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LEXICOGRAPHIC PRODUCT DECOMPOSITIONS OF CYCLICALLY ORDERED GROUPS

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In this paper the notion of cyclically ordered set will be used in the same sense as by Novák and Novotný [6], [7].

Next, the concept of cyclically ordered group is understood as by Zheleva [9] (this concept is more general than that applied in the fundamental Rieger's paper [8] and in some other articles).

A particular type of cyclically ordered groups, denoted as *dc*-groups, has been defined by the author [4] and investigated by Černák [1]. Roughly speaking, the relation between cyclically ordered groups and *dc*-groups is analogous to the relation between partially ordered groups and directed groups.

In the present paper it will be proved that any two lexicographic product decompositions of a *dc*-group have isomorphic refinements.

This generalizes the main theorem of [1] concerning finite lexicographic product decompositions of a *dc*-group.

Analogous results on lexicographic product decompositions of linearly ordered groups, directed groups or directed groupoids were proved by Malcev [5], Fuchs [2] and the author [3], respectively.

The methods which have been used [3] will be adapted and applied in the present paper.

1. Preliminaries

First we recall some basic definitions.

1.1. Definition. (Cf. [6].) A nonempty set M endowed with a ternary relation C is said to be *cyclically ordered* if the following conditions are satisfied:

(I) If $(x, y, z) \in C$, then $(y, x, z) \notin C$.

(II) If $(x,y,z)\in C$, then $(z,x,y)\in C$.

(III) If $(x,y,z)\in C$ and $(x,z,u)\in C$, then $(x,y,u)\in C$.

The relation C is called a *cyclic order* on M.

If M_1 is a nonempty subset of M, then we consider M_1 to be cyclically ordered by the relation of cyclic order which is inherited from C.

It is easy to verify that if $(x,y,z)\in C$, then the elements x,y and z must be distinct. Hence if $\operatorname{card} M\leqslant 2$, then the set C must be empty.

1.2. Definition. Assume that G is a group (with the group operation written additively, the commutativity of this operation being not assumed) and that, at the same time, it is a cyclically ordered set such that the following condition is satisfied:

(IV) If $(x_1,x_2,x_3)\in C$, $a\in G$, $y_i=a+x_i$, $z_i=x_i+a$ $(i=1,2,3)$, then $(y_1,y_2,y_3)\in C$ and $(z_1,z_2,z_3)\in C$.

Under these assumption G is called a *cyclically ordered group.*

1.2.1. Remark. In [3], [5], [8] and some other papers the term "cyclically ordered group" means a structure G satisfying the conditions from Definition 1.2 and the following additional condition: if x, y and z are distinct elements of G, then either $(x,y,z)\in C$ or $(y,x,z)\in C$.

1.3. Definition. (Cf. [1] and [4].) A cyclically ordered group is said to be a *dc-group* if for each $x,y\in G$ with $x\neq y$ there exists $z\in G$ such that either $(x,y,z)\in C$ or $(y,x,z)\in C$.

Let I be a linearly ordered set and for each $i\in I$ let G_i be a *dc*-group. We denote by G_0 the cartesian product of groups G_i $(i\in I)$. For $x=(x_i)_{i\in I}\in G_0$ we put

$$I(x)=\{i\in I\colon\ x_i\neq 0\}.$$

Let G be the set of all $x\in G_0$ such that the set $I(x)$ is well-ordered. Then G is a subgroup of the group G_0.

Let x,y and z be distinct elements of G. We put $(x,y,z)\in C$ if there is $i(1)\in I$ such that (a) $(x_{i(1)},y_{i(1)},z_{i(1)})\in C$, and (b) for each $i\in I$ with $i<i(1)$ the relation $x_i=y_i=z_i$ is valid. Then G turns out to be a cyclically ordered group.

Let a and b be distinct elements of G. There is $i(2)\in I$ such that $a_{i(2)}\neq b_{i(2)}$ and $a_i=b_i$ for each $i\in I$ with $i<i(2)$. Since $G_{i(2)}$ is a a *dc*-group there exists $c^{i(2)}\in G_{i(2)}$ such that either $(a_{i(2)},b_{i(2)},c^{i(2)})\in C$ or $(b_{i(2)},a_{i(2)},c^{i(2)})\in C$. Next, there is $c\in G$ such that $c_{i(2)}=c^{i(2)}$ and $c_i=0$ for each $i\in I$ with $i\neq i(2)$. Then we have either $(a,b,c)\in C$ or $(b,a,c)\in C$. Hence G is a *dc*-group.

1.4. Definition. Under the assumption as above we write

$$G = [\Gamma_{i \in I} G_i] \tag{1}$$

and say that G is an *external lexicographic product of dc-groups* G_i. The *dc*-groups G_i are called lexicographic factors of G. If I is the set $\{1, 2, \ldots, n\}$ with the natural linear order, then we write also $G = [G_1 \circ G_2 \circ \ldots \circ G_n]$. Next, if $I(1)$ is a subset of I such that $G_i = \{0\}$ for each $i \in I \setminus I(1)$, then $[\Gamma_{i \in I} G_i]$ will be identified with $[\Gamma_{i \in I(1)} G_i]$; in the case $I = \emptyset$ we put $\Gamma_{i \in I} G_i = \{0\}$.

The notion of an isomorphism of *dc*-groups is defined in the obvious way. If we have an isomorphism

$$\alpha \colon G \longrightarrow [\Gamma_{i \in I} G_i], \tag{2}$$

then α is said to be a *lexicographic product decomposition* of G.

Let us have another lexicographic product decomposition of G

$$\beta \colon G \longrightarrow [\Gamma_{j \in J} G'_j]. \tag{3}$$

We say that α and β are isomorphic if there exists an isomorphism φ of I onto J such that for each $i \in I$, G_i is isomorphic to $G'_{\varphi(i)}$.

1.5. Definition. Let (2) and (3) be valid. Suppose that to each $i \in I$ there corresponds a subset $\psi(i)$ of J such that the following conditions are satisfied:

(a) If $i(1), i(2) \in I$ and $i(1) < i(2)$, then $j_1 < j_2$ for each $j_1 \in \psi(i(1))$ and each $j_2 \in \psi(i(2))$.

(b) $\bigcup\limits_{i \in I} \psi(i) = J$.

(c) For each $i \in I$ the relation $G_i = [\Gamma_{j \in \psi(i)} G'_j]$ is valid.

Under these assumptions β is said to be *a refinement of* α.

1.6. Example. This example shows that if the relations (2), (3) are valid and if, moreover, $I = J$ and $G_i = G'_i$ for each $i \in I$, then the mappings α and β need not coincide.

Let $G_0 = \{0, 1, 2\}$ with the operation $+$ denoting the addition mod 3. Put $C = \{(0, 1, 2), (1, 2, 0), (2, 0, 1)\}$. Then G_0 with the ternary relation C is a *dc*-group. Let I be the set of all integers, $J = I$. For each $i \in I$ let $G_i = G'_i = G_0$. Put $G = [\Gamma_{i \in I} G_i]$ and let α be the identity on G. For $x \in G$ let y be the element of G such that $y_i = x_{i-1}$ for each $i \in I$; put $\beta(x) = y$. Then (2) and (3) are valid, but α is not equal to β.

Again, let G be a dc-group.

Let (2) be valid. For $i \in I$ and $x_i \in G_i$ we denote by $\bar{x}_i$ the element of G such that

$$\alpha(\bar{x}_i)_i = x_i,$$
$$\alpha(\bar{x}_i)_{i(1)} = 0 \text{ for each } i(1) \in I \text{ with } i(1) \neq i.$$

Next we put $\overline{G}_i = \{\bar{x}_i \colon x_i \in G_i\}$. Then $\overline{G}_i$ is a dc-group. The mapping $x_i \longrightarrow \bar{x}_i$ is an isomorphism of G_i onto $\overline{G}_i$.

For each $g \in G$ with $\alpha(g) = (g_i)_{i\in I}$ we put $\bar{\alpha}(g) = (\bar{g}_i)_{i\in I}$. We obtain an isomorphism

$$\bar{\alpha}\colon G \longrightarrow [\Gamma_{i\in I}\overline{G}_i].$$

2.1. Definition. Under the above assumptions we write

$$\bar{\alpha}\colon G = \Gamma_{i\in I}\overline{G}_i;$$

$\bar{\alpha}$ is said to be an *internal lexicographic product decomposition* of G. If $I = \{1, 2, \ldots, n\}$ with the natural linear order, then we write $\overline{\alpha}\colon G = G_1 \circ G_2 \circ \ldots \circ G_n$; in this case the lexicographic product decomposition $\overline{\alpha}$ is said to be finite.

In [1] a formally different definition of finite interal lexicographic product decomposition was given. For the sake of completeness and also in view of applications we recall this definition (for the case of a dc-group).

2.1.1. Definition. (Cf. [1].) Let G be a dc-group. Let A and B be subgroups of G such that the following conditions hold:

(i) for each $g \in G$ there exist uniquely determined elements $a \in A$, $b \in B$ such that $g = a + b$;

(ii) if $g_i = a_i + b_i$, $a_i \in A$, $b_i \in B$ $(i = 1, 2)$, then $g_1 + g_2 = (a_1 + a_2) + (b_1 + b_2)$;

(iii) if g_1, g_2, g_3 are distinct elements of G, $g_i = a_i + b_i$, $a_i \in A$, $b_i \in B$ $(i = 1, 2, 3)$, then $(g_1, g_2, g_3) \in C$ iff either $(a_1, a_2, a_3) \in C$, or $a_1 = a_2 = a_3$ and $(b_1, b_2, b_3) \in C$.

Under these assumptions we write $G = A \circ B$; this equation is said to be an *internal lexicographic product decomposition* of G with factors A and B. Next, for $n > 2$ the relation $G = G_1 \circ G_2 \circ \ldots \circ G_n$ is defined by induction; it expresses the fact that $G = (G_1 \circ G_2 \circ \ldots \circ G_{n-1}) \circ G_n$.

It can be easily verified that for a finite set I Definition 2.1.1 is equivalent with Definition 2.1. This implies that for finite I the symbol $\bar{\alpha}$ in 2.1 can be omitted.

The natural question arises whether the symbol $\bar{\alpha}$ can be omitted also for the case of infinite I, i.e., whether the "pathological" situation described in Example 1.6 can occur in the case when α and β are internal lexicographic product decompositions.

In this section we shall show that in the "internal" case such a situation cannot occur.

First we recall that in [1] it was proved that the operation of forming finite internal lexicographic products is associative, i.e., we need not apply brackets.

If g, a, b are as in 2.1.1, then a is called the *component* of g in A; similarly, b is the component of g in B (with respect to the internal lexicographic decomposition $G = A \circ B$). Analogously, by applying 2.1.1, we define the component of g in G_i ($i \in \{1, 2, \ldots, n\}$) in the case when the relation $G = G_1 \circ G_2 \circ \ldots \circ G_n$ is valid. According to 2.1.1, in the case $n = 2$ the components of g in G_1 and G_2 are uniquely determined. By applying induction on n we obtain

2.2. Lemma. *Let $G = G_1 \circ G_2 \circ \ldots \circ G_n$ and let $g \in G$, $i \in \{1, 2, \ldots, n\}$. Then the component of g in G_i is uniquely determined.*

Assume that the relation

$$\alpha\colon\ G = \Gamma_{i \in I} G_i \tag{4}$$

is valid. Let I_1 and I_2 be subsets of I such that

(a) whenever $i_1 \in I_1$ and $i_2 \in I_2$, then $i_1 < i_2$;
(b) $I_1 \cup I_2 = I$.

If $j \in \{1, 2\}$ and $I_j \neq \emptyset$, then we denote by P_j the set of all $g \in G$ such that $(\alpha(g))_i = 0$ for each $i \in I \setminus I_j$; in the case $I_j = \emptyset$ we set $P_j = \{0\}$. It is clear that P_1 and P_2 are subgroups of the group G. Let $x \in G$ and $j \in \{1, 2\}$. There exists a uniquely determined element $x^j \in P_j$ such that

$$(\alpha(x))_i = (\alpha(x^j))_i \quad \text{for each} \quad i \in I_j.$$

Put $\chi(x) = (x^1, x^2)$.

2.3. Lemma. $G = P_1 \circ P_2$. *If $x \in G, \chi(x) = (x^1, x^2)$, then x^j is the component of x in P_j ($j = 1, 2$).*

The proof is simple and will be omitted.

Let $i(1) \in I$. Put

$$I^1 = \{i \in I\colon\ i < i(1)\}, \quad I^2 = \{i \in I\colon\ i > i(1)\}.$$

If $I^1 \neq \emptyset$, then we define $H_{i(1)}$ to be the set of all $g \in G$ such that $(\alpha(g))_i = 0$ for each $i \in I \setminus I^1$; the set $D_{i(1)}$ is defined analogously with I^1 replaced by I^2.

2.4. Lemma. *Under the above notation we have $G = H_{i(1)} \circ G_{i(1)} \circ D_{i(1)}$. If $g \in G$, $x \in H_{i(1)}$, $y \in G_{i(1)}$, $z \in D_{i(1)}$, $g = x + y + z$, then*

$$(\alpha(x))_i = (\alpha(g))_i \quad \textit{for each} \quad i > i(1).$$

P r o o f. This is a consequence of 2.3 and of the definitions of $H_{i(1)}, D_{i(1)}$. $\square$

2.5. Lemma. *Let $i(1) \in I$, $G_{i(1)} \neq \{0\}$, $0 \neq g \in G_{i(1)} \circ D_{i(1)}$. Then the following conditions are equivalent:*

(a) $g \in D_{i(1)}$.
(b) *If $x, y, z \in G_{i(1)}, (x, y, z) \in C$, then $(x + g, y, z) \in C$.*

P r o o f. The implication (a)$\Rightarrow$(b) is an immediate consequence of the relation $g \in G_{i(1)} \circ D_{i(1)}$. Assume that (b) is valid and suppose that (a) does not hold. Hence there are $g_1 \in G_{i(1)}$ and $g_2 \in D_{i(1)}$ such that $g = g_1 + g_2$ and $g_1 \neq 0$. Since $G_{i(1)}$ is a *dc*-group there exists $g_3 \in G_{i(1)}$ such that either $(-g_1, 0, g_3) \in C$ or $(-g_1, g_3, 0) \in C$. Thus $g_3 \neq 0$. Let the first case be valid (in the opposite case we proceed analogously). In view of (b) we infer that $(-g_1 + g, 0, g_3) \in C$, hence $(g_2, 0, g_3) \in C$. Therefore according to 2.1.1 (iii) we have arrived at a contradiction. $\square$

Now suppose that we are given (together with (4)) another internal lexicographic product decomposition

$$\beta\colon\ G = \Gamma_{j \in J} G'_j. \tag{5}$$

For $j(1) \in J$ we can apply analogous notation as in 2.4 obtaining

$$G = H'_{j(1)} \circ G'_{j(1)} \circ D'_{j(1)}. \tag{6}$$

2.6. Lemma. *Assume that there are $i(1) \in I$ and $j(1) \in J$ such that $G_{i(1)} = G_{j(1)}$ and $G_{i(1)} \neq \{0\}$. Then $D_{i(1)} = D'_{j(1)}$.*

P r o o f. This follows from 2.5. $\square$

The following lemma improves Theorem 3.8 of [1].

2.7. Lemma. *Let $G = A \circ B$ and $G = D \circ B$. If $d \in D$, $a \in A$, $b \in B$, $d = a + b$, then we put $\varphi(d) = a$. The mapping φ is an isomorphism of D onto A.*

P r o o f. Let $d \in D$. There are uniquely determined elements $a \in A$ and $b \in B$ such that $d = a + b$. Hence $\varphi(d) = a$. It is obvious that $\varphi(d + d_1) = \varphi(d) + \varphi(d_1)$ for $d_1 \in D$. If $a' \in A$, then there are $d' \in D'$ and $b' \in B'$ such that $a' = d' + b'$; this yields that $\varphi(d') = a'$. If $\varphi(d) = 0$, then $d = b$, thus $d \in B$. Since $D \cap B = \{0\}$ we get $d = 0$. Hence φ is an isomorphism of the group $(D; +)$ onto the group $(A; +)$.

Let $d_1 \in D$, $d_i = a_i + b_i$, $a_i \in A$, $b_i \in B$ $(i = 1, 2, 3)$. Suppose that $(d_1, d_2, d_3) \in C$. Then d_1, d_2 and d_3 are distinct, whence a_1, a_2 and a_3 are distinct. Thus from $G = A \circ B$ we obtain that $(a_1, a_2, a_3) \in C$.

Similarly we can verify that if $(a_1, a_2, a_3) \in C$, then $(d_1, d_2, d_3) \in C$. Hence φ is an isomorphism of D onto A. $\square$

2.8. Lemma. *Let $I = J$ and $G_i = G'_i$ for each $i \in I$. Then $H_{i(1)} = H'_{i(1)}$ for each $i(1) \in I$.*

P r o o f. Let $i(1) \in I$. In view of 2.6, we have

$$G = H_{i(1)} \circ (G_{i(1)} \circ D_{i(1)}),$$
$$G = H'_{i(1)} \circ (G_{i(1)} \circ D_{i(1)}).$$

Let $h \in H_{i(1)}$. There exist uniquely determined elements $a \in H'_{i(1)}$ and $b \in G_{i(1)} \circ D_{i(1)}$ such that $h = a + b$. We put $\varphi(h) = a$. According to 2.7, φ is an isomorphism of $H_{i(1)}$ onto $H'_{i(1)}$.

In view of the definition of $H'_{i(1)}$ and by the assumption the relation $H'_{i(1)} = \Gamma_{i<i(1)} G_i$ is valid, hence $G_i \subseteq H'_{i(1)}$ for each $i < i(1)$. Thus $\varphi(G_i) = G_i$ for each $i < i(1)$.

Next, from the isomorphism φ and from

$$H_{i(1)} = \Gamma_{i<i(1)} G_i$$

we obtain

$$H'_{i(1)} = \varphi(H_{i(1)}) = \Gamma_{i<i(1)} \varphi(G_i) = \Gamma_{i<i(1)} G_i = H_{i(1)}.$$

$\square$

2.9. Theorem. *Let G be a dc-group and let* (4), (5) *be valid. Assume that $J = I$ and that $G_i = G'_i$ for each $i \in I$. Then $\alpha = \beta$.*

P r o o f. This is a consequence of 2.6, 2.8 and 2.4. (Cf. also 2.1.1.) $\square$

In view of 2.9, the symbol α in (4) can be omitted. Thus when (4) is fixed then we often write $(\alpha(g))_i = g_i = g(G_i)$; next, for $X \subseteq G$ we put $X(G_i) = \{x_i \colon x \in X\}$.

Again, let us consider the relations (4) and (5). We can ask whether the following assertion is valid:

(∗) If $G_{i(1)} = G'_{j(1)}$ for some fixed $i(1) \in I$ and some fixed $j(1) \in J$, then $(\alpha(g))_{i(1)} = (\beta(g))_{j(1)}$ for each $g \in G$.

It can be shown by examples that the answer to this question is “No”. Let us remark that for internal direct product decompositions of lattice ordered groups the assertion analogous to (∗) is valid.

3. Auxiliary results

In this section we apply the same assumptions and notation as above.

3.1. Lemma. (Cf. [1], 3.3 and 3.4.) *Assume that $G = A \circ B$ and $G = A_1 \circ B_1$. Then either $B \subseteq B_1$ or $B_1 \subseteq B$. If $B_1 \subseteq B$, then $B = D \circ B_1$, where $D = A_1 \cap B$. Moreover, $D = B(A)$.*

3.2. Lemma. *Let $G = A \circ B, G = C_1 \circ C_2 \circ \ldots \circ C_n, A \neq \{0\}$. Then $A = C_1(A) \circ C_2(A) \circ \ldots \circ C_n(A)$.*

P r o o f. If suffices to apply the same steps as in the proof of [3], 16 with the distinction that [3], 13.4 is replaced by 2.7. □

In the following lemma the symbol $A \circ D \cap C_1$ denotes $(A \circ D) \cap C_1$, and analogously in other places below.

3.3. Lemma. *Let $G = H \circ A \circ D$, $G = C_1 \circ C_2 \circ \ldots \circ C_n$. Then*

$$A = (A \circ D \cap C_1)(A) \circ \ldots \circ (A \circ D \cap C_n)(A).$$

P r o o f. Cf. the proof of 16.1 in [3] (we replace [3], 11 and [3], 16 by 3.1 and 3.2, respectively). □

Now let us assume that we are given two internal lexicographic product decompositions

$$G = H \circ A \circ D, \quad G = H' \circ B \circ D'.$$

3.4. Lemma. *The dc-groups $(A \circ D \cap B)(A)$ and $(B \circ D' \cap A)(B)$ are isomorphic.*

P r o o f. Cf. [3], 16.2–18 (we replace [3], 9, [3], 16.1 and [3], 13.4 by 3.1, 3.3 or 2.7, respectively); in fact, in the proof of [3], 17 we should have 16.1 instead of 6.1. □

Now let (4) be valid. Then the following conditions hold:

(a) $G = H_i \circ G_i \circ D_i$ for each $i \in I$;

(b) if $x \in G$, then the set $\{i \in I \colon x(G_i) \neq 0\}$ is well-ordered;

(c) if I_1 is a well-ordered subset of I and if $x^i \in G_i$ for each $i \in I$, then there exists a uniquely determined element $x \in G$ such that $x(G_i) = x^i$ for each $i \in I_1$ and $x(G_i) = 0$ otherwise;

(d) if $i, j \in I$, $i < j$, then $G_j \circ D_j \subseteq D_i$, $H_i \circ G_i \subseteq H_j$.

For a), cf. 2.4; the conditions b)–d) are immediate consequences of (4).

3.5. Lemma. *Let I be a linearly ordered set. For each $i \in I$ let H_i, G_i and D_i be subgroups of a dc-group G such that the conditions* a)–d) *are valid. Then* (4) *holds.*

P r o o f. We proceed analogously as in the proof of [3], 22.1. The modifications which are due to the fact that we are now dealing with the internal case are obvious. The only place in the proof which is to be essentially changed is the assertion (δ) in [3], p. 290; it is to be replaced by the following argument:

(δ) Let x, y and z be distinct elements of G and let $i \in I$. Suppose that the elements $x(G_i), y(G_i), z(G_i)$ are distinct and that $x(G_j) = y(G_j)$ for each $j \in I$ with $j < i$. Then $(x, y, z) \in C$ iff $(x(G_i), y(G_i), z(G_i)) \in C$.

Proof of (δ): There exists $t \in G$ such that $t(G_j) = x(G_j)$ for each $j \in I$ with $j < i$, and $t(G_j) = 0$ otherwise (cf. the assertion (α) in [3], p. 290). Denote $x' = x - t$, $y' = y - t$, $z' = z - t$. Then $x', y', z' \in G_i \circ D_i$ and $x'(G_i), y'(G_i), z'(G_i)$ are distinct. Hence $(x', y', z') \in C$ iff $(x'(G_i), y'(G_i), z'(G_i)) \in C$. We obtain that $(x, y, z) \in C$ iff $(x(G_i), y(G_i), z(G_i)) \in C$. □

Again, let (4) be valid. Suppose that for each $i \in I$ a lexicographic product decomposition

$$G_i = \Gamma_{j \in J_i} G_{ij}$$

is given. Let Q be the set of all pairs (i, j) with $i \in I$, $j \in J_i$. For $q_1, q_2 \in Q$ with $q_1 = (i_1, j_1)$, $q_2 = (i_2, j_2)$ we put $q_1 < q_2$ if either $i_1 < i_2$, or $i_1 = i_2$ and $j_1 < j_2$. For each $i \in I$ and each $j \in J_i$ we have (under analogous notation as in 2.4 above)

$$G_i = H^0_{ij} \circ G_{ij} \circ D^0_{i,j},$$

hence

$$G = H_i \circ H^0_{ij} \circ G_{ij} \circ D^0_{ij} \circ D_i.$$

Denote

$$H_i \circ H^0_{ij} = H_{ij}, \quad D^0_{ij} \circ D_i = D_{ij}.$$

Therefore

$$G = H_q \circ G_q \circ D_q \quad \text{for each} \quad q \in Q. \tag{7}$$

3.6. Lemma. $G = \Gamma_{q \in Q} G_q$.

P r o o f. The validity of the conditions a)–d) for H_q, G_q, D_q ($q \in Q$) can be easily verified. Now it suffices to apply 3.5. $\square$

It is obvious that the lexicographic product decomposition given in 3.6 is a refinement of the lexicographic product decomposition (4).

3.7. Lemma. *Let* (4) *be valid,* $\emptyset \neq I_1 \subseteq I$*. For each* $i \in I_1$ *let* H_1 *be a subgroup of* G_i*. Let* $H = \{x \in G\colon x(G_i) \in H_i$ *for each* $i \in I_1$*, and* $x(G_i) = 0$ *otherwise*$\}$*. Then* $H = \Gamma_{i \in I_1} H_i$.

P r o o f. This is a consequence of 3.5. $\square$

4. On the lexicographic product decomposition $G = A \circ B$

In this section we assume that (4) is valid and that, at the same time, $G = A \circ B$.

4.1. Lemma. $B = \Gamma_{i \in I} B(G_i)$*; moreover,* $B(G_i) = B \cap G_i$ *for each* $i \in I$.

P r o o f. In view of 3.7 we can construct the dc-group $B' = \Gamma_{i \in I} B(G_i)$ and B' is a subgroup of B. Let $x \in B$. By the same method as in part a) of the proof of [3], 26 (where [3], 11 is replaced by 3.1) we obtain that $x \in B'$. Therefore $B = B'$.

Next, in view of 3.1 the relation $x(G_i \circ D_i) \in B$ is valid for each $x \in B$ and each $i \in I$. This yields that $B(G_i) \subseteq B \cap G_i$. Conversely, $B \cap G_i = (B \cap G_i)(G_i) \subseteq B(G_i)$, thus $B(G_i) = B \cap G_i$. $\square$

In 4.2–4.4 we assume that $G_i \neq \{0\}$ for each $i \in I$.

Put $I(B) = \{i \in I\colon\ B \cap G_i \neq \{0\}\}$.

4.2. Lemma. *Let* $i_1 \in I(B)$*,* $i_2 \in I$*,* $i_1 < i_2$*. Then* $G_{i_2} \subseteq B$.

P r o o f. By 3.1 we have either $D_{i_1} \supset B$ or $D_{i_1} \subseteq B$. In the first case we would have

$$G_{i_1} \cap B \subseteq G_{i_1} \cap D_{i_1} = \{0\},$$

which is a contradiction. Hence $G_{i_2} \subseteq D_{i_1} \subseteq B$. $\square$

4.3. Corollary. *Let $B \neq \{0\}$. Then $I(B)$ is a dual ideal of I.*

P r o o f. There exists $x \in B$ with $x \neq 0$. In view of (4) there is $i(1) \in I$ such that $x(G_{i(1)}) \neq 0$ and $x(G_i) = 0$ for each $i \in I$ with $i < i(1)$. Put $x' = x(G_{i(1)})$. According to 4.1, $x' \in B \cap G_{i(1)}$, hence $I(B) \neq \emptyset$. Now it suffices to apply 4.2. $\square$

Now we distinguish two cases.

a) First suppose that $I(B)$ has no least element. Then in view of 4.2, $B(G_i) = G_i$ for each $i \in I(B)$. Clearly $B(G_i) = \{0\}$ for each $i \in I \setminus I(B)$. Hence 4.1 yields

$$B = \Gamma_{i \in I(B)} G_i. \tag{8}$$

b) Next suppose that $i(0)$ is the least element of $I(B)$. Then from 4.1 and 4.2 we get

$$B = (G_{i(0)} \cap B) \circ \Gamma_{i > i(0)} G_i. \tag{9}$$

4.4. Lemma. $A = \Gamma_{i \in I} G_i(A)$.

P r o o f. We apply (8) and (9). It suffices to use the same steps as in the proof of [3], 29–31 (where [3], 13.4 and [3], 11 are replaced by 2.7 and 3.1, respectively). $\square$

5. Isomorphic refinements

In the present section we suppose that G is a *dc*-group which has two internal lexicographic product decompositions

$$G = \Gamma_{i \in I} A_i, \tag{10}$$

$$G = \Gamma_{j \in J} B_j. \tag{11}$$

For $i \in I$ and $j \in J$ the symbols H_i, D_i, H_j, D_j have analgous meanings as above. Without loss of generality we can suppose that $I \cap J = \emptyset$ and that $A_i \neq \{0\} \neq B_j$ for each $i \in I$, $j \in J$.

Let $i \in I$. In view of 4.1 we have

$$A_i \circ D_i = \Gamma_{j \in J}\ (A_i \circ D_i \cap B_j).$$

Thus according to 4.4

$$A_i = \Gamma_{j \in J}\ (A_i \circ D_i \cap B_j)(A_i).$$

Put $(A_i \circ D_i \cap B_j)(A_i) = E_{ij}$, and let $I \circ J$ have the same meaning as Q in 3.7. Hence

$$G = \Gamma_{(i,j) \in I \circ J} E_{ij} \tag{12}$$

and the lexicographic product decomposition (12) is a refinement of (10).

Analogously we obtain

$$G = \Gamma_{(j,i) \in J \circ I} E_{ji}, \tag{13}$$

where $E_{ji} = (B_j \circ D_j \cap A_i)(B_j)$. The lexicographic product decomposition (13) is a refinement of (11).

5.1. Lemma. *Let (i,j), $(i(1), j(1)) \in I \circ J$, $i(1) < i$, $j(1) > j$, $E_{ij} \neq \{0\}$. Then $E_{i(1),j(1)} = \{0\}$.*

Proof. From $j(1) > j$ we obtain

$$B_{j(1)} \circ D_{j(1)} \subseteq D_j.$$

If $B_{j(1)} \circ D_{j(1)} \supseteq A_i \circ D_i$, then $D_j \supseteq A_i \circ D_i$, whence

$$A_i \circ D_i \cap B_j \subseteq D_j \cap B_j = \{0\}$$

and thus $E_{ij} = \{0\}$, which is a contradiction. Therefore according to 3.1 the relation $B_{j(1)} \circ D_{j(1)} \subseteq A_i \circ D_i \subseteq D_{i(1)}$ is valid. Hence

$$E_{i(1)j(1)} = (A_{i(1)} \circ D_{j(1)} \cap B_{j(1)})(A_{i(1)}) \subseteq B_{j(1)}(A_{i(1)}) \subseteq D_{i(1)}(A_{i(1)}) = \{0\}.$$

□

5.2. Lemma. *E_{ij} is isomorphic to E_{ji} for each $(i,j) \in I \circ J$.*

Proof. This is a consequence of 3.4. □

Let $(I \circ J)^0$ be the set of all elements (i,j) of $I \circ J$ such that $E_{ij} \neq \{0\}$ and let $(J \circ I)^0$ be defined analogously. For each $(i,j) \in (I \circ J)^0$ put $\varphi((i,j)) = (j,i)$.

5.3. Lemma. *φ is an isomorphism of $(I \circ J)^0$ onto $(J \circ I)^0$.*

Proof. This follows from 5.1 and 5.2. □

In view of (12) and (13) we have

$$G = \Gamma_{(i,j)\in(I\circ J)^0} E_{ij}, \tag{12'}$$

$$G = \Gamma_{(j,i)\in(J\circ I)^0} E_{ji}. \tag{13'}$$

Next, (12′) is a refinement of (10), and (13′) is a refinement of (11). According to 5.2 and 5.3, the lexicographic product decompositions (12′) and (13′) are isomorphic. Summarizing, we have

5.4. Theorem. *Any two internal lexicographic product decompositions of a dc-group have isomorphic refinements.*

To each lexicographic product decomposition of a *dc*-group G we can construct the corresponding internal lexicographic product decomposition of G (cf. Section 2); hence in 5.4 the word "internal" can be omitted.

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