\mathbb{L}})$ is complete. □

Theorem 3.3. *Let $(X, \mathbb{L})$ be an $\mathcal{L}_0^*$-ring in $\mathcal{B}$ and let $(\hat{X}, \hat{\mathbb{L}})$ be its categorical $\mathcal{L}_0^*$-ring completion. Then $\omega\text{-cl}\, X = \hat{X}$.*

P r o o f. The assertion follows from Lemma 3.2. Putting $\hat{X} = Y$ and $\hat{\mathbb{L}} = \mathbb{K}$, we easily infer that $\overline{X} = \hat{X}$ and $\overline{\mathbb{L}} = \hat{\mathbb{L}}$. □

Corollary 3.4. *$B(\mathbb{R})$ carrying the pointwise convergence fails to be the categorical completion of $C(\mathbb{R})$.*

P r o o f. Indeed, $\omega\text{-cl}\, C(\mathbb{R}) = B_\omega(\mathbb{R}) \subsetneqq B(\mathbb{R})$, while $C(\mathbb{R})$ is ω-dense in its categorical $\mathcal{L}_0^*$-ring completion. □

Problem. Describe the categorical $\mathcal{L}_0^*$-ring completion of $C(\mathbb{R})$.

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