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Józef Dudek

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SMALL IDEMPOTENT CLONES I

Józef Dudek, Wrocław

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Abstract. G. Grätzer and A. Kisielewicz devoted one section of their survey paper concerning p_n-sequences and free spectra of algebras to the topic "Small idempotent clones" (see Section 6 of [18]). Many authors, e.g., [8], [14, 15], [22], [25] and [29, 30] were interested in p_n-sequences of idempotent algebras with small rates of growth. In this paper we continue this topic and characterize all idempotent groupoids $(G,\cdot)$ with $p_2(G,\cdot) \leqslant 2$ (see Section 7). Such groupoids appear in many papers see, e.g. [1], [4], [21], [26, 27], [25], [28, 30, 31, 32] and [34].

1. Introduction

Recall that by $p_n = p_n(\mathcal{A})$ we denote the number of all essentially n-ary polynomials (term functions) of a given algebra $\mathcal{A}$. Note that $p_0(\mathcal{A})$ denotes the number of unary constant polynomials over $\mathcal{A}$. The notation and notions used in this paper are standard and we refer to [17] and [16, 18].

In [28] J. Płonka characterized all groupoids $(G,\cdot)$ (algebras) with $p_n(G,\cdot) = n$ for all n (see also [17]).

P.P. Pálfy described all groupoids $(G,\cdot)$ whose clones are minimal and $p_2(G,\cdot) \leqslant 2$ (see [26]).

In [19] G. Grätzer and R. Padmanabhan described all groupoids $(G,\cdot)$ representing the sequence $(0,1,1,3,5)$. They proved that any groupoid $(G,\cdot)$ representing this sequence is a nontrivial affine space over $GF(3)$.

Recall that a sequence $a = (a_0,\ldots)$ (finite or infinite) of cardinals is representable if there exists an algebra $\mathcal{A}_0$ such that $p(\mathcal{A}_0) = a$, i.e., $p_i(\mathcal{A}_0) = a_i$ for $i = 0,1,\ldots$

In [24] A. Mischke and H. Werner among other described by means of identities the variety of groupoids $(G,\cdot)$ which are polynomially equivalent to affine spaces over $GF(4)$. Obviously for such groupoids $(G,\cdot)$ we have $p_2(G,\cdot) \leqslant 2$. Recall that two algebras $(A, F_1), (A, F_2)$ are polynomially equivalent if $\mathbf{A}(F_1) = \mathbf{A}(F_2)$, where $\mathbf{A}(F)$ denotes the set of all polynomials over (A, F).

In [12] we proved that a groupoid $(G, \cdot)$ represents the sequence $(0, 1, 2, 7)$ if and only if $(G, \cdot)$ is a nontrivial affine space over $GF(4)$.

For more details see also [3],[7] and [8]. Note that in [28] R. Park considered groupoids $(G, \cdot)$ with $p_2(G, \cdot) = 1$ which are not finitely based.

The simplest nontrivial groupoids $(G, \cdot)$ with $p_2(G, \cdot) \leqslant 2$ are obviously the following ones: $T_1 = (\{0, 1\}, e_1)$, $T_2 = (\{0, 1\}, e_2)$, $S_0 = (\{0, 1\}, \vee)$, where e_1 is the first projection, e_2 is the second projection and also $A_0 = (\{0, 1, 2\},\ 2x +_3 2y)$. We have $p_2(T_1) = p_2(T_2) = 0$ and $p_2(S_0) = p_2(A_0) = 1$. It is easy to see that $T_1 \times T_2$ is a groupoid satisfying $x^2 = x$, $(xy)z = x(yz) = xz$. Groupoids which satisfy these identities are called diagonal semigroups or rectangular bands. Note that $p(T_1 \times T_2) = (0, 1, 2, 0, 0, \ldots)$.

For a given groupoid $(G, \cdot)$ we put $xy^1 = xy$ and $xy^{n+1} = (xy^n)y$ and dually we define ${}^n yx$ $(n = 1, \ldots)$. We also write $x_1x_2x_3$ instead of $(x_1x_2)x_3$ and in general $x_1x_2 \ldots x_{n-1}x_n$ stands for $(x_1x_2 \ldots x_{n-1})x_n (n \geqslant 3)$.

A commutative idempotent groupoid $(G, \cdot)$ satisfying $xy^2 = x$ is called a Steiner quasigroup and if $xy^2 = xy$, then $(G, \cdot)$ is called a near-semilattice. Further, a groupoid $(G, \cdot)$ is distributive if $(G, \cdot)$ satisfies $(xy)z = (xz)(yz)$ and $z(xy) = (zx)(zy)$ for $x, y, z \in G$, and $(G, \cdot)$ is medial if $(G, \cdot)$ satisfies the medial law, i.e., $(xy)(uv) = (xu)(yv)$ for all $x, y, u, v \in G$.

We recall the groupoids introduced in [9], namely for a nonnegative integer n by N_n we mean a groupoid $(\{-1, 0, 1, \ldots, n-1, n\}, \cdot)$, where the fundamental operation $\cdot$ is defined as follows:

$$xy = \begin{cases} x \quad \text{if} \quad x = y, \\ 1 + \max\,(x, y) \quad \text{if} \quad x \neq y \quad \text{and} \quad x, y \leqslant n - 1, \\ n \quad \text{otherwise.} \end{cases}$$

By $\mathbf{N}_n$ we denote the variety of all commutative idempotent groupoids $(G, \cdot)$ satisfying $xy^2 = yx^2$ and $xy^n = xy^{n+1}$ for a fixed n.

Recall also that a groupoid $(G, \cdot)$ is totally commutative if every essentially binary polynomial f over $(G, \cdot)$ is commutative, i.e., $f(x, y) = f(y, x)$ for all $x, y \in G$.

Following [31], we say that an identity is regular if the sets of variables on both sides coincide.

2. General remarks

We start with a result from [9].

Theorem 2.1. *If $(G,\cdot)$ is an idempotent groupoid, then $p_2(G,\cdot)=1$ if and only if $(G,\cdot)$ is either a nontrivial Steiner quasigroup or a nontrivial near-semilattice.*

Concerning commutative idempotent groupoids we have

Theorem 2.2. *Let $(G,\cdot)$ be a commutative idempotent groupoid. Then*

(i) *If $p_2(G,\cdot)=2$, then $(G,\cdot)$ contains isomorphically as a subgroupoid the groupoid N_2 (see [9]).*

(ii) *$p_2(G,\cdot)=2$ if and only if $(G,\cdot)$ satisfies $xy^2=yx^2$, $xy^2=xy^3$ (i.e., $(G,\cdot)\in \mathbf{N}_2$) and $(G,\cdot)$ is not a near-semilattice (see [9]).*

(iii) *If $(G,\cdot)$ satisfies $xy^2=yx^2$, then $(G,\cdot)$ is totally commutative (see Theorem 4 of [6]) and if $\operatorname{card} G>1$, then such groupoids $(G,\cdot)$ satisfy only regular identities.*

We also have (easy to prove)

Theorem 2.3. *For every n, $N_n\in \mathbf{N}_n$ and every nontrivial member from the variety $\mathbf{N}_n$ satisfies only regular identities.*

Recall also the results from [10] and [11].

Theorem 2.4. *If $(G,\cdot)$ is a commutative idempotent groupoid, then $(G,\cdot)$ is either a semilattice (here $p(G,\cdot)=(0,1,1,1,\ldots)$ provided $\operatorname{card} G>1$) or $(G,\cdot)$ is an affine space over $GF(3)$ (with $p(G,\cdot)=(0,1,1,3,\ldots,\frac{2^n-(-1)^n}{3},\ldots)$ or else $p_n(G,\cdot)\geqslant 3^{n-1}$ for all $n\geqslant 4$.*

Moreover, a commutative groupoid $(G,\cdot)$ represents the sequence $(0,1,3,\ldots,3^{n-1},\ldots)$ if and only if $(G,\cdot)$ is a nontrivial Płonka sum of affine spaces over $GF(3)$ which are not all singletons (for details see [10] and [31]).

Theorem 2.5. *There are no distributive, commutative and idempotent groupoids $(G,\cdot)$ with $p_2(G,\cdot)=2$.*

Note that all distributive noncommutative idempotent groupoids $(G,\cdot)$ with $p_2(G,\cdot)\leqslant 2$ are described in [5].

According to the formula of the description of the set of all n-ary polynomials over a given algebra $\mathcal{A}=(A,F)$ we have $\mathbf{A}^{(n)}(\mathcal{A})=\bigcup\limits_{k=0}^{\infty}\mathbf{A}_k^{(n)}(\mathcal{A})$, where $\mathbf{A}_0^{(n)}=\mathbf{A}_0^{(n)}(\mathcal{A})=\{e_1^{(n)},\ldots,e_n^{(n)}\}$, $e_i^{(n)}(x_1,\ldots,x_n)=x_i$, $i=1,\ldots,n$, $\mathbf{A}_{k+1}^{(n)}=\mathbf{A}_{k+1}^{(n)}(\mathcal{A})=\mathbf{A}_k^{(n)}(\mathcal{A})\cup\{f(f_1,\ldots,f_m)\colon\ f_1,\ldots,f_m\in\mathbf{A}_k^{(n)}(\mathcal{A}),f\in F\}$ (for details see [23]). For an idempotent groupoid $(G,\cdot)$ with $e_1^{(2)}(x,y)=x$, $e_2^{(2)}(x,y)=y$ we have $\mathbf{A}_2^{(2)}(G,\cdot)=\{x,y,xy,yx,xy^2,yx^2,{}^2yx,{}^2xy,(xy)x,\ (yx)y,x(yx),y(xy),(xy)(yx),(yx)(xy)\}$. So, dealing with idempotent groupoids $(G,\cdot)$ satisfying $p_2(G,\cdot)=2$ we have to consider the set $\mathbf{A}_2^{(2)}(G,\cdot)$.

According to Theorems 2.1 and 2.2 we mainly deal with noncommutative idempotent groupoids $(G,\cdot)$ satisfying $p_2(G,\cdot)=2$. (Keep in mind that our main aim is to describe such groupoids by means of identities). The main role in our considerations play by the polynomial xy^2.

3. Groupoids with $XY^2=X$.

We start with

Lemma 3.1. *Let $(G,\cdot)$ be an idempotent groupoid satisfying $xy^2=x$. Then we have*

(i) *$(G,\cdot)$ satisfies $xy=x$ iff $(G,\cdot)$ satisfies $(xy)x=x$.*

(ii) *$(G,\cdot)$ is a Steiner quasigroup iff $(G,\cdot)$ satisfies $(xy)x=y$ (or dually $x(yx)=y$).*

(iii) *$(G,\cdot)$ satisfies $(xy)x=yx$ iff* $\operatorname{card} G=1$.

(iv) *$(G,\cdot)$ is a left–sided quasigroup, i.e., the equation $xa=b$ has a unique solution for $a,b\in G$.*

(v) *If $p_2(G,\cdot)=2$, then $(G,\cdot)$ is a noncommutative groupoid satisfying $(xy)x=xy$.*

(vi) *If $(G,\cdot)$ satisfies $(xy)x=xy$, then $(G,\cdot)$ satisfies $x(xy)=x$.*

P r o o f. We prove only (v). If $(G,\cdot)$ is commutative, then $(G,\cdot)$ is a Steiner quasigroup and therefore $p_2(G,\cdot)\leqslant 1$, which contradicts the assumption $p_2(G,\cdot)=2$. Take the polynomial $(xy)x$. Using the assumption $p_2(G,\cdot)=2$ and the conditions (i)–(iii) we get $(xy)x=xy$, completing the proof of the lemma. □

Similarly we get

Lemma 3.2. *Let $(G,\cdot)$ be an idempotent groupoid satisfying $xy^2=x$. Then we have*

(i) *$(G,\cdot)$ satisfies $x(yx)=x$ iff $(G,\cdot)$ satisfies $xy=x$.*

(ii) *$(G,\cdot)$ satisfies $x(yx)=y$ iff $(G,\cdot)$ is a Steiner quasigroup.*

(iii) *$(G,\cdot)$ satisfies $x(yx)=yx$ iff* $\operatorname{card} G=1$.

(iv) *If $p_2(G,\cdot)=2$, then $(G,\cdot)$ satisfies $x(yx)=xy$ and $(xy)(yx)=x$.*

Note that there exist idempotent groupoids $(G,\cdot)$ satisfying $xy^2=x$ in which the polynomial $(xy)x=x(yx)$ is commutative. Take e.g. an abelian group $(G,+)$ of exponent 5. Then $(G,\cdot)$ where $xy=4x+2y$ is the required groupoid. Such groupoids are obviously polynomially equivalent to affine spaces over $GF(5)$.

For a given groupoid $(G, \cdot)$ with the fundamental polynomial xy we consider the groupoid $(G, \circ)$ where $x \circ y = yx$ for all $x, y \in G$; it is called the dual groupoid of $(G, \cdot)$. If K is a class of groupoids, then by K^d we denote the class of all dual groupoids $(G, \cdot)$ from K. Note that a groupoid $(G, \cdot)$ is called proper if the fundamental operation xy is essentially binary.

We have

Lemma 3.3. *Let $(G, \cdot)$ be an idempotent groupoid satisfying $xy^2 = x$. Then $p_2(G, \cdot) = 2$ iff $(G, \cdot)$ belongs to the variety $K_1\colon$ $xy = (xy)x = x(yx)$ and $(G, \cdot)$ is proper.*

(The dual class K_1^d of the class K_1 is defined by $x = y(yx)$, $yx = x(yx) = (xy)x$).

Proof. If $p_2(G, \cdot) = 2$, then the proof follows from the last two lemmas. If $(G, \cdot) \in K_1$ and $(G, \cdot)$ is proper, then obviously $(G, \cdot)$ is noncommutative and xy is essentially binary. Using the identities of the groupoid $(G, \cdot)$ one can compute the elements of the set $\mathbf{A}^{(2)}(G, \cdot)$. We simply show that $\mathbf{A}^{(2)}(G, \cdot) = \{x, y, xy, yx\}$ and hence $p_2(G, \cdot) = 2$, completing the proof. □

Note that there exist proper (noncommutative) idempotent groupoids $(G, \cdot)$ belonging to the variety K_1. For example, let $(G, +)$ be an abelian group of exponent 4. Take $(G, \cdot)$ with $xy = 3x + 2y$. Then $(G, \cdot) \in K_1$.

4. Groupoids with $XY^2 = Y$

Concerning $(xy)x$ we have

Lemma 4.1. *Let $(G, \cdot)$ be an idempotent groupoid satisfying $xy^2 = y$. Then $(G, \cdot)$ is one-element iff one of the following conditions holds: either $(G, \cdot)$ is commutative or $(xy)x = y$ or $(xy)x = xy$.*

Proof. If e.g. $(xy)x = xy$, then using $xy^2 = y$ we get $xy = (xy)x = ((xy)x)x = x$, which gives $x = xy = (xy)y = y$ as required, completing the proof. □

According to this lemma we have the following possibilities for the polynomial $(xy)x$: $(xy)x = x$ or $(xy)x = yx$.

Lemma 4.2. *If $(G, \cdot)$ is a proper idempotent groupoid satisfying $xy^2 = y$ and $(xy)x = x$, then $(G, \cdot)$ satisfies $x(xy) = y(xy) = xy$ and $(xy)(yx) \notin \{y, xy\}$.*

Proof. Putting xy for x in $(xy)x = x$ we get $xy = ((xy)y)(xy) = y(xy)$. If e.g. $(xy)(yx) = y$, then $y = ((xy)y)(y(xy)) = y(y(xy)) = y(xy) = xy$, which is impossible. □

From this lemma we get

Lemma 4.3. *Let $(G,\cdot)$ be a proper idempotent groupoid satisfying $y = xy^2 = (yx)y$. Then $p_2(G,\cdot) = 2$ iff $(G,\cdot)$ belongs to one of the following varieties:*
$K_2\colon$ $xy = x(xy) = y(xy)$ *and* $(xy)(yx) = x$,
$K_3\colon$ $xy = x(xy) = y(xy) = (yx)(xy)$.

Recall that the dual classes of K_2 and K_3 are defined by
$K_2^d\colon$ $y = y(yx) = y(xy)$, $yx = (yx)x = (yx)y$ and $(xy)(yx) = x$,
$K_3^d\colon$ $y = y(yx) = y(xy)$, $yx = (yx)x = (yx)y = (yx)(xy)$.
Further we consider idempotent groupoids $(G,\cdot)$ with $p_2(G,\cdot) = 2$ satisfying $xy^2 = y$ and $(xy)x = yx$. We have

Lemma 4.4. *Let $(G,\cdot)$ be an idempotent groupoid satisfying $xy^2 = y$ and $(xy)x = yx$. Then we have*
(i) $\operatorname{card} G = 1$ *iff* $x(xy) \in \{x, yx\}$ *or* $x(yx) \in \{y, xy\}$.
(ii) $(G,\cdot)$ *satisfies* $xy = y$ *iff* $x(yx) = x$.

P r o o f. (i) If e.g. $x(xy) = yx$, then $x = (yx)x = (x(xy))x = (xy)x = yx$. The identities $xy = y$ and $x(xy) = yx$ give $\operatorname{card} G = 1$.
(ii) If $x(yx) = x$, then $x = x(yx) = (x(yx))(yx) = yx$, completing the proof. □

As a corollary of this lemma we get

Lemma 4.5. *Let $(G,\cdot)$ be a proper idempotent groupoid satisfying $xy^2 = y$ and $(xy)x = yx$. Then $p_2(G,\cdot) = 2$ iff $(G,\cdot)$ belongs to one of the following varieties:*

$$K_1^d\colon\ x(xy) = y \text{ and } x(yx) = yx,$$
$$K_4\colon\ x(xy) = xy \text{ and } x(yx) = yx.$$

Moreover, the varieties K_1^d (and also K_1), K_4 have proper models (see e.g. p. 394 [17]).

From Lemmas 4.3 and 4.5 we get

Lemma 4.6. *Let $(G,\cdot)$ be a proper idempotent groupoid satisfying $xy^2 = y$. Then $p_2(G,\cdot) = 2$ iff $(G,\cdot)$ belongs to one of the varieties*

$$K_2\colon\ xy = x(xy) = y(xy),\ x = (xy)x = (xy)(yx),$$
$$K_3\colon\ xy = x(xy) = y(xy) = (yx)(xy) \text{ and } (xy)x = x,$$
$$K_4\colon\ xy = x(xy),\ (xy)x = x(yx) = yx,$$

and to the dual of K_1.

We continue our characterization and deal with idempotent groupoids $(G,\cdot)$ satisfying $p_2(G,\cdot)=2$ and either $xy^2=xy$ or $xy^2=yx$. It is obvious that such groupoids are proper and noncommutative.

5. Groupoids with $XY^2 = XY$

First we consider the polynomial $(xy)x$. We have

Lemma 5.1. *Let $(G,\cdot)$ be a proper idempotent groupoid satisfying $xy^2=xy$ and $(xy)x=x$. Then $p_2(G,\cdot)=2$ iff $(G,\cdot)$ belongs to the variety*
$K_5\colon\ x=x(yx)=(xy)(yx)$.

Proof. First observe that the identity $(xy)x=x$ implies $x(xy)=xy$. Assume now that $(G,\cdot)$ satisfies $xy^2=xy$, $(xy)x=x$ and $p_2(G,\cdot)=2$. Take $x(yx)$. Obviously $x(yx)\neq y$. If $x(yx)=xy$, then putting yx for x in this identity we get $y=(yx)y=(yx)(y(yx))=yx$, a contradiction. If $(G,\cdot)$ satisfies $x(yx)=yx$, then $yx=(yx)x=(x(yx))x=x$, which is impossible. Thus we have proved that $(G,\cdot)$ satisfies $x(yx)=x$. In this case we consider the polynomial $(xy)(yx)$. If $(xy)(yx)=y$, then using $yx=y(yx)$ we get $yx=y(yx)=((xy)(yx))(yx)=(xy)(yx)=y$, a contradiction. If $(xy)(yx)=xy$, then putting xy for y we get $xy=x(xy)=(x(xy))((xy)x)=(xy)x=x$, again a contradiction. Analogously we get $(xy)(yx)\neq yx$ and therefore $(G,\cdot)$ satisfies $(xy)(yx)=x$ as required.

If $(G,\cdot)$ is a proper member of K_5, then the standard method shows that $p_2(G,\cdot)=2$, completing the proof of the lemma. □

Note that $K_5=K_5^d$ and any near-rectangular band, i.e. an idempotent groupoid $(G,\cdot)$ satisfying $(xy)z=xz$ and $x(yx)=x$, is a member of K_5 (for details see [2]).

Now we deal with groupoids $(G,\cdot)$ satisfying either $(xy)x=xy$ or $(xy)x=yx$.

It is easy to prove

Lemma 5.2. *Let $(G,\cdot)$ be an idempotent groupoid satisfying $xy=xy^2=(xy)x$. Then we have*

(i) *$(G,\cdot)$ satisfies $x(yx)=y$ iff* $\operatorname{card} G=1$.

(ii) *If $p_2(G,\cdot)=2$, then $(G,\cdot)$ satisfies either $x(yx)=x$ or $x(yx)=xy$ or $x(yx)=yx$.*

Further we have

Lemma 5.3. *Let $(G,\cdot)$ be a proper idempotent groupoid satisfying $xy=xy^2=(xy)x$ and $x(yx)=x$. Then $p_2(G,\cdot)=2$ iff $(G,\cdot)$ belongs either to K_2^d or to K_3^d.*

P r o o f. Recall that K_2^d and K_3^d are defined by

K_2^d: $xy^2 = xy = (xy)x$, $x(yx) = (xy)(yx) = x$, and

K_3^d: $xy^2 = xy = (xy)x = (xy)(yx)$, $x(yx) = x$

(see Lemma 4.6). Assume that $p_2(G, \cdot) = 2$. Consider the polynomial $(xy)(yx)$. If $(xy)(yx) = y$, then $y = (xy^2)(y(xy)) = xy$, a contradiction. Analogously we have $(xy)(yx) \neq yx$. Thus we see that $(G, \cdot)$ satisfies either $(xy)(yx) = x$ or $(xy)(yx) = xy$ and therefore $(G, \cdot) \in K_2^d \cup K_3^d$. It is routine to prove the converse, completing the proof. □

Concerning Lemma 5.2 we have

Lemma 5.4. *Let $(G, \cdot)$ be a proper noncommutative idempotent groupoid satisfying $xy = xy^2 = (xy)x = x(yx)$. Then $p_2(G, \cdot) = 2$ iff $(G, \cdot)$ belongs either to K_4^d or to the variety K_6: $x(xy) = xy$.*

P r o o f. Assume that $p_2(G, \cdot) = 2$. Take $x(xy)$. If $x(xy) = y$, then $y = x(xy) = x(x(yx)) = yx$, which is impossible. If $x(xy) = yx$, then putting xy for x we get $yx = y(xy) = (xy)(xy^2) = xy$ which proves that $(G, \cdot)$ is commutative, a contradiction. Since $p_2(G, \cdot) = 2$ we infer that $(G, \cdot)$ satisfies either $x(xy) = x$ or $x(xy) = xy$. If the first case occurs, then $(G, \cdot) \in K_4^d$, i.e. $(G, \cdot)$ satisfies $xy = xy^2 = (xy)x = x(yx) = (xy)(yx)$ and $x(xy) = x$. If the second case occurs, then $(G, \cdot)$ satisfies $xy = xy^2 = (xy)x = x(yx) = x(xy) = (xy)(yx)$, i.e. $(G, \cdot) \in K_6$, completing the proof. □

According to Lemma 5.2 it remains to consider yet in this case the identity $x(yx) = yx$. We have

Lemma 5.5. *There is no idempotent groupoid with $p_2(G, \cdot) = 2$ satisfying $xy = xy^2 = (xy)x = y(xy)$.*

P r o o f. First we prove that $x(xy) = xy$. Indeed, we have $x(xy) = x((xy)x) = (xy)x = xy$. Take now the polynomial $(xy)(yx)$. If $(xy)(yx) = x$, then $x = (x(xy))((xy)x) = (xy)(xy) = xy$, a contradiction. Analogously we get $(xy)(yx) \neq y$. If $(xy)(yx) = yx$, then $xy = (yx)(xy) = ((xy)(yx))(xy) = (xy)(yx) = yx$ and hence $(G, \cdot)$ is a near-semilattice, which is impossible. Analogously, using $xy = y(xy)$ we infer that $(xy)(yx) \neq xy$, which completes the proof of the lemma. □

According to Lemmas 5.1 and 5.2 to complete the case $xy^2 = xy$ it remains to consider yet the identity $(xy)x = yx$. Note that if $(G, \cdot)$ satisfies $(xy)x = y$, then $(G, \cdot)$ is a quasigroup and hence $(G, \cdot)$ is cancellative. In this case $xy^2 = xy$ gives $x = y$. Further we have.

Lemma 5.6. *Let $(G,\cdot)$ be a noncommutative idempotent groupoid satisfying $xy^2 = xy$ and $(xy)x = yx$. Then $p_2(G,\cdot) = 2$ iff $(G,\cdot)$ belongs to the dual of K_6, i.e., $(G,\cdot)$ satisfies K_6^d: $xy = xy^2 = (yx)y = y(xy) = x(xy) = (yx)(xy)$.*

Proof. Putting xy for x in $(xy)x = yx$ we get $y(xy) = (xy^2)(xy) = xy$ and we have $(yx)(xy) = ((xy)x)(xy) = x(xy)$. Consider now the polynomial $x(xy)$. We see that $x(xy) \neq y$. If $x(xy) = x$, then using $(xy)x = yx$ we get $(xy)(yx) = (xy)((xy)x) = xy$ and hence $xy = (xy)(yx) = y(yx) = y$, a contradiction. Let $x(xy) = yx$. Then $(yx)(xy) = (x(xy))(xy) = x(xy) = yx$ and $xy = (xy)(yx) = (xy)((yx)(xy)) = (yx)(xy) = yx$, which is impossible. Thus we have proved that $(G,\cdot)$ satisfies $x(xy) = xy$ and consequently $(G,\cdot) \in K_6^d$; the fact that if $(G,\cdot) \in K_6^d$, then $p_2(G,\cdot) = 2$ is obvious, which completes the proof of the lemma. □

To complete our characterization of idempotent groupoids $(G,\cdot)$ with $p_2(G,\cdot) = 2$ it remains to consider the last case, namely the identity $xy^2 = yx$.

6. Groupoids with $XY^2 = YX$

We begin with

Lemma 6.1. *If $(G,\cdot)$ is a groupoid satisfying $xy^2 = yx$, then $(G,\cdot)$ satisfies $(xy)x = x(yx)$.*

Proof. Using $xy^2 = yx$ we have $(yx)y = ((xy)y)y = y(xy)$ as required. □

Lemma 6.2. *Let $(G,\cdot)$ be a proper idempotent groupoid satisfying $xy^2 = yx$. Then $p_2(G,\cdot) = 2$ iff $(G,\cdot)$ belongs to K_7: $(xy)x = y$.*

Moreover, if $(G,\cdot) \in K_7$ and $a, b \in G$ $a \neq b$, then the subgroupoid $G(a,b)$ generated by the set $\{a,b\}$ is a four-element affine space over $GF(4)$.

Proof. Let $p_2(G,\cdot) = 2$. Using the preceding lemma we get $(xy)x = x(yx)$. Take the polynomial $(xy)x$. If $(xy)x = x$, then $x = x(yx) = (x(yx))(yx) = (yx)x = xy$, a contradiction. If $(xy)x = yx$, then $xy = (yx)x = ((xy)x)x = x(xy)$. Using $xy = x(xy)$ and $(xy)x = yx$ we get $(yx)(xy) = ((xy)x)(xy) = x(xy) = xy$. The identity $(xy)(yx) = yx$ gives $xy = ((yx)(xy))(xy) = (xy)(yx) = yx$ and therefore $(G,\cdot)$ is a near-semilattice, a contradiction. If $(xy)x = xy$, then $(xy)(yx) = ((xy)x)(yx) = (x(yx))(yx) = (yx)x = xy$. Using $xy = (xy)(yx)$ we get $xy = (xy)(yx) = ((xy)(yx))(yx) = (yx)(xy) = yx$, a contradiction. Thus we have proved that $(G,\cdot)$ satisfies $(xy)x = y$.

Let now $(G,\cdot) \in K_7$ i.e., $(G,\cdot)$ satisfies $xy^2 = yx$, $(xy)x = y$ and also $x(xy) = yx$, $x(yx) = y$, $(xy)(yx) = y$, $xy^3 = x$ and ${}^3yx = x$. Let $G(a,b)$ be the subgroupoid generated by the set $\{a, b\}$ where $a \neq b$ and $a, b \in G$. One can check that $G(a, b) = \{a, b, ab, ba\}$ and the groupoid $G(a, b)$ is an affine space over $GF(4)$, which completes the proof of the lemma. $\square$

7. A characterization theorem

As a corollary of Lemmas 3.3, 4.6, 5.3, 5.4, 5.5, 5.6, 6.2 and Theorems 2.1, 2.2 we get

Theorem. *Let $(G, \cdot)$ be an idempotent groupoid. Then $p_2(G, \cdot) \leqslant 2$ if and only if $(G, \cdot)$ belongs to one of the following varieties:*

$K_1\colon\ xy^2 = x,\ xy = (xy)x = x(yx),\ x(xy) = (xy)(yx) = x;$

$K_2\colon\ xy^2 = y,\ (xy)(yx) = (xy)x = x,\ xy = x(xy) = y(xy);$

$K_3\colon\ xy^2 = y,\ (xy)x = x,\ xy = x(xy) = y(xy) = (yx)(xy);$

$K_4\colon\ xy^2 = y,\ xy = (yx)y = y(xy) = x(xy) = (yx)(xy);$

$K_5\colon\ xy^2 = xy,\ (xy)x = x(yx) = (xy)(yx) = x;$

$K_6\colon\ xy^2 = xy = (xy)x = x(yx) = x(xy) = (xy)(yx);$

$K_7\colon\ xy^2 = yx,\ (xy)x = x(yx) = y,\ x(xy) = yx,\ (xy)(yx) = y;$

$K_8\colon\ xy^2 = x,\ xy = yx$*—the variety of Steiner quasigroups;*

$K_9\colon\ xy = yx,\ xy^2 = yx^2,\ xy^2 = xy^3$*—the variety* $\mathbf{N}_2$

and to the varieties K_i^d $(i = 1, \dots, 9)$.

Note that in the above varieties some identities can be omitted (they are inserted only for the sake of symmetry). Proper models were given earlier from each variety K_i $(i = 1, \dots, 9)$, except for $i = 2$ and $i = 3$. The next two sections are devoted to these varieties. Note also that the variety K_9 plays an important role in Grätzer's problem of the minimal extension of sequences (see [16]).

On the variety K_2. In this section we shall deal with the variety K_2, i.e. the variety of all idempotent groupoids $(G, \cdot)$ defined by $xy^2 = y$, $(xy)x = (xy)(yx) = x$ and $xy = y(xy)$. Note that $xy = x(xy)$ is a consequence of $(xy)x = x$.

We define the four-element groupoid $k_2 = (\{1, 2, 3, 4\}, \cdot)$, where $1 \cdot 2 = 3$, $2 \cdot 3 = 4$, $3 \cdot 4 = 1$, $4 \cdot 3 = 2$ and otherwise the fundamental operation xy is the second projection. It is not hard to prove that k_2 is a proper groupoid belonging to the variety K_2. We have

Lemma 8.1. *If $(G, \cdot)$ is a proper groupoid from the variety K_2, then $(G, \cdot)$ contains isomorphically the groupoid k_2 as a subgroupoid.*

P r o o f. Since $(G,\cdot)$ is proper we infer that there exist elements $a,b\in G$ such that $ab\neq b$ and $a\neq b$. Take the subgroupoid $G(a,b)$ generated by the set $\{a,b\}$. We have $G(a,b)=\{a,b,ab,ba\}$. Now we show that $\operatorname{card} G(a,b)=4$. If $a=ab$, then $a=ab=(ab)b=b$, which is impossible. If $a=ba$, then $ab=(ba)b=b$. If $b=ba$, then $b=ba=(ba)a=a$. If again $ab=ba$, then $a=(ab)(ba)=(ba)(ab)=b$. The required isomorphism between $G(a,b)$ and the groupoid k_2 is given by the mapping $a\to 1$, $b\to 2$, $ab\to 3$ and $ba\to 4$, which completes the proof of the lemma. □

Further we try to estimate the number p_n of proper groupoids $(G,\cdot)$ from K_2. In this connection we recall some definitions.

For a given function $f=f(x_1,\ldots,x_n)$ on a set A we denote by $G(f)$ the symmetry group of the function f, i.e., the subgroup of the group S_n of all permutations $\sigma\in S_n$ such that $f(x_1,\ldots,x_n)=f(x_{\sigma_1},\ldots,x_{\sigma_n})$ for all $x_1,\ldots x_n\in A$ (for more details see [20]). It is clear that if $f=f(x_1,\ldots,x_n)$ is essentially n-ary, then f^σ where $f^\sigma(x_1,\ldots,x_n)=f(x_{\sigma_1},\ldots,x_{\sigma_n})$ is also essentially n-ary. So, from an essentially n-ary function f on a set A we can get (permuting its variables) $\frac{n!}{\operatorname{card} G(f)}$ different essentially n-ary functions on A.

Lemma 8.2. *If $(G,\cdot)$ is a proper groupoid from K_2, then the polynomials $(xy)z$, $x(yz)$ are essentially ternary, their symmetry groups are one-element and $(xy)z\neq\sigma x(\sigma y\sigma z)$ for every permutation σ.*

P r o o f. Let $(G,\cdot)\in K_2$. Using the identities $xy^2=y$, $(xy)x=(xy)(yx)=x$ and $xy=x(xy)=y(xy)$ one can prove that both $(xy)z$ and $x(yz)$ are essentially ternary. Since $(G,\cdot)$ is not a semilattice we infer that $(xy)z\neq(yz)x$ and $x(yz)\neq y(zx)$ (see e.g. Theorem 8 [7]). If $(xy)z=(yx)z$, then $xy=(xy)(xy)=(yx)(xy)=y$, a contradiction.

If $(xy)z=(zy)x$, then $y=(xy)y=yx$, a contradiction. If $(xy)z=(xz)y$, then $xy=(xy)x=x$, which is again impossible. Thus we have proved that $\operatorname{card} G((xy)z)=1$. Consider now the polynomial $x(yz)$. If $x(yz)=x(zy)$, then $yx=x(yx)=x(xy)=xy$, a contradiction. If $x(yz)=z(yx)$, then putting $y=z$ we get $xy=yx$. If $x(yz)=y(xz)$, then $x=(xy)(yx)=y((xy)x)=yx$ and this proves that $\operatorname{card} G(x(yz))=1$. Let now $(xy)z\in\{x(yz),y(zx),z(xy),y(xz),z(yx),x(zy)\}$. We see that in any case if $(xy)z=\sigma x(\sigma y\sigma z)$ holds, then putting $y=z$ we get a contradiction, which completes the proof. □

Analogously as above we prove

Lemma 8.3. *If $(G,\cdot)$ is a proper groupoid from K_2, then the polynomial $f(x,y,z)=((xy)z)x$ is essentially ternary, its symmetry group is one-element, $f\notin\{(xy)z,(yz)x,(zx)y,(yx)z,(zy)x\}$ and $f\notin\{x(yz),y(zx),z(xy),y(xz),z(yx),x(zy)\}$.*

As a corollary from Lemmas 8.1 and 8.2 we get

Proposition 8.4. *If $(G,\cdot)$ is a proper groupoid from the variety K_2, then $p_n(G,\cdot) \geqslant p_n(k_2)$ for all n and $p_3(G,\cdot) \geqslant 12$.*

9. On the variety K_3

Recall that K_3 is a variety of idempotent groupoids $(G,\cdot)$ defined by the following identities:

$$xy^2 = y,\ (xy)x = x,\ xy = x(xy) = y(xy) = (yx)(xy).$$

We start with an easy

Lemma 9.1. *The groupoid $k_3 = (\{0,1,2\},\cdot)$, where $0\cdot 1 = 1\cdot 0 = 2$ and otherwise the operation xy is the second projection, belongs to the variety K_3. Moreover, this groupoid satisfies the identity $x(yz) = y(xz)$.*

Lemma 9.2. *If $(G,\cdot)$ is a proper groupoid from K_3, then the polynomials $(xy)z, x(yz)$ are essentially ternary and the symmetry group of $(xy)z$ is one-element. If the symmetry group of $x(yz)$ is nontrivial, then $(G,\cdot)$ satisfies $x(yz) = y(xz)$ and the transposition (x,y) is the only admissible nontrivial permutation of $x(yz)$.*

Proof. Analogously as in Lemma 8.2 we prove that $(xy)z, x(yz)$ are essentially ternary. Now we examine the symmetry groups of these polynomials. According to Theorem 8 of [7] we infer that $(xy)z \neq (yz)x$ and $x(yz) \neq y(zx)$. If $(xy)z = (yx)z$, then $yz = ((xy)y)z = (y(xy))z = (xy)z$, which contradicts the fact that $(xy)z$ is essentially ternary. If $(xy)z = (zy)x$, then $y = (xy)y = yx$, a contradiction. Similarly $(xy)z \neq (xz)y$. Thus we get $\operatorname{card} G((xy)z) = 1$. We also have $x(yz) \neq x(zy)$ and $x(yz) \neq z(yx)$. If e.g. $x(yz) = x(zy)$, then $xy = (yx)(xy) = yx$ and hence $x = (xy)x = x(xy) = xy$, a contradiction which completes the proof. $\square$

We have (similarly as in Lemma 8.3)

Lemma 9.3. *If $(G,\cdot)$ is a proper groupoid from K_3, then the polynomial $f(x,y,z) = ((xy)z)x$ is essentially ternary, its symmetry group is trivial and the groupoid k_3 satisfies the identity $((xy)z)x = yzx$.*

Proof. Since $f(x,y,y) = yx$ we infer that f depends on x. If f does not depend on y, then $(G,\cdot)$ satisfies $x = (xz)x = zx$, a contradiction. If f does not depend on z, then $x = ((xy)x)x = ((xy)y)x = yx$, again a contradiction. If $((xy)z)x = ((yx)z)y$ or $((xy)z)x = ((yx)z)y$, then putting $x = z$ we get $x = xy$, a contradiction. If

$((xy)z)x = ((xz)y)x$, then putting $z = xy$ we obtain $x = (xy)x = ((x(xy))y)x = ((xy)y)x = yx$, again a contradiction. If $((xy)z)x = ((zy)x)z$, then $y = z$ gives $yx = y$. Thus we have proved that $\operatorname{card} G((xy)z)) = 1$. Further observe that if $(G, \cdot) \in K_3$, then $(G, \cdot)$ satisfies $xyzx = yzx$ under any identification of variables and now it is not difficult to check that the groupoid k_2 satisfies this identity, completing the proof of the lemma. $\square$

As a corollary from Lemmas 9.1 and 9.2 we get

Proposition 9.4. *If $(G, \cdot)$ is a proper groupoid from the variety K_3, then* $p_3(G, \cdot) \geqslant 9$.

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Author's address: Mathematical Institute, University of Wrocław, Pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland.