Alexander Fischer

Approximation of almost periodic functions by periodic ones

Czechoslovak Mathematical Journal, Vol. 48 (1998), No. 2, 193–205

Persistent URL: `http://dml.cz/dmlcz/127411`

Czechoslovak Mathematical Journal, 48 (123) (1998), 193–205

APPROXIMATION OF ALMOST PERIODIC FUNCTIONS BY PERIODIC ONES

Alexander Fischer, Praha

(Received March 23, 1993)

Abstract. It is not the purpose of this paper to construct approximations but to establish a class of almost periodic functions which can be approximated, with an arbitrarily prescribed accuracy, by continuous periodic functions uniformly on $\mathbb{R} = (-\infty; +\infty)$.

Keywords: almost periodic function

1. Introduction

In many technical as well as purely theoretical domains an important role is played by periodic functions representing periodic motions or periodic processes. The sum of periodic functions need not be a periodic function, hence the space of periodic functions is not linear. Therefore, some attempts to generalize the notion of the periodic function appeared already towards the end of the last century. This effort was successfully completed in the twenties of this century when the Danish mathematician Harald Bohr published his theory of almost periodic functions the space of which is linear and which generalize periodic functions. Since then this theory has been developed by a number of outstanding mathematicians. In 1933 Salomon Bochner presented his important work extending the theory of almost periodic functions to abstract functions with range in a Banach space.

It is not the purpose of this paper to construct approximations but to establish a class of almost periodic functions which can be approximated, with an arbitrarily prescribed accuracy, by continuous periodic functions uniformly on $\mathbb{R} = (-\infty; +\infty)$.

This paper may be looked upon also as an argument against the controversial opinion, namely, that the introduction of almost periodic functions is unnecessary and that it suffices to consider only continuous periodic functions by means of which any almost periodic function on any finite interval can be uniformly approximated with an arbitrary accuracy. Nowadays, however, there exist devices with everlasting schedule; indeed, such an everlasting schedule can be exemplified by the motion of celestial bodies. In view of this we must abandon the hypothesis about the uniform approximation on finite intervals in favour of studying possibilities of the uniform approximation on $\mathbb{R}$.

This paper presents an original approach even though some reasoning against the negative point of view on almost periodic functions may be found already in Bohr's and Bochner's works.

2. Notation and definitions

We will use the following notation: $\mathbb{N}$—the set of all positive integers, $\mathbb{Z}$—the set of all integers, $\mathbb{Q}$—the set of all rational numbers, $\mathbb{R}$—the set of all real numbers, $\mathbb{C}$—the set of all complex numbers. Further, $\mathbf{X}$ will stand for a Banach space (B-space) with the norm $|.|_{\mathbf{X}}$, i.e. a complete linear normed space (such as $\mathbb{R}$ or $\mathbb{C}$ with the norm given by the absolute value of the number).

Here we will deal only with (abstract) functions $\mathbb{R} \to \mathbf{X}$, i.e. functions defined on $\mathbb{R}$ with their ranges in a B-space $\mathbf{X}$.

Let us denote by $C(\mathbf{X})$ the set of all functions $\mathbb{R} \to \mathbf{X}$ that are continuous on $\mathbb{R}$. A functions $f \in C(\mathbf{X})$ is said to be bounded if its range $R_f = \{f(t)\colon\ t \in \mathbb{R}\}$ is a bounded set in $\mathbf{X}$. In the space $CB(\mathbf{X})$ of all bounded functions from $C(\mathbf{X})$ we introduce the norm $\|f\| = \sup\{|f(t)|_X\colon\ t \in \mathbb{R}\}$, $f \in CB(\mathbf{X})$, yielding the uniform convergence on $\mathbb{R}$. Under this norm, $CB(\mathbf{X})$ becomes a B-space. The points from $\mathbf{X}$ may be identified with constant functions from $CB(\mathbf{X})$ and we can use the norm $\|.\|$ for them which is equal in this case to the norm $|.|_X$.

3. Periodic functions

We say that a real number ω is a period of a function f if $f(t+\omega) = f(t)$ for all $t \in \mathbb{R}$.

A function f is said to be periodic if there exists its non-vanishing period. Let us denote by $CP(\mathbf{X})$ the class of all periodic functions from $CB(\mathbf{X})$.

If $f \in CP(\mathbf{X})$ then the infimum of the set of all positive periods of a function f will be denoted by ω_f. The number ω_f is called the primitive period of the function f and it is its period. If $\omega_f = 0$ then the function f is constant and *vice versa* if a function f is constant then $\omega_f = 0$.

4. Trigonometric polynomials and almost periodic functions

Besides the symbol e^{α} for the value of the exponential function we will use also the notation $\exp(\alpha)$.

If $a_1, \ldots, a_N$ are elements from $\mathbf{X}$ and $\lambda_1, \ldots, \lambda_N$ are (mutually different) real numbers then the function

$$Q(t) = \sum_{k=1}^{N} a_k \exp(\mathrm{i}\lambda_k t), \quad t \in \mathbb{R},$$

is called an $\mathbf{X}$-trigonometric polynomial or shortly a trigonometric polynomial. Obviously, $Q \in CB(\mathbf{X})$.

The class of all $\mathbf{X}$-trigonometric polynomials is linear and its closure, which we denote by $AP(\mathbf{X})$, is a subspace of the space $CB(\mathbf{X})$, that is, $AP(\mathbf{X})$ is a B-space with the norm $\|.\|$. The elements of the space $AP(\mathbf{X})$ are called $\mathbf{X}$-almost periodic functions or shortly almost periodic functions.

The inclusion $CP(\mathbf{X}) \subset AP(\mathbf{X})$ is valid, though, as we shall see later, $CP(\mathbf{X})$ is not dense in $AP(\mathbf{X})$, i.e. its closure does not contain $AP(\mathbf{X})$.

5. Properties of almost periodic functions

Almost periodic functions exhibit a number of properties similar to the periodic ones. Before stating them we adopt two definitions.

A real number τ is said to be an ε-almost period of a function $f\colon \mathbb{R} \to \mathbf{X}$, where ε is a positive number, if

$$|f(t+\tau) - f(t)|_X \leqslant \varepsilon \quad \text{for all } t \in \mathbb{R}.$$

The set of all ε-almost periods of a function f is denoted by $T(\varepsilon, f)$.

A set $\Lambda \subset \mathbb{R}$ is said to be relatively dense (in $\mathbb{R}$) if there exists a positive number l, the so-called inclusive length of the relative density, such that the intersection of Λ and any closed interval of length l is non-empty.

Now we are in position to state some of the basic properties of almost periodic functions. If $f \in AP(\mathbf{X})$ then

i) the function f is uniformly continuous on $\mathbb{R}$;

ii) the range R_f of the function f is a relatively compact set, i.e. any sequence of points from R_f contains a subsequence convergent in $\mathbf{X}$;

iii) for any $\varepsilon > 0$ the set $T(\varepsilon, f)$ of all ε-almost periods of the function f is relatively dense.

If $f \in AP(\mathbf{X})$ then the limit

$$M(f) = M_t\{f(t)\} = \lim_{T\to+\infty} \frac{1}{T} \int_s^{s+T} f(t)\,\mathrm{d}t$$

exists uniformly with respect to the parameter $s \in \mathbb{R}$, and it is called the mean value of the function f. The function

$$a(\lambda) = a(\lambda, f) = M_t\{f(t)\exp(-\mathrm{i}\lambda t)\}, \quad \lambda \in \mathbb{R},$$

is called the Bohr transform of the function f. If $a(\lambda) \neq 0$ then λ is called the Fourier exponent and $a(\lambda)$ the Fourier coefficient of the function f. The set of all Fourier exponents of the function f will be denoted by Λ_f. This set is at most countable (finite or can be arranged into a sequence).

From the harmonic analysis of periodic functions it is known that each Fourier exponent of any non-constant function $g \in CP(\mathbf{X})$ is equal to the product of the number $2\pi/\omega_g$ by an integer.

Remark. Here and hereafter the symbol 0 denotes both zero and the zero element in $\mathbf{X}$. The meaning of the symbol 0 is always clear from the context.

The trigonometric series

$$\sum_\lambda a(\lambda)\exp(\mathrm{i}\lambda t), \quad \lambda \in \Lambda_f,$$

is called the Fourier series of the function f. It is uniquely determined up to the order of summation.

For any $\lambda \in \mathbb{R}$ the inequality $|a(\lambda)|_{\mathbf{X}} = |a(\lambda, f)|_{\mathbf{X}} \leqslant \|f\|$ holds.

If $f \in AP(\mathbf{X})$ and $\varepsilon > 0$ then it is possible to construct the so-called Bochner-Fejer approximation (trigonometric) polynomial Q_ε such that $\Lambda_{Q_\varepsilon} \subset \Lambda_f$, $\|f - Q_\varepsilon\| \leqslant \varepsilon$ and for all $\lambda \in \Lambda_f$ we have $a(\lambda, Q_\varepsilon) = r(\lambda, \varepsilon)a(\lambda, f)$, where $0 \leqslant r(\lambda, \varepsilon) \leqslant 1$ and $\lim r(\lambda, \varepsilon) = 1$ for $\varepsilon \to 0^+$.

7. Kronecker's theorem

(Congruent) equalities $A = \delta \pmod{2\pi}$ and $|A| = \delta \pmod{2\pi}$ mean that there exists an integer m such that $A - 2\pi m = \delta$ and $|A - 2\pi m| = \delta$, respectively. (Congruent) inequalities are defined in the analogous manner.

In the sequel we will use the following theorem.

Theorem 1 (Kronecker). *If $\lambda_1, \ldots, \lambda_N$ and $\Theta_1, \ldots, \Theta_N$ are real numbers ($N \in \mathbb{N}$) then a necessary and sufficient condition for the system of inequalities (for the unknown $t \in \mathbb{R}$)*

$$|\lambda_j t - \Theta_j| \leqslant \delta \pmod{2\pi}, \quad j = 1, \ldots, N, \tag{1}$$

to have a solution for any positive number δ is that each equality $m_1\lambda_1 + \ldots + m_N\lambda_N = 0$, where $m_1, \ldots, m_N$ are integers, implies the equality $m_1\Theta_1 + \ldots + m_N\Theta_N = 0 \pmod{2\pi}$.

The proof of this theorem may be found in [1], [4], [5].

Real numbers $\lambda_1, \ldots, \lambda_N$ are said to be linearly dependent (over $\mathbb{Q}$) if there are rational number $r_1, \ldots, r_N$ not all vanishing such that $r_1\lambda_1 + \ldots + r_N\lambda_N = 0$. If $\lambda_1, \ldots, \lambda_N$ are not linearly dependent we call them linearly independent.

The conditions of the Kronecker theorem are fulfilled, for instance, if $\lambda_1, \ldots, \lambda_N$ are linearly independent numbers.

8. Diameter of the range

If $M \subset \mathbf{X}$ is a non-empty set then the diameter of M is defined by $d(M) = \sup\{|x - y|_{\mathbf{X}} : x, y \in \mathbf{M}\}$.

Theorem 2. *If $f \in AP(\mathbf{X})$ is a non-constant function and its mean value vanishes, i.e. $M(f) = O \in \mathbf{X}$, then the inequality*

$$d(R_f) > \|f\| \tag{2}$$

holds.

P r o o f. The definition of the norm $\|f\|$ yields the existence of a sequence $\{t_n\} \subset \mathbb{R}$ such that

$$|f(t_n)|_X > \|f\| - \frac{1}{n}, \quad n = 1, 2, \ldots \tag{3}$$

Owing to the relative compactness of the set R_f we can assume that the sequence $\{f(t_n)\} \subset R_f$ is convergent in $\mathbf{X}$ (otherwise we would pass to a convergent subsequence) and that $z_0 = \lim f(t_n)$. In view of inequalities (3), $|z_0|_{\mathbf{X}} = \|f\|$ must hold.

Let us introduce an auxiliary quantity $d_1 = d_1(f) = \sup\{|f(t) - z_0|_{\mathbf{X}} \colon t \in \mathbb{R}\}$. It is evident that $d_1 \leqslant d(R_f)$ so that it suffices to prove the inequality $\|f\| < d_1$. We will proceed by contradiction. Assume $d_1 \leqslant \|f\|$. On account of $\lim f(t_n) = z_0$, for any $\varepsilon > 0$ there exists a point $s \in \{t_n\}$ such that $|f(s) - z_0|_X < \varepsilon$. Further, for this ε there exists an inclusive length $l = l(\varepsilon)$ of the relative density of the set $T(\varepsilon, f)$ and there exists a positive number $\delta = \delta(\varepsilon)$ such that $|f(t) - f(t')|_{\mathbf{X}} \leqslant \varepsilon$ for $|t - t'| \leqslant \delta$ (the uniform continuity of the function f on $\mathbb{R}$).

For any closed interval $J_k = \langle kl, (k+1)L\rangle$, where $k \in \mathbb{Z}$ and $L = l + 2\delta$, there exists an ε -almost period τ_k such that $\Delta_k = \langle s + \tau_k - \delta, s + \tau_k + \delta\rangle \subset J_k$. Indeed, it is sufficient to take $\tau_k \in \langle kL + \delta - s, kL + \delta - s + l\rangle \cap T(\varepsilon, f) \neq \emptyset$. For any $k \in \mathbb{Z}$ we have $|s + \tau_k - t| \leqslant \delta$ for all $t \in \Delta_k$ so that for these t the inequality $|f(t) - z_0|_{\mathbf{X}} \leqslant |f(t) - f(s + \tau_k)|_{\mathbf{X}} + |f(s + \tau_k) - f(s)|_{\mathbf{X}} + |f(s) - z_0|_{\mathbf{X}} < 3\varepsilon$ holds, which implies

$$\begin{aligned}\frac{1}{L}\int_{J_k} |f(t) - z_0|_{\mathbf{X}}\,\mathrm{d}t &\leqslant \frac{1}{L}\left[\int_{J_k - \Delta_k} d_1\,\mathrm{d}t + \int_{\Delta_k} 3\varepsilon\,\mathrm{d}t\right]\\ &= \frac{1}{L}[(L - 2\delta)d_1 + 6\varepsilon\delta] = d_1 - \frac{2\delta}{L}(d_1 - 3\varepsilon) < d_1\end{aligned}$$

for $0 < \varepsilon < (d_1/3)$ (f is non-constant and thus $d_1 > 0$). From these relations we obtain the inequality

$$\begin{aligned}M(|f - z_0|_{\mathbf{X}}) &= \lim_{n\to\infty} \frac{1}{nL}\int_0^{nL} |f(t) - z_0|_{\mathbf{X}}\,\mathrm{d}t\\ &= \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \frac{1}{L}\int_{J_k} |f(t) - z_0|_{\mathbf{X}}\,\mathrm{d}t\\ &\leqslant \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n}\Big[d_1 - \frac{2\delta}{L}(d_1 - 3\varepsilon)\Big] = d_1 - \frac{2\delta}{L}(d_1 - 3\varepsilon) < d_1.\end{aligned}$$

However this is a contradiction because

$$\begin{aligned}0 = |M(f)|_{\mathbf{X}} &= |M(z_0) + M(f - z_0)|_{\mathbf{X}} = |z_0 + M(f - z_0)|_{\mathbf{X}}\\ &\geqslant |z_0|_{\mathbf{X}} - |M(f - z_0)|_{\mathbf{X}} > |z_0|_{\mathbf{X}} - d_1 = \|f\| - d_1 \geqslant 0.\end{aligned}$$

One must therefore have $\|f\| < d_1 \leqslant d$. $\square$

Corollary 3. *If $f \in AP(\mathbf{X})$ is a non-constant function then*

$$d(R_f) > \|f - M(f)\|. \tag{4}$$

P r o o f. In virtue of $|f(s) - f(t)|_{\mathbf{X}} = |(f(s) - M(f)) - (f(t) - M(t))|_{\mathbf{X}}$, which is valid for any real numbers s, t, we have $d(R_f) = d(R_{f-M(f)})$. Further, the function $f - M(f)$ has a vanishing mean value and is non-constant. Thus, it fulfils the conditions of Theorem 2 and consequently, using (4), we conclude $d(R_f) = d(R_{f-M(f)}) > \|f - M(f)\|$. □

9. Approximation of almost periodic functions

In this section we determine a class of **X**-almost periodic functions which can be uniformly approximated on $\mathbb{R}$ by functions from $CP(\mathbf{X})$ with an arbitrary accuracy.

Theorem 4. *Let f and g be functions from $CP(\mathbf{X})$. If their primitive periods ω_f and ω_g are linearly independent then the estimate*

$$\|f - g\| \geqslant \frac{1}{2}\|f - M(f)\| \tag{5}$$

holds.

P r o o f. First, we assume $M(f) = 0$ and verify the validity of (5) by contradiction.

Let $\|f - g\| < \frac{1}{2}\|f\|$, which means that $|f(t) - g(t)|_{\mathbf{X}} < \frac{1}{2}\|f\|$ for all $t \in \mathbb{R}$. If we choose $\varepsilon = \frac{1}{4}(d(R_f) - \|f\|) > 0$, then $d(R_f) > \|f\| + 3\varepsilon$, and there are real numbers t_1, t_2 such that $|f(t_1) - f(t_2)|_{\mathbf{X}} > \|f\| + 3\varepsilon$.

By assumption the numbers ω_f and ω_g are linearly independent and so the numbers $2\pi/\omega_f$, $2\pi/\omega_g$ are linearly independent as well. By virtue of the Kronecker theorem, for any $\delta > 0$ there exists a solution $\tau = \tau(\delta)$ of the system of inequalities

$$|(2\pi/\omega_f)t - (2\pi/\omega_f)(t_2 - t_1)| < (2\pi/\omega)\delta \pmod{2\pi},$$
$$|(2\pi/\omega_g)t| < (2\pi/\omega)\delta \pmod{2\pi},$$
$$(\lambda_1 = 2\pi/\omega_f, \quad \lambda_2 = 2\pi/\omega_g, \quad \Theta_1 = (t_2 - t_1)2\pi/\omega_f, \quad \Theta_2 = 0),$$

where $\omega = \max\{\omega_f, \omega_g\}$. This means that there exists a real number τ and integer numbers m_f, m_g such that the inequalities

$$|(2\pi/\omega_f)\tau - (2\pi/\omega_f)(t_2 - t_1) - 2\pi m_f| < (2\pi/\omega)\delta,$$
$$|(2\pi/\omega_g)\tau - 2\pi m_g| < (2\pi/\omega)\delta$$

are fulfilled.

Multiplying the first inequality by the number $\omega_f/2\pi$ and the second by the number $\omega_g/2\pi$ we get the system of inequalities

$$\begin{aligned} &|t_1+\tau-t_2-m_f\omega_f| < \delta\omega_f/\omega \leqslant \delta, \\ &|t_1+\tau-t_1-m_g\omega_g| < \delta\omega_g/\omega \leqslant \delta. \end{aligned} \tag{6}$$

The uniform continuity of the functions f and g on $\mathbb{R}$ yields the existence of a positive constant $\delta = \delta(\varepsilon)$ such that $|f(t)-f(t')|_{\mathbf{X}} \leqslant \varepsilon$ and $|g(t)-g(t')|_{\mathbf{X}} \leqslant \varepsilon$ for $|t-t'| \leqslant \delta$. From this and on account of (6) we obtain

$$\begin{aligned} |f(t_2)-f(t_1+\tau)|_{\mathbf{X}} &= |f(t_2+m_f\omega_f)-f(t_1+\tau)|_{\mathbf{X}} \leqslant \varepsilon, \\ |g(t_1)-g(t_1+\tau)|_{\mathbf{X}} &= |g(t_1+m_g\omega_g)-g(t_1+\tau)|_{\mathbf{X}} \leqslant \varepsilon. \end{aligned}$$

This leads to a contradiction since

$$\begin{aligned} \|f\|/2 &> \|f-g\| \geqslant |f(t_1+\tau)-g(t_1+\tau)|_{\mathbf{X}} \\ &\geqslant |f(t_2)-g(t_1)|_{\mathbf{X}} - |f(t_2)-f(t_1+\tau)|_{\mathbf{X}} - |g(t_1)-g(t_1+\tau)|_{\mathbf{X}} \\ &\geqslant |f(t_2)-f(t_1)|_{\mathbf{X}} - |f(t_1)-g(t_1)|_{\mathbf{X}} - 2\varepsilon \\ &> \|f\| + 3\varepsilon - \|f\|/2 - 2\varepsilon = \|f\|/2 + \varepsilon. \end{aligned}$$

One must therefore have $\|f-g\| \geqslant \frac{1}{2}\|f\|$.

Next, we turn to the case $M(f) \neq 0$. Then the mean value of the function $f-M(f)$ vanishes and its primitive period is ω_f, so that by the above $\|f-g\| = \|(f-M(f))-(g-M(f))\| \geqslant \frac{1}{2}\|f-M(f)\|$ (the primitive period of the function $g-M(f)$ is ω_g). The proof is complete. □

A set $\Lambda \subset \mathbb{R}$ is said to have a one-point basis if there exists a real number β such that $\Lambda \subset \beta\mathbb{Q} = \{\beta r \colon r \in \mathbb{Q}\}$. A function $f \in AP(\mathbf{X})$ is said to have a one-point basis if the set Λ_f has a one-point basis.

The validity of the following lemma is obvious.

Lemma 5. *If an* $\mathbf{X}$*-trigonometric polynomial has a one-point basis then it is a periodic function.*

Theorem 6. *A necessary and sufficient condition that an* $\mathbf{X}$*-almost periodic function might be uniformly approximated on* $\mathbb{R}$ *by functions from* $CP(\mathbf{X})$ *with an arbitrary accuracy (i.e. that the distance* $d_P(f)$ *of* f *from* $CP(\mathbf{X})$ *be zero,* $d_P(f)$ *being defined by* $\sup\{\|f-g\| \colon g \in CP(\mathbf{X})\}$*) is that the function* f *have a one-point basis.*

P r o o f. The statement of Theorem 6 is immediate for a constant function f. So, let us assume that f is a non-constant function and without restricting generality we may assume that $M(f) = 0$.

Necessity. Let $d_P(f) = 0$. We prove that Λ_f has a one-point basis. Choose $\varepsilon = \|f\|/6 > 0$. In view of $d_P(f) = 0$ there exists a function $g \in CP(\mathbf{X})$ such that $\|f - g\| < \varepsilon$. The function g satisfies the relations

$$\|M(g)\| = |M(g)|_{\mathbf{X}} = |M(g) - M(f)|_{\mathbf{X}} = |M(g - f)|_{\mathbf{X}} \leqslant \|g - f\| < \varepsilon$$

so that

$$\|g - M(g)\| \geqslant \|f\| - \|f - g\| - \|M(f - g)\| \geqslant \|f\| - 2\|f - g\| = 4\varepsilon > 0.$$

Hence, the function g is non-constant and $\omega_g > 0$. Putting $\beta = 2\pi/\omega_g$, one has $\Lambda_g \subset \beta\mathbb{Q}$ (see Sec. 6).

If η is an arbitrary positive number less than ε then there exists a function $h \in CP(\mathbf{X})$ such that $\|f - h\| < \eta < \varepsilon$. Such a function h exists thanks to $d_P(f) = 0$. Let us assume that the primitive periods ω_g, ω_h are linearly independent. By Theorem 4 the estimate (5) is then valid, so that $2\varepsilon > \|f - g\| + \|f - h\| \geqslant \|g - h\| \geqslant \|g - M(g)\|/2 > 2\varepsilon$. This is a contradiction. Hence ω_f, ω_g must be linearly dependent and $\Lambda_h \subset \beta\mathbb{Q}$. For $\lambda \notin \beta\mathbb{Q}$ we have $a(\lambda, h) = 0$ so that $|a(\lambda, f)|_{\mathbf{X}} = |a(\lambda, f - h)|_{\mathbf{X}} \leqslant \|f - h\| < \eta$. Due to the arbitrariness of $\eta \in (0, \varepsilon), a(\lambda, f) = 0$ must hold, hence $\Lambda_f \subset \beta\mathbb{Q}$, i.e. Λ_f has a one-point basis.

Sufficiency. Let a function f have a one-point basis, i.e. let there exist a real number β such that $\Lambda_f \subset \beta\mathbb{Q}$. This means that each Bochner-Fejer polynomial of the function f has a one-point basis and therefore it is periodic. Since the function f is approximated uniformly on $\mathbb{R}$ with an arbitrary accuracy by these polynomials, we have $d_P(f) = 0$. □

The class of all **X**-almost periodic functions with a one-point basis forms the closure of $CP(\mathbf{X})$. Provided $\mathbf{X} \neq \{0\}$ this closure does not contain $AP(\mathbf{X})$ since there exist **X**-almost periodic functions that have no one-point basis. For instance, the numbers 1 and π are linearly independent so that for any non-zero element $a \in \mathbf{X}$ the function $f(t) = [\exp(\mathrm{i}t) + \exp(\mathrm{i}\pi t)]a$, $t \in \mathbb{R}$, is **X**-almost periodic and has no one-point basis.

Almost periodic functions enjoy similar properties as continuous periodic functions but some properties can be altogether different. For example, any periodic function assumes each function value infinitely many times whereas the almost periodic function $f(t) = \cos t + \cos \pi t$, $t \in \mathbb{R}$, assumes the value 2 only at one point, namely at $t = 0$. Indeed, provided $f(t) = 2$ for $t \neq 0$ then $t = 2\pi k$, $\pi t = 2\pi l$, where k, l are integer non-vanishing numbers, so that $\pi = \pi t/t = 2\pi l/2\pi k = l/k$, which leads to a contradiction with the fact that π is an irrational number. The function indeed assumes the value 2 only for $t = 0$.

Moreover, there exist one-to-one almost periodic functions, that is M functions which assign different values to different arguments. These functions are invertible. In what follows we will construct such a one-to-one almost periodic function.

Let λ and μ be two linearly independent real numbers and let a be a positive number less than 1. We define a function $f \in AP(\mathbb{C})$ by

$$f(t) = f(t, \lambda, \mu) = \exp(\mathrm{i}\lambda t) + a \exp(\mathrm{i}\mu t), \quad t \in \mathbb{R}. \tag{7}$$

Denote $\omega = \mu - \lambda$. If there exist real numbers s, τ such that $s \neq \tau$ and $f(s) = f(\tau)$ then

$$\begin{aligned} |f(s)|^2 &= |1 + a \exp(\mathrm{i}\omega s)|^2 = 1 + a^2 + 2a \cos \omega s \\ &= |f(\tau)|^2 = |1 + a \exp(i\omega\tau)|^2 = 1 + a^2 + 2a \cos \omega\tau, \end{aligned}$$

that is,

$$\cos \omega s - \cos \omega\tau = -2 \sin \frac{\omega(s-\tau)}{2} \sin \frac{\omega(s+\tau)}{2} = 0.$$

This means that either $\omega(s - \tau)$ or $\omega(s + \tau)$ is an integer multiple of the number 2π.

First, assume that $\omega(s-\tau) = 2\pi k$, where k is a non-vanishing integer since $\mu - \lambda \neq 0$, $s - \tau \neq 0$. It follows that $\omega s = \omega\tau + 2\pi k$ and $\exp(\mathrm{i}\omega s) = \exp(\mathrm{i}\omega\tau)$ and, moreover, $f(s) = \exp(\mathrm{i}\lambda s)(1 + a \exp(\mathrm{i}\omega s)) = \exp(\mathrm{i}\lambda s)(1 + a \exp(\mathrm{i}\omega\tau)) = f(\tau) = \exp(\mathrm{i}\lambda\tau)(1 + a \exp(\mathrm{i}\omega\tau))$.

Since $1 + a \exp(\mathrm{i}\omega t) \neq 0$ for all $t \in \mathbb{R}$ $(0 < a < 1)$, $\exp(\mathrm{i}\lambda s) = \exp(\mathrm{i}\lambda\tau)$ must hold. Hence there exists an integer l such that $\lambda(s - \tau) = 2\pi l$. But then $\mu(s - \tau) = \lambda(s - \tau) + 2\pi k = 2\pi(k + l)$, $l\mu(s - \tau) - (k + l)\lambda(s - \tau) = 2\pi(l(k + l) - (k + l)l) = 0$. Since $s - \tau \neq 0$ we have $l\mu - (k + l)\lambda = 0$. Due to the fact that the numbers λ and μ are linearly independent we get $l = k + l = 0$, hence $k = 0$, which is a contradiction with $k \neq 0$. The number $\omega(s - \tau)$ cannot be an integer multiple of 2π and

$$\sin \frac{\omega(s-\tau)}{2} \neq 0 \tag{8}$$

must hold.

The remaining case is $\omega(s+\tau) = 2\pi k$, where k is an integer (possibly zero). Then we get $\omega s = -\omega\tau + 2\pi k$, $\exp(i\omega s) = \exp(-i\omega\tau)$ and $f(s) = \exp(i\lambda s)(1 + a\exp(-i\omega\tau)) = \exp(i\lambda\tau)(1 + a\exp(i\omega\tau)) = f(\tau)$. Taking the real and imaginary parts of the latter relation we obtain a system of equalities

$$\begin{aligned}(1 + a\cos\omega\tau)\cos\lambda s + a\sin\lambda s\sin\omega\tau \\ = (1 + a\cos\omega\tau)\cos\lambda\tau - a\sin\lambda\tau\sin\omega\tau \\ (1 + a\cos\omega\tau)\sin\lambda s - a\cos\lambda s\sin\omega\tau \\ = (1 + a\cos\omega\tau)\sin\lambda\tau + a\cos\lambda\tau\sin\omega\tau.\end{aligned} \tag{9}$$

If $\sin\omega\tau = 0$ then there exists a non-vanishing integer l such that $\omega\tau = \pi l$,which yields $\omega(s-\tau) = \omega(s+\tau) - 2\omega\tau = 2\pi(k-l)$, and this contradicts (8). Hence

$$\sin\omega\tau \neq 0 \tag{10}$$

must hold. Now, the system (9) can be arranged to the form

$$\begin{aligned}(1 + a\cos\omega\tau)\sin\frac{\lambda(s-\tau)}{2}\sin\frac{\lambda(s+\tau)}{2} \\ = a\cos\frac{\lambda(s-\tau)}{2}\sin\frac{\lambda(s+\tau)}{2}\sin\omega\tau, \\ (1 + a\cos\omega\tau)\sin\frac{\lambda(s-\tau)}{2}\cos\frac{\lambda(s+\tau)}{2} \\ = a\cos\frac{\lambda(s-\tau)}{2}\cos\frac{\lambda(s+\tau)}{2}\sin\omega\tau.\end{aligned} \tag{11}$$

If $\sin\frac{1}{2}\lambda(s-\tau) = 0$ then $|\cos\frac{1}{2}\lambda(s-\tau)| = 1$ and the first equality implies that $\sin\frac{1}{2}\lambda(s+\tau) = 0$, $|\cos\frac{1}{2}\lambda(s+\tau)| = 1$. But this leads to a contradiction in the second equality from (11): $0 = a\cos\frac{1}{2}\lambda(s-\tau)\cos\frac{1}{2}\lambda(s+\tau)\sin\omega\tau \neq 0$. Consequently,

$$\sin\frac{\lambda(s-\tau)}{2} \neq 0. \tag{12}$$

As $|\sin\frac{1}{2}\lambda(s+\tau)| + |\cos\frac{1}{2}\lambda(s+\tau)| > 0$, either of the equalities (11) implies the equality

$$\frac{1 + a\cos\omega\tau}{a\sin\omega\tau} = \frac{\cos\frac{1}{2}\lambda(s-\tau)}{\sin\frac{1}{2}\lambda(s-\tau)} \neq 0. \tag{13}$$

Now, let us return to the function $g(t) = f(t,\mu,\lambda)$, $t \in \mathbb{R}$, see (7). If there are real numbers s, τ such that $g(s) = g(\tau)$ and $s \neq \tau$ then by the above considerations $\omega(s+$

τ) is an integer multiple of 2π and the equality (obtained from (13) by interchanging λ and μ)

$$\frac{1+a\cos\omega\tau}{-a\sin\omega\tau}=\frac{\cos\frac{1}{2}\mu(s-\tau)}{\sin\frac{1}{2}\mu(s-\tau)}\neq 0 \tag{14}$$

holds.

We define a function $F\in(AP(\mathbb{C}^2)$ by the formula $F(t)=(f(t),g(t)),t\in\mathbb{R}$. If there are real numbers s,τ such that $F(s)=F(\tau)$ and $s\neq\tau$ then $\omega(s+\tau)$ is an integer multiple of 2π, the equalities (13) and (14) are valid and thus also the equality

$$\frac{\cos\frac{1}{2}\lambda(s-\tau)}{\sin\frac{1}{2}\lambda(s-\tau)}=-\frac{\cos\frac{1}{2}\mu(s-\tau)}{\sin\frac{1}{2}\mu(s-\tau)},$$

holds and enables us to conclude

$$\cos\frac{\lambda(s-\tau)}{2}\sin\frac{\mu(s-\tau)}{2}+\cos\frac{\mu(s-\tau)}{2}\sin\frac{\lambda(s-\tau)}{2}=\sin\frac{(\lambda+\mu)(s-\tau)}{2}=0.$$

This means that in addition to $(\mu-\lambda)(s+\tau),(\mu+\lambda)(s-\tau)$ must be an integer multiple of 2π as well.

The function $G\in AP(\mathbb{C}^2)$, defined by the formula $G(t)=(f(t,-\lambda,\mu),f(t,\mu,-\lambda))$, $t\in\mathbb{R}$, possesses the following property which is a consequence of the preceding considerations on substituting $-\lambda$ instead of λ: the equality $G(s)=G(\tau)$ while $s\neq\tau$ implies that both $(\mu+\lambda)(s+\tau),(\mu-\lambda)(s-\tau)$ are integer multiples of 2π.

Eventually, we define a function $\Phi\in CP(\mathbb{C}^4)$ by the formula $\Phi(t)=(F(t),G(t))$, $t\in\mathbb{R}$. The equality $\Phi(s)=\Phi(\tau)$ and $s\neq\tau$ then implies that

$$\begin{aligned}(\mu-\lambda)(s+\tau)&=2\pi k,\\(\mu+\lambda)(s+\tau)&=2\pi l,\\(\mu-\lambda)(s-\tau)&=2\pi m,\\(\mu+\lambda)(s-\tau)&=2\pi n,\end{aligned}\tag{15}$$

where k, l, m, n are integers. Since λ, μ are linearly independent and $s\neq\tau$ we get that m, n are non-vanishing numbers. The latter two equalities in (15) yield the equality $\frac{\mu-\lambda}{\mu+\lambda}=m/n$, i.e., $(m+n)\lambda+(m-n)\mu=0$. The fact that λ and μ are linearly independent implies $m+n=0$ and $m-n=0$ which gives $m=n=0$, and this is a contradiction. Thus, there do not exist real numbers s, τ such that $\Phi(s)=\Phi(\tau)$ and $s\neq\tau$. The function Φ assigns different values to different arguments, that is, Φ is a one-to-one function and the inverse function to Φ exists.

11. Conclusion

The crux of the paper lies in Sections 8 and 9 where the original assertions of Theorems 2, 4 and 6 and Corollary 3 state that the class of almost periodic functions which can be approximated uniformly on $\mathbb{R}$ by continuous periodic functions with an arbitrary accuracy forms a relatively narrow class of almost periodic functions with a one-point basis.

The idea of existence and construction of an invertible almost periodic function occurs in the theory of almost periodic function for the first time and suggests the wealth and diversity of the space of almost periodic functions.

For background material concerning the theory of almost periodic functions and the proof of the Kronecker theorem we refer the reader to the publications listed in the References.

References

[1] *Amerio, L. – Prouse, G.*: Almost Periodic Functions and Functional Equations. N.Y. Van Nostrand Reihold Company, 1971.

[2] *Bohr, H.*: Zur Theorie der fastperiodischen Funktionen, I, II, III Teil. 1925.

[3] *Coppel, W. A.*: Almost periodic properties of ordinary differential equations. Ann. Mat. Pura Appl. *76* (1963).

[4] *Levitan, B. M.*: Almost Periodic Functions. G.I.T.L. Moscow, 1953. (In Russian.)

[5] *Levitan, B. M. – Zikov, V. V.*: Almost Periodic Functions and Differential Equations. I. M. U. Moscow, 1978. (In Russian.)

Author's address: Strojní fakulta ČVUT Praha, Karlovo nám. 13, 121 35 Praha 1, Czech Republic.