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SOME CARDINAL CHARACTERISTICS OF ORDERED SETS

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Abstract. For ordered (= partially ordered) sets we introduce certain cardinal characteristics of them (some of those are known). We show that these characteristics—with one exception—coincide.

0. PRELIMINARIES

An *ordered set* is a pair $(G,<)$ where G is a set and $<$ is an irreflexive and transitive binary relation on G. We shall write briefly G instead of $(G,<)$. Such a set will be always assumed to be nonempty. The symbol $x \prec y$ means that y is a *cover* of x, i.e. $x < y$ and $x < z < y$ holds for no $z \in G$. If $x \leqslant y$ or $y \leqslant x$ then the elements x, y are *comparable*; otherwise they are *incomparable*, notation $x \parallel y$. A *chain* is an ordered set any two elements of which are comparable; an *antichain* is an ordered set any two distinct elements of which are incomparable. By the symbol $\mathbf{2}$ we denote the two-element chain, i.e. $\mathbf{2} = (\{0,1\};\ 0<1)$.

An *ideal* in an ordered set G is such a subset $A \subseteq G$ that the following holds: $y \in A$, $x \in G$, $x \leqslant y \Rightarrow x \in A$. The empty set $\emptyset$ will be also assumed to be an ideal in G. If $x \in G$, then $(x] = \{t \in G;\ t \leqslant x\}$ is an ideal in G, called the *principal ideal* generated by the element x. If G, H are ordered sets then the *cardinal power* G^H ([1]) is the set of all order preserving mappings $f\colon H \to G$ ordered by $f \leqslant g \iff f(x) \leqslant g(x)$ for all $x \in H$. Especially, if H is an antichain, then G^H is the set of all mappings $f\colon H \to G$ ordered by this rule. The symbol $\max G$ ($\min G$) denotes the *greatest (least)* element of G, if this element exists.

Let G be an ordered set. The *dimension* of G ([3]) can be defined in the following manner:

dim $G = \min\{\operatorname{card} T$; there exists a system $(L_t;\ t \in T)$ of chains and a system $(f_t;\ t \in T)$ where $f_t\colon G \to L_t$ is injective and order preserving for any $t \in T$ such that $x \leqslant y \iff f_t(x) \leqslant f_t(y)$ for all $t \in T\}$.

If all chains L_t have the same order type α we get the definition of the $\alpha-$*dimension* of G ([5], this cardinal need not exist). By a slight modification we get the definition of the α-*pseudodimension* of G ([7], this cardinal always exists). We describe here especially the definition of the **2**-pseudodimension of G.

Let G be an ordered set, let $T \neq \emptyset$ be a set and let $f_t\colon G \to \mathbf{2}$ be a mapping for any $t \in T$. The system $(f_t;\ t \in T)$ will be called a **2**-*realizer* of G iff for any $x, y \in G$ the following holds:

$$x \leqslant y \iff f_t(x) \leqslant f_t(y) \text{ for all } t \in T. \tag{1}$$

Evidently, the condition (1) can be reformulated in the following way:

(2) (i) $$x < y \Rightarrow f_t(x) \leqslant f_t(y) \text{ for all } t \in T \text{ and there exists } t_0 \in T \text{ with } f_{t_0}(x) = 0 < 1 = f_{t_0}(y),$$

(ii) $$x \parallel y \Rightarrow \text{ there exist } t_1, t_2 \in T \text{ such that } f_{t_1}(x) = 0, f_{t_1}(y) = 1, f_{t_2}(x) = 1, f_{t_2}(y) = 0.$$

Let G be an ordered set, let $T \neq \emptyset$ be a set, let $(A_t;\ t \in T)$ be a system of ideals in G. This system is called an *order base* in G ([10]) iff for any $x, y \in G$ the following holds:

(3) (i) $$x < y \Rightarrow \text{ there exists } t_0 \in T \text{ such that } x \in A_{t_0}, y \notin A_{t_0},$$

(ii) $$x \parallel y \Rightarrow \text{ there exist } t_1, t_2 \in T \text{ such that } x \in A_{t_1}, y \notin A_{t_1}, x \notin A_{t_2}, y \in A_{t_2}.$$

The condition (3) can be reformulated in the following way:

$$x \not\leqslant y \Rightarrow \text{ there exists } t_0 \in T \text{ such that } y \in A_{t_0}, x \notin A_{t_0}. \tag{4}$$

Theorem 1.1. *Let G be an ordered set, let $T \neq \emptyset$ be a set. Then the following statements are equivalent:*

(i) *For any* $t \in T$ *there exists a mapping* $f_t\colon G \to \mathbf{2}$ *such that* $(f_t;\ t \in T)$ *is a* $\mathbf{2}$*-realizer of* G.

(ii) *For any* $t \in T$ *there exists an ideal* $A_t \subseteq G$ *such that* $(A_t;\ t \in T)$ *is an order base in* G.

Proof. (i) $\Rightarrow$ (ii): Let (i) hold and put $A_t = f_t^{-1}(0)$ for any $t \in T$. If $y \in A_t$, $x \in G$, $x \leqslant y$ then $f_t(y) = 0$, thus $f_t(x) = 0$ and $x \in A_t$. Hence A_t is an ideal in G. If $x, y \in G$, $x < y$ then by (2) there exists $t_0 \in T$ with $f_{t_0}(x) = 0$, $f_{t_0}(y) = 1$; thus $x \in A_{t_0}$, $y \notin A_{t_0}$. If $x \parallel y$ then there exist $t_1, t_2 \in T$ such that $f_{t_1}(x) = 0$, $f_{t_1}(y) = 1$, $f_{t_2}(x) = 1$, $f_{t_2}(y) = 0$. Then $x \in A_{t_1}$, $y \notin A_{t_1}$, $x \notin A_{t_2}$, $y \in A_{t_2}$. By (3) $(A_t;\ t \in T)$ is an order base in G and (ii) holds.

(ii) $\Rightarrow$ (i): Let (ii) hold and let $(A_t;\ t \in T)$ be an order base in G. Let us define a mapping $f_t\colon G \to \mathbf{2}$ for any $t \in T$ by $f_t(x) = 0$ if $x \in A_t$, $f_t(x) = 1$ if $x \notin A_t$. We show that $(f_t;\ t \in T)$ is a $\mathbf{2}$-realizer of G. Let $x, y \in G$, $x < y$. If $f_t(y) = 0$ then $y \in A_t$ and as A_t is an ideal, $x \in A_t$ so that $f_t(x) = 0$. Thus $f_t(x) \leqslant f_t(y)$ for all $t \in T$. Further, by (3) there exists $t_0 \in T$ such that $x \in A_{t_0}$, $y \notin A_{t_0}$. Then $f_{t_0}(x) = 0$, $f_{t_0}(y) = 1$. Let $x, y \in G$, $x \parallel y$. Then there exist $t_1, t_2 \in T$ such that $x \in A_{t_1}$, $y \notin A_{t_1}$, $x \notin A_{t_2}$, $y \in A_{t_2}$. Hence $f_{t_1}(x) = 0$, $f_{t_1}(y) = 1$, $f_{t_2}(x) = 1$, $f_{t_2}(y) = 0$. By (2), $(f_t;\ t \in T)$ is a $\mathbf{2}$-realizer of G and (i) holds. □

Corollary. *Let* G *be an ordered set. Then there exists a* $\mathbf{2}$*-realizer of* G.

Proof. The system of all principal ideals is trivially an order base in G. □

Definition. Let G be an ordered set. We put

$$\mathbf{2}\text{-pdim}\, G = \min\{\operatorname{card} T;\ (f_t;\ t \in T) \text{ is a } \mathbf{2}\text{-realizer of } G\};$$

this cardinal is called the **2**-*pseudodimension* of G.

Theorem 1.2. *Let* G *be an ordered set, let* $T \neq \emptyset$ *be a set. Then the following statements are equivalent:*

(i) *For any* $t \in T$ *there exists a mapping* $f_t\colon G \to \mathbf{2}$ *such that* $(f_t;\ t \in T)$ *is a* $\mathbf{2}$*-realizer of* G.

(ii) *There exists an isomorphic embedding of* G *into* $\mathbf{2}^T$.

Proof. (i) $\Rightarrow$ (ii): Let (i) hold. Define for any $x \in G$ a mapping $\varphi(x)\colon T \to \mathbf{2}$ by the rule $\varphi(x)(t) = f_t(x)$. We show that φ is an isomorphic embedding of G into $\mathbf{2}^T$. Indeed, for $x, y \in G$ we have $x \leqslant y \iff f_t(x) \leqslant f_t(y)$ for all $t \in T \iff \varphi(x)(t) \leqslant \varphi(y)(t)$ for all $t \in T \iff \varphi(x) \leqslant \varphi(y)$ in $\mathbf{2}^T$. Therefore (ii) holds.

(ii) $\Rightarrow$ (i): Let (ii) hold and let φ be an isomorphism of G into $\mathbf{2}^T$. Let us define for any $t \in T$ a mapping $f_t\colon G \to \mathbf{2}$ by $f_t(x) = \varphi(x)(t)$. For $x, y \in G$ we have:

$x \leqslant y \Longleftrightarrow \varphi(x) \leqslant \varphi(y) \Longleftrightarrow \varphi(x)(t) \leqslant \varphi(y)(t)$ for all $t \in T \Longleftrightarrow f_t(x) \leqslant f_t(y)$ for all $t \in T$. Thus $(f_t;\ t \in T)$ is a $\mathbf{2}$-realizer of G and (i) holds. $\square$

Corollary. *Let G be an ordered set. Then the following cardinals are equal:*
(i) $\mathbf{2}$-*pdim* G,
(ii) *the least cardinal m such that G can be isomorphically embedded into a set of type $\mathbf{2}^m$,*
(iii) *the least cardinal n such that in G there exists an order base of cardinality n.*

2. Rings of sets

Let $G \neq \emptyset$ be a set, $A \subseteq G$, $x, y \in G$, $x \neq y$. We say that the set A *separates elements* x, y iff either $x \in A$, $y \notin A$ or $x \notin A$, $y \in A$.

Let $\mathscr{A}$ be a system of subsets of G , $x, y \in G$, $x \neq y$. We say that the system $\mathscr{A}$ *separates elements* x, y iff there exists a set $A \in \mathscr{A}$ which separates x, y.

Let $\mathscr{A}$, $\mathscr{B}$ be systems of subsets of G. We say that $\mathscr{A}$, $\mathscr{B}$ *similarly separate elements* of G iff for any two elements $x, y \in G$ the following holds:

(5) $$\mathscr{A} \text{ separates } x, y \Longleftrightarrow \mathscr{B} \text{ separates } x, y.$$

Example 2.1. Let $G = \{a, b, c\}$, $\mathscr{A} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b, c\}\}$, $\mathscr{B} = \{\{a, b\}, \{a, c\}\}$. Then $\mathscr{A}$, $\mathscr{B}$ similarly separate elements of G.

Indeed, as $\mathscr{A}$ contains all one-element subsets of G, it separates any two elements of G. Thus it suffices to show that $\mathscr{B}$ separates any two elements of G. The set $\{a, c\} \in \mathscr{B}$ separates elements a, b and the set $\{a, b\} \in \mathscr{B}$ separates both a, c and b, c.

Let $\mathscr{A}$ be a nonempty system of sets. $\mathscr{A}$ is called a *ring of sets* ([2], p. 12) iff $A \cup B \in \mathscr{A}$, $A \cap B \in \mathscr{A}$ for any $A, B \in \mathscr{A}$. If $\bigcup\{X;\ X \in \mathscr{A}\} = G$ then we will say that $\mathscr{A}$ is a ring of sets on G.

Let $\mathscr{B}$ be a nonempty system of sets and $\bigcup\{X;\ X \in \mathscr{B}\} = G$. As the system of all rings of sets on G is a closure system on G, there exists the least ring of sets $\mathscr{A}$ on G such that $\mathscr{B} \subseteq \mathscr{A}$. We say that $\mathscr{B}$ *generates* the ring $\mathscr{A}$.

Theorem 2.1. *Let $\mathscr{B}$ be a nonempty system of sets, $\bigcup\{X;\ X \in \mathscr{B}\} = G$, let $\mathscr{A}$ be a ring of sets on G and let $\mathscr{B} \subseteq \mathscr{A}$. If $\mathscr{B}$ generates $\mathscr{A}$ then $\mathscr{A}$, $\mathscr{B}$ similarly separate elements of G.*

P r o o f. Suppose that $\mathscr{B}$ generates $\mathscr{A}$ and the assertion does not hold. As $\mathscr{B} \subseteq \mathscr{A}$, there must exist $x, y \in G$ such that $\mathscr{A}$ separates them, $\mathscr{B}$ does not. Thus there exists $A \in \mathscr{A}$ which separates x, y and no $B \in \mathscr{B}$ separates x, y. Put

$\mathscr{C} = \{X \in \mathscr{A};\ X$ does not separate $x, y\}$. Then $\mathscr{B} \subseteq \mathscr{C} \subseteq \mathscr{A}$, $\mathscr{C} \neq \mathscr{A}$ as $A \notin \mathscr{C}$. We show that $\mathscr{C}$ is a ring of sets. Let $X, Y \in \mathscr{C}$. Then $X, Y \in \mathscr{A}$ and we have either $x, y \in X$ or $x, y \notin X$ and also either $x, y \in Y$ or $x, y \notin Y$. If $x, y \in X$ then $x, y \in X \cup Y$; the same holds if $x, y \in Y$. If both $x, y \notin X$ and $x, y \notin Y$ then $x, y \notin X \cup Y$. Thus $X \cup Y \in \mathscr{A}$ and it does not separate x, y, i.e. $X \cup Y \in \mathscr{C}$. If $x, y \notin X$ then $x, y \notin X \cap Y$ and the same if $x, y \notin Y$. If both $x, y \in X$ and $x, y \in Y$ then $x, y \in X \cap Y$. Thus $X \cap Y \in \mathscr{A}$ and $X \cap Y$ does not separate x, y, i.e. $X \cap Y \in \mathscr{C}$. Hence $\mathscr{C}$ is a ring on G, $\mathscr{C} \supseteq \mathscr{B}$, $\mathscr{C} \subseteq \mathscr{A}$, $\mathscr{C} \neq \mathscr{A}$, a contradiction with the assumption that $\mathscr{B}$ generates $\mathscr{A}$. □

Let $G \neq \emptyset$ be a set, let $\mathscr{A}$, $\mathscr{B}$ be systems of subsets of G. We will say that $\mathscr{B}$ *separates elements of G better than $\mathscr{A}$* iff for any two elements $x, y \in G$ the following holds:

$$\text{there exists } A \in \mathscr{A} \text{ such that } x \in A, y \notin A \Longrightarrow \text{ there exists } B \in \mathscr{B} \text{ such that } x \in B,\ y \notin B. \tag{6}$$

We will say that $\mathscr{A}$, $\mathscr{B}$ *equally separate elements* of G iff $\mathscr{A}$ separates elements of G better than $\mathscr{B}$ and $\mathscr{B}$ separates elements of G better than $\mathscr{A}$.

Example 2.2. Let $G = \{a, b, c\}$ and $\mathscr{A} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b, c\}\}$, $\mathscr{B} = \{\{a, b\}, \{a, c\}, \{b, c\}\}$. Then $\mathscr{A}$, $\mathscr{B}$ equally separate elements of G.

Indeed, as $\mathscr{A}$ contains all one-element subsets of G, it suffices to show: for any $x, y \in G$, $x \neq y$ there exists $B \in \mathscr{B}$ such that $x \in B$, $y \notin B$. This is really so: $a \in \{a, c\}$, $b \notin \{a, c\}$, $b \in \{b, c\}$, $a \notin \{b, c\}$ a.s.o.

The relation of better separating is transitive in the following sense: If $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ are systems of subsets of a set G such that $\mathscr{B}$ separates elements of G better than $\mathscr{A}$ and $\mathscr{C}$ separates elements of G better than $\mathscr{B}$ then $\mathscr{C}$ separates elements of G better than $\mathscr{A}$. It is also reflexive. The relation of equal separating is reflexive, symmetric and transitive. Further, we have: If $\mathscr{A}$, $\mathscr{B}$ equally separate elements of G, $\mathscr{C}$, $\mathscr{D}$ equally separate elements of G and $\mathscr{A}$ separates elements of G better than $\mathscr{C}$ then $\mathscr{B}$ separates elements of G better than $\mathscr{D}$.

Let $\mathscr{A}$ be a system of subsets of a set G. We will say that $\mathscr{A}$ is a *complete ring of sets* on G iff for any set I and any $A_i \in \mathscr{A}$ $(i \in I)$ we have $\bigcup_{i \in I} A_i \in \mathscr{A}$, $\bigcap_{i \in I} A_i \in \mathscr{A}$.

Note that if $\mathscr{A}$ is a complete ring of sets on G then $\emptyset \in \mathscr{A}$, $G \in \mathscr{A}$.

Let $\mathscr{B}$ be a system of subsets of a set G. Then there exists the least complete ring of sets $\mathscr{A}$ on G such that $\mathscr{B} \subseteq \mathscr{A}$; we will say that $\mathscr{B}$ *generates* the complete ring $\mathscr{A}$.

Definition. Let $\mathscr{A}$ be a complete ring of sets on a set G. We put

$$w(\mathscr{A}) = \min\{\operatorname{card} \mathscr{B};\ \mathscr{B} \subseteq \mathscr{A},\ \mathscr{B} \text{ generates } \mathscr{A}\};$$

this cardinal will be called the *weight* of the complete ring $\mathscr{A}$.

Theorem 2.2. *Let $\mathscr{A}$ be a complete ring of sets on a set G, let $\mathscr{B} \subseteq \mathscr{A}$ be a system of subsets of G. If $\mathscr{B}$ generates $\mathscr{A}$ then $\mathscr{A}$, $\mathscr{B}$ equally separate elements of G.*

Proof is similar to the proof of Theorem 2.1. Thus let $\mathscr{B}$ generate $\mathscr{A}$ and suppose that there exist $x, y \in G$, $A \in \mathscr{A}$, $x \in A$, $y \notin A$ such that there exists no $B \in \mathscr{B}$ with $x \in B$, $y \notin B$. Denote $\mathscr{C} = \{X \in \mathscr{A};$ neither $x \in X$ nor $y \notin X$ holds$\} = \{X \in \mathscr{A};$ either $x \notin X$ or $y \in X$ holds $\}$. Then $\mathscr{B} \subseteq \mathscr{C} \subseteq \mathscr{A}$, $C \neq \mathscr{A}$ as $A \notin \mathscr{C}$ and we show that $\mathscr{C}$ is a complete ring on G. Clearly $\emptyset \in \mathscr{C}$. Let $I \neq \emptyset$ be a set and $X_i \in \mathscr{C}$ for $i \in I$. For any $i \in I$ we have $x \notin X_i$ or $y \in X_i$. If $y \in X_i$ for some $i \in I$, we have $y \in \bigcup\limits_{i \in I} X_i$; in the other case $x \notin X_i$ for all $i \in I$ and then $x \notin \bigcup\limits_{i \in I} X_i$. Thus $\bigcup\limits_{i \in I} X_i \in \mathscr{A}$ and $x \notin \bigcup\limits_{i \in I} X_i$ or $y \in \bigcup\limits_{i \in I} X_i$, i.e. $\bigcup\limits_{i \in I} X_i \in \mathscr{C}$. If $x \notin X_i$ for some $i \in I$, then $x \notin \bigcap\limits_{i \in I} X_i$; otherwise $y \in X_i$ for all $i \in I$ and then $y \in \bigcap\limits_{i \in I} X_i$. Thus $\bigcap\limits_{i \in I} X_i \in \mathscr{A}$, $x \notin \bigcap\limits_{i \in I} X_i$ or $y \in \bigcap\limits_{i \in I} X_i$, i.e. $\bigcap\limits_{i \in I} X_i \in \mathscr{C}$. Further, clearly $G \in \mathscr{C}$. Thus $\mathscr{C}$ is a complete ring on G, $\mathscr{B} \subseteq \mathscr{C} \subseteq \mathscr{A}$, $\mathscr{C} \neq \mathscr{A}$, a contradiction. □

Theorem 2.3. *Let $\mathscr{A}$, $\mathscr{B}$ be complete rings of sets on a set G. Then $\mathscr{A} \subseteq \mathscr{B}$ if and only if $\mathscr{B}$ separates elements of G better than $\mathscr{A}$.*

Proof. If $\mathscr{A} \subseteq \mathscr{B}$ then trivially $\mathscr{B}$ separates elements of G better than $\mathscr{A}$. Suppose that $\mathscr{B}$ separates elements of G better than $\mathscr{A}$. For any $x \in G$ there exists the least element $B(x) \in \mathscr{B}$ which contains x, namely $B(x) = \bigcap\{B \in \mathscr{B};\ x \in B\}$. Let $A \in \mathscr{A}$ be any element, $A \neq \emptyset$. We show $A = \bigcup\{B(x);\ x \in A\}$. Trivially, $A \subseteq \bigcup\{B(x);\ x \in A\}$. Suppose the existence of an element $y \in \bigcup\{B(x);\ x \in A\} - A$. Then $y \notin A$ and there exists an element $z \in A$ such that $y \in B(z) = \bigcap\{B \in \mathscr{B};\ z \in B\}$. As $z \in A$, $y \notin A$, there exists $B \in \mathscr{B}$ such that $z \in B$, $y \notin B$. Then $y \notin \bigcap\{B \in \mathscr{B};\ z \in B\} = B(z)$, which is a contradiction. Thus $A = \bigcup\{B(x);\ x \in A\}$, which implies $A \in \mathscr{B}$. Hence $\mathscr{A} \subseteq \mathscr{B}$. □

Corollary. *Let $\mathscr{A}$, $\mathscr{B}$ be complete rings of sets on a set G. Then $\mathscr{A} = \mathscr{B}$ iff $\mathscr{A}$, $\mathscr{B}$ equally separate elements of G.*

Theorem 2.4. *Let $\mathscr{B}_1$, $\mathscr{B}_2$ be systems of subsets of a set G, let $\mathscr{A}_1$, $\mathscr{A}_2$ be complete rings of sets on G and let $\mathscr{B}_1$ generate $\mathscr{A}_1$, $\mathscr{B}_2$ generate $\mathscr{A}_2$. Then $\mathscr{A}_1 \subseteq \mathscr{A}_2$ iff $\mathscr{B}_2$ separates elements of G better than $\mathscr{B}_1$.*

Proof. If $\mathscr{A}_1 \subseteq \mathscr{A}_2$ then $\mathscr{A}_2$ separates elements of G better than $\mathscr{A}_1$. By Theorem 2.2, $\mathscr{A}_1$, $\mathscr{B}_1$ equally separate elements of G, and $\mathscr{A}_2$, $\mathscr{B}_2$ equally separate elements of G. Thus $\mathscr{B}_2$ separates elements of G better than $\mathscr{B}_1$. If $\mathscr{B}_2$ separates elements of G better than $\mathscr{B}_1$ then $\mathscr{A}_2$ separates elements of G better than $\mathscr{A}_1$. By Theorem 2.3 we have $\mathscr{A}_1 \subseteq \mathscr{A}_2$. □

As a corollary, we obtain

Theorem 2.5. *Let $\mathscr{B}_1$, $\mathscr{B}_2$ be systems of subsets of a set G. Then $\mathscr{B}_1$, $\mathscr{B}_2$ generate the same complete ring of sets on G iff $\mathscr{B}_1$, $\mathscr{B}_2$ equally separate elements of G.*

Further, we have

Theorem 2.6. *Let $G \neq \emptyset$ be a set, $\mathscr{A}$ a complete ring of sets on G and $\mathscr{B} \subseteq \mathscr{A}$ a system of subsets of G. Then $\mathscr{B}$ generates $\mathscr{A}$ iff $\mathscr{A}$, $\mathscr{B}$ equally separate elements of G.*

P r o o f. The necessity of the given condition follows from Theorem 2.2, its sufficiency follows from Theorem 2.5, as trivially $\mathscr{A}$ generates $\mathscr{A}$. □

Let G be an ordered set. Then the system of all its ideals is a complete ring of sets on G. Now, we prove

Theorem 2.7. *Let G be an ordered set, let $\mathscr{A}$ be the complete ring of all its ideals and let $\mathscr{B}$ be some system of its ideals. Then $\mathscr{B}$ generates $\mathscr{A}$ iff $\mathscr{B}$ is an order base in G.*

P r o o f. 1. Let $\mathscr{B}$ generate $\mathscr{A}$. By Theorem 2.6, $\mathscr{A}$, $\mathscr{B}$ equally separate elements of G. Let $x, y \in G$, $x \nleq y$. Then $(y] \in \mathscr{A}$, $y \in (y]$, $x \notin (y]$. Thus there exists $B \in \mathscr{B}$ such that $y \in B$, $x \notin B$. By (4), $\mathscr{B}$ is an order base in G.

2. Let $\mathscr{B}$ be an order base in G. Let $x, y \in G$, $A \in \mathscr{A}$ be such elements that $x \in A$, $y \notin A$. As A is an ideal in G, necessarily $y \nleq x$. By (4) there exists $B \in \mathscr{B}$ such that $x \in B$, $y \notin B$. Thus $\mathscr{A}$, $\mathscr{B}$ equally separate elements of G and by Theorem 2.6 $\mathscr{B}$ generates $\mathscr{A}$. □

A similar result is proved in [11], Hilfsatz 3.2.

Corollary. *Let G be an ordered set. Then the following cardinals are equal:*

(i) **2**-*pdim* G,

(ii) *the least cardinal m such that G can be isomorphically embedded into a set of type $\mathbf{2}^m$,*

(iii) *the least cardinal n such that in G there exists an order base of cardinality n,*

(iv) $w(\mathscr{A})$ *where $\mathscr{A}$ is the complete ring of all ideals in G.*

Let G be an ordered set and $H \subseteq G$. We will say that H is *dense* in G iff the following holds:

$$\text{(7)}\qquad \begin{aligned} &\text{(i)}\quad x, y \in G, x < y \Longrightarrow \text{ there exist } u, v \in H \text{ such that } x \leqslant u < v \leqslant y,\\ &\text{(ii)}\quad x, y \in G, x \parallel y \text{ and } z > y \text{ for any } z \in G, z > x \Longrightarrow x \in H.\end{aligned}$$

The condition (i) was formulated already in [4], p. 89, for linearly ordered sets, the condition (ii) can be found—in a modified form—in [9].

Clearly, any ordered set is dense in itself.

Definition. Let G be an ordered set. We put

$$\operatorname{sep} G = \min\{\operatorname{card} H\,;\ H \subseteq G \text{ is dense in } G\};$$

this cardinal will be called the *separability* of G.

Lemma 3.1. *Let G be an ordered set, let $H \subseteq G$ be dense in G and let $x, y \in G$. If $u \geqslant y$ for any $u \in H$, $u \geqslant x$, then $x \geqslant y$.*

Proof. Let the condition be satisfied. If $x \in H$, then $x \geqslant y$ for $x \geqslant x$. Thus let $x \notin H$. Assume $x \parallel y$. If $z \in G$, $z > x$ then by (7) there exist $u, v \in H$ such that $x \leqslant u < v \leqslant z$; by assumption then $u \geqslant y$ and thus $z > y$. By (ii) in (7) we have $x \in H$, a contradiction. Thus the elements x, y must be comparable. If $x < y$, then there exist $u, v \in H$ such that $x \leqslant u < v \leqslant y$ so that $u \geqslant x$, $u \ngeqslant y$, a contradiction. Hence $x \geqslant y$. □

Theorem 3.1. *Let G be an ordered set, let $H \subseteq G$ be dense in G. Then $((u]\,;\ u \in H)$ is an order base in G.*

Proof. Let $x, y \in G$, $x < y$. By (7) there exist $u, v \in H$ such that $x \leqslant u < v \leqslant y$. Then $x \in (u]$, $y \notin (u]$ and condition (i) from (3) is satisfied. Let $x, y \in G$, $x \parallel y$. If $x, y \in H$ then $x \in (x]$, $y \notin (x]$, $x \notin (y]$, $y \in (y]$. Suppose $x \notin H$, $y \in H$. Then $x \notin (y]$, $y \in (y]$. If $u \geqslant y$ for any $u \in H$ with $u \geqslant x$, then by Lemma 3.1 $x \geqslant y$, a contradiction. Thus there exists $u \in H$ such that $u \geqslant x$, $u \ngeqslant y$ and then $x \in (u]$, $y \notin (u]$. Similarly in the case $x \in H$, $y \notin H$. Finally, let $x \notin H$, $y \notin H$. If $u \geqslant y$ for any $u \in H$ with $u \geqslant x$, then $x \geqslant y$ by Lemma 3.1, a contradiction. Hence there exists $u \in H$ such that $u \geqslant x$, $u \ngeqslant y$; then $x \in (u]$, $y \notin (u]$. For the same reason there exists $v \in H$ such that $v \geqslant y$, $v \ngeqslant x$ and then $x \notin (v]$, $y \in (v]$. Thus the condition (ii) from (3) is satisfied and $((u]\,;\ u \in H)$ is an order base in G. □

Corollary. *Let G be an ordered set. Then* $\mathbf{2}$*-pdim* $G \leqslant$ *sep* G.

By examples we can show that $\mathbf{2}$-pdim $G =$ sep G need not hold. If, e.g., G is a finite chain with m elements, then sep $G = m$; as the cardinal power $\mathbf{2}^{m-1}$ contains an m-element chain and $\mathbf{2}^{m-2}$ contains no such chain, we see that $\mathbf{2}$-pdim $G = m-1$. If G is a finite m-element antichain, then sep $G = m$ and $\mathbf{2}$-pdim $G = n$ where n is the least positive integer with $\binom{n}{[\frac{n}{2}]} \geqslant m$, for a maximal antichain in $\mathbf{2}^n$ contains $\binom{n}{[\frac{n}{2}]}$ elements ([13], [6]). If G is an infinite antichain, $\operatorname{card} G = m$ then sep $G = m$ and $\mathbf{2}$-pdim $G = n$ where n is the least cardinal with $2^n \geqslant m$, for the cardinal power $\mathbf{2}^n$ (n infinite) contains an antichain of cardinality 2^n ([12], p. 450, Theorem 1, [8], Theorem 8). We show that if G is an infinite chain then $\mathbf{2}$-pdim $G =$ sep G. In contrast to all preceding theorems and lemmas, for a proof of this assertion we need the axiom of choice (AC).

Theorem 3.2. (AC). *Let G be an infinite chain. Then* $\mathbf{2}$-pdim $G =$ sep G.

P r o o f. It suffices to show sep $G \leqslant \mathbf{2}$-pdim G. Let $\mathbf{2}$-pdim $G = m$; clearly $m \geqslant \aleph_0$. By Corollary to Theorem 1.2, in G there exists an order base $(A_t;\ t \in T)$ with $\operatorname{card} T = m$. Denote $T_0 = \{t \in T;\ A_t$ contains the greatest element, $G - A_t$ contains the least element$\}$, $T_{12} = \{(t_1,t_2) \in T^2;\ A_{t_2} - A_{t_1} \neq \emptyset\}$, $H_0 = \bigcup\limits_{t \in T_0} \{\max A_t, \min(G - A_t)\}$ and let $H_{12} \subseteq \bigcup\limits_{(t_1,t_2) \in T_{12}} (A_{t_2} - A_{t_1})$ be such a set that $H_{12} \cap (A_{t_2} - A_{t_1})$ is a one-point set for any $(t_1,t_2) \in T_{12}$. Put $H = H_0 \cup H_{12}$; then $\operatorname{card} H \leqslant m$ and we show that H is dense in G. First, we show: If $x, y \in G$, $x \prec y$, then $x, y \in H$. Indeed, there exists $t \in T$ such that $x \in A_t$, $y \notin A_t$. Then necessarily $x = \max A_t$, $y = \min(G - A_t)$ so that $x, y \in H$. Now let $x, y \in G$, $x < y$. If $x \prec y$ or if there exists $z \in G$ such that $x < z \prec y$, then $x, y \in H$ and condition (i) from (7) is satisfied. Let there exist $w, z \in G$ such that $x < w < z < y$. By (i) in (3) there exist $t_1, t_2, t_3 \in T$ such that $x \in A_{t_1}$, $w \notin A_{t_1}$, $w \in A_{t_2}$, $z \notin A_{t_2}$, $z \in A_{t_3}$, $y \notin A_{t_3}$. Thus $A_{t_2} - A_{t_1} \neq \emptyset$, $A_{t_3} - A_{t_2} \neq \emptyset$ and there exist $u, v \in H$ such that $u \in A_{t_2} - A_{t_1}$, $v \in A_{t_3} - A_{t_2}$. Then $x < u < v < y$ and condition (i) in (7) is satisfied. Hence H is dense in G. □

Corollary. *Let G be an infinite chain. Then the following cardinals are equal:*

(i) $\mathbf{2}$*-pdim* G,

(ii) *the least cardinal m such that G can be isomorphically embedded into a set of type* $\mathbf{2}^m$,

(iii) *the least cardinal n such that in G there exists an order base of cardinality n,*

(iv) $w(\mathscr{A})$ *where $\mathscr{A}$ is the complete ring of all ideals in G,*

(v) *sep* G.

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