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ON LIPSCHITZ CONDITIONS FOR ORDINARY DIFFERENTIAL EQUATIONS IN FRÉCHET SPACES

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Abstract. We will give an existence and uniqueness theorem for ordinary differential equations in Fréchet spaces using Lipschitz conditions formulated with a generalized distance and row-finite matrices.

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1. Introduction

Let $K = \mathbb{R}$ or $\mathbb{C}$ and F be a vector space over K. A mapping $\|\cdot\|\colon F \to [0,\infty)^{\mathbb{N}}$ is called a *polynorm* on F if $\|\cdot\|_n$ is a seminorm on F for each $n \in \mathbb{N}$ and $\|x\| = 0$ if and only if $x = 0$. Inequalities between elements of $\mathbb{R}^{\mathbb{N}}$ are intended componentwise. We have:

(a) $\|x\| \geqslant 0$, $x \in F$.
(b) $\|x+y\| \leqslant \|x\| + \|y\|$, $x, y \in F$.
(c) $\|\lambda x\| = |\lambda|\,\|x\|$, $x \in F$, $\lambda \in K$.

$\bigl(F, \|\cdot\|\bigr)$ is a Fréchet space if the locally convex topology induced by the seminorms $\|\cdot\|_n$, $n \in \mathbb{N}$, is complete. A polynorm is a generalized distance (e.g. according to Schröder [12]), and this concept allows to study Lipschitz mappings on F with generalized Lipschitz constants which are row-finite matrices. In this paper we want to study Lipschitz conditions for ordinary differential equations in Fréchet spaces continuing the work of Lemmert [9]. For related concepts see also [2], [3] and [11].

We consider the Fréchet space $(\mathbb{C}^{\mathbb{N}}, \|\cdot\|)$, $\|x\| = (|x_n|)_{n=1}^{\infty}$ and its topological dual space

$$\mathbb{C}_{\mathbb{N}} = \{y \in \mathbb{C}^{\mathbb{N}} : \text{ at most finitely many } y_n \text{ are different from zero}\}$$

together with the duality

$$\langle x, y\rangle = \sum_{n=1}^{\infty} x_n y_n, \quad (x,y) \in \mathbb{C}^{\mathbb{N}} \times \mathbb{C}_{\mathbb{N}}.$$

A matrix $L = (l_{ij})_{i,j\in\mathbb{N}}$, $l_{ij} \in \mathbb{C}$, is called *row-finite* if every row is in $\mathbb{C}_{\mathbb{N}}$. Correspondingly, L is called *column-finite* if every column is in $\mathbb{C}_{\mathbb{N}}$. The row-finite matrices are exactly the continuous endomorphisms of $\mathbb{C}^{\mathbb{N}}$, and the column-finite matrices are exactly the endomorphisms of $\mathbb{C}_{\mathbb{N}}$. If L is row-finite, then the matrix ${}^{\top}L$ is column-finite, and it holds that $\langle x, {}^{\top}Ly\rangle = \langle Lx, y\rangle$, $(x,y) \in \mathbb{C}^{\mathbb{N}} \times \mathbb{C}_{\mathbb{N}}$.

A column-finite matrix L is called *locally algebraic* if for every $y \in \mathbb{C}_{\mathbb{N}}$ there is a polynomial $p \in \mathbb{C}[\lambda] \setminus \{0\}$ such that $p(L)y = 0$.

The spectrum σ of a row-finite resp. column-finite matrix L is defined as

$$\sigma(L) = \{\lambda \in \mathbb{C}\colon\ L - \lambda I \text{ is not invertible}\}.$$

It holds that $\sigma(L) = \sigma({}^{\top}L) \neq \emptyset$ and that either $\sigma(L)$ or $\mathbb{C}\setminus\sigma(L)$ is at most countable (see e. g. [7], [13]). For the following proposition compare [5], [7], [8], [13] and [14].

Proposition 1. *Let $L = (l_{ij})_{i,j\in\mathbb{N}}$, $l_{ij} \in \mathbb{C}$, be row-finite. Then the following assertions are equivalent:*

1. *${}^{\top}L$ is locally algebraic.*
2. *$\sigma(L)$ is at most countable.*
3. $\limsup_{k\to\infty} \sqrt[k]{|\langle L^k x, y\rangle|} < \infty$, $(x,y) \in \mathbb{C}^{\mathbb{N}} \times \mathbb{C}_{\mathbb{N}}$.
4. *For every entire function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ it holds that $\sum_{k=0}^{\infty} a_k L^k x$ converges in $\mathbb{C}^{\mathbb{N}}$ for all $x \in \mathbb{C}^{\mathbb{N}}$ (by that a row-finite matrix is defined which is denoted by $f(L)$ and $\sigma(f(L))$ is at most countable).*
5. *The initial value problem $x'(t) = Lx(t)$, $x(0) = x_0$ is uniquely solvable in $\mathbb{C}^{\mathbb{N}}$ for every $x_0 \in \mathbb{C}^{\mathbb{N}}$ (the solution is $\mathrm{e}^{Lt}x_0$, $t \in \mathbb{R}$).*

3. Lipschitz conditions

Let $(F, \|\cdot\|)$ be a Fréchet space, $f\colon [0,T]\times F\to F$ continuous and $x_0\in F$. We consider the initial value problem

$$\begin{cases} x'(t)=f\big(t,x(t)\big), & t\in[0,T],\\ x(0)=x_0. \end{cases} \tag{1}$$

Furthermore, let f satisfy the Lipschitz condition

$$\big\|f(t,u)-f(t,v)\big\|\leqslant L\big\|u-v\big\|,\qquad (t,u),(t,v)\in[0,T]\times F. \tag{2}$$

Here L is a row-finite matrix with nonnegative entries. Condition (2) in general implies neither uniqueness nor existence of solutions of (1) even in the case that the right-hand side in (1) is linear (see [4], [5], [8] and [10]). Lemmert [9] proved the following theorem.

Theorem 1. *If $\sigma(L)$ is at most countable then* (1) *is uniquely solvable for every $x_0\in F$.*

If f is bounded, i. e. there is a $b\in[0,\infty)^{\mathbb{N}}$ such that $\big\|f(t,x)\big\|\leqslant b$, $(t,x)\in[0,T]\times F$, we have

Theorem 2. *If*

$$\limsup_{k\to\infty}\sqrt[k]{\big\langle L^k b, y\big\rangle}<\infty,\qquad y\in[0,\infty)_{\mathbb{N}}, \tag{3}$$

then (1) *is uniquely solvable for every $x_0\in F$.*

Condition (3) is satisfied, for example, if $Lb\leqslant cb$ for some $c\geqslant 0$ (see Deimling [1], p. 86 and [11]).

We will now generalize these theorems in the following way (for another generalization of Theorem 2 see [6]).

Let $g,h\colon [0,T]\times F\to F$ be continuous and $f=g+h$. Furthermore, let g and h satisfy a Lipschitz condition of the form (2) with L_1 and L_2 as Lipschitz matrices, and let h be bounded by $b\in[0,\infty)^{\mathbb{N}}$. Then we have

Theorem 3. *If $\sigma(L_1)$ is at most countable and*

$$\limsup_{k\to\infty}\sqrt[k]{\big\langle (\mathrm{e}^{TL_1}L_2)^k\mathrm{e}^{TL_1}b, y\big\rangle}<\infty,\qquad y\in[0,\infty)_{\mathbb{N}}, \tag{4}$$

then (1) *is uniquely solvable for every $x_0\in F$.*

Remarks.

1) f is satisfying (2) with $L = L_1 + L_2$.

2) If $L_2 = 0$, (4) is satisfied, and we have Theorem 1.

3) If $L_1 = 0$, (4) is condition (3) of Theorem 2.

4) e^{TL_1} is a row-finite matrix with nonnegative entries.

5) To check condition (4), it is sufficient to show (4) for $y = e_n$, $n \in \mathbb{N}$; $e_n \in \mathbb{C}_{\mathbb{N}}$ denotes the vector with 1 in the n-th coordinate and 0 elsewhere.

6) Condition (4) holds e. g. if, for some $c \geqslant 0$, $(\mathrm{e}^{TL_1} L_2)\mathrm{e}^{TL_1} b \leqslant c\mathrm{e}^{TL_1} b$, which is implied by

$$L_2 \mathrm{e}^{TL_1} b \leqslant cb. \tag{5}$$

7) If L_1 and L_2 commute, condition (4) reduces to

$$\limsup_{k\to\infty} \sqrt[k]{\langle L_2^k b, y\rangle} < \infty, \qquad y \in [0,\infty)_{\mathbb{N}},$$

for the following reason: Since ${}^{\top}\mathrm{e}^{TL_1}$ is locally algebraic, the subspace $U = \operatorname{span}\{{}^{\top}\mathrm{e}^{kTL_1} b\colon\ k \in \mathbb{N}_0\}$ of $\mathbb{C}_{\mathbb{N}}$ is finite-dimensional. For every $y \in [0,\infty)_{\mathbb{N}}$ there is $\gamma > 0$ and $z \in [0,\infty)_{\mathbb{N}}$ such that ${}^{\top}\mathrm{e}^{kTL_1} y \leqslant \gamma^k z$, $k \in \mathbb{N}$, which implies

$$\limsup_{k\to\infty} \sqrt[k]{\langle L_2^k b, {}^{\top}\mathrm{e}^{(k+1)TL_1} y\rangle} \leqslant \gamma \limsup_{k\to\infty} \sqrt[k]{\langle L_2^k b, z\rangle}.$$

We will use the following propositions to prove Theorem 3:

Proposition 2. *Let $A = (a_{ij})_{i,j\in\mathbb{N}}$ be a real row-finite quasimonotone matrix (i. e. $a_{ii} \in \mathbb{R}$, $i \in \mathbb{N}$ and $a_{ij} \geqslant 0$, $i,j \in \mathbb{N}$, $i \neq j$) with $\sigma(A)$ at most countable. If $u\colon\ [0,T) \to \mathbb{R}^{\mathbb{N}}$ is continuous, right-hand side differentiable and*

$$\begin{cases} u'_+(t) \geqslant Au(t), & t \in [0,T), \\ u(0) \geqslant 0, \end{cases}$$

then $u(t) \geqslant 0$, $t \in [0,T)$.

For the proof of this Proposition see Lemmert [9], p. 1387.

Proposition 3. *Let $\sigma(L_1)$ be at most countable, $u \in C^1([0,T],F)$, and $v \in C([0,T],F)$ such that*

$$\|u'(t)\| \leqslant L_1\|u(t)\| + \|v(t)\|.$$

Then

$$\|u(t)\| \leqslant \mathrm{e}^{tL_1}\|u(0)\| + \int_0^t \mathrm{e}^{(t-s)L_1}\|v(s)\|\,\mathrm{d}s, \qquad t \in [0,T].$$

P r o o f. The function $\delta\colon [0,T] \to [0,\infty)^{\mathbb{N}}$, $\delta(t) = \|u(t)\|$ is right-hand side differentiable on $[0,T)$ and

$$\delta'_+(t) \leqslant \|u'(t)\| \leqslant L_1\delta(t) + \|v(t)\|.$$

According to Theorem 1, the initial value problem

$$\begin{cases} z'(t) = L_1 z(t) + \|v(t)\|, & t \in [0,T], \\ z(0) = \|u(0)\| \end{cases}$$

is uniquely solvable on $[0,T]$, and the solution is

$$z(t) = \mathrm{e}^{tL_1}\|u(0)\| + \int_0^t \mathrm{e}^{(t-s)L_1}\|v(s)\|\,\mathrm{d}s.$$

Therefore

$$\begin{aligned}(z-\delta)'_+(t)& \\ &\geqslant L_1 z(t) + \|v(t)\| - L_1\delta(t) - \|v(t)\| \\ &= L_1(z-\delta)(t), \quad t \in [0,T)\end{aligned}$$

and $z(0) - \delta(0) = 0$. According to Proposition 2, this implies that $z(t) - \delta(t) \geqslant 0$ on $[0,T]$ which is

$$\delta(t) \leqslant \mathrm{e}^{tL_1}\|u(0)\| + \int_0^t \mathrm{e}^{(t-s)L_1}\|v(s)\|\,\mathrm{d}s.$$

□

P r o o f o f T h e o r e m 3. Let $u_1 \in C^1([0,T],F)$, $u_1(0) = x_0$. Since $\sigma(L_1)$ is at most countable, there is, according to Theorem 1, a sequence $(u_k)_{k=1}^{\infty}$ in $C^1([0,T],F)$ such that

$$\begin{cases} u'_{k+1}(t) = g\big(t, u_{k+1}(t)\big) + h\big(t, u_k(t)\big), & t \in [0,T],\ k \in \mathbb{N}, \\ u_{k+1}(0) = x_0. \end{cases}$$

It holds that

$$\|u'_{k+1}(t) - u'_k(t)\| \leqslant L_1\|u_{k+1}(t) - u_k(t)\| + \|h(t,u_k(t)) - h(t,u_{k-1}(t))\|,$$

$t \in [0,T]$, $k \geqslant 2$. From Proposition 3 we get

$$\|u_{k+1}(t) - u_k(t)\| \leqslant \int_0^t \mathrm{e}^{TL_1}\|h(s,u_k(s)) - h(s,u_{k-1}(s))\|\,\mathrm{d}s,$$

$t\in[0,T]$, $k\geqslant 2$. Therefore

$$\big\|u_{k+1}(t)-u_k(t)\big\|\leqslant \mathrm{e}^{TL_1}L_2\int_0^t\big\|u_k(s)-u_{k-1}(s)\big\|\,\mathrm{d}s, \tag{6}$$

$t\in[0,T]$, $k\geqslant 2$, and

$$\big\|u_3(t)-u_2(t)\big\|\leqslant 2T\mathrm{e}^{TL_1}b,\qquad t\in[0,T]. \tag{7}$$

Successive application of inequality (6) and (7) leads to

$$\big\|u_{k+1}(t)-u_k(t)\big\|\leqslant \frac{2T^{k-1}}{(k-2)!}\big(\mathrm{e}^{TL_1}L_2\big)^{k-2}\mathrm{e}^{TL_1}b,\quad t\in[0,T],\quad k\geqslant 2.$$

Condition (4) implies the convergence of $\sum\limits_{k=2}^{\infty}\frac{2T^{k-1}}{(k-2)!}\,(\mathrm{e}^{TL_1}L_2)^{k-2}\mathrm{e}^{TL_1}b$ in $\mathbb{C}^{\mathbb{N}}$.

Therefore $(u_k)_{k=1}^{\infty}$ is a Cauchy sequence in the Fréchet space $\big(C([0,T],F),\|\!|\cdot|\!\|\big)$, $\|\!|u|\!\|=\Big(\max\limits_{t\in[0,T]}\big\|u(t)\big\|_n\Big)_{n=1}^{\infty}$, and $x=\lim\limits_{k\to\infty}u_k$ is a solution of (1): It holds that

$$\begin{aligned}
&\bigg\|x(t)-x_0-\int_0^t g\big(s,x(s)\big)+h\big(s,x(s)\big)\,\mathrm{d}s\bigg\|\\
&\quad\leqslant\big\|x(t)-u_{k+1}(t)\big\|+\bigg\|\int_0^t g\big(s,u_{k+1}(s)\big)+h\big(s,u_k(s)\big)-g\big(s,x(s)\big)-h\big(s,x(s)\big)\,\mathrm{d}s\bigg\|\\
&\quad\leqslant\|\!|x-u_{k+1}|\!\|+TL_1\|\!|x-u_{k+1}|\!\|+TL_2\|\!|x-u_k|\!\|\to 0
\end{aligned}$$

in $\mathbb{C}^{\mathbb{N}}$ as $k\to\infty$, $t\in[0,T]$.

Now let $x_1,x_2\in C^1([0,T],F)$ be solutions of (1). With a similar calculation as above we get

$$\big\|x_1(t)-x_2(t)\big\|\leqslant \frac{2T^{k-1}}{(k-2)!}\big(\mathrm{e}^{TL_1}L_2\big)^{k-2}\mathrm{e}^{TL_1}b,\quad t\in[0,T],\quad k\geqslant 2.$$

Since the right-hand side of this inequality tends to 0 in $\mathbb{C}^{\mathbb{N}}$ as $k\to\infty$, we have $x_1=x_2$ and therefore the solution of (1) is unique. □

The solution of (1) is continuously depending on x_0. The following theorem holds.

Theorem 4. *Let $\sigma(L_1)$ be at most countable and provide (4). If $(x_k)_{k=1}^{\infty}$ is a sequence in $C^1([0,T],F)$ such that*

$$\lim_{k\to\infty}x_k(0)=x_0\ \ \text{and}\ \ x_k'(t)=f\big(t,x_k(t)\big),\qquad t\in[0,T],\ k\in\mathbb{N},$$

then $(x_k)_{k=1}^{\infty}$ is tending to the solution of (1) in $\big(C([0,T],F),\|\!|\cdot|\!\|\big)$.

P r o o f. Let x be the solution of (1). It holds for every $k \in \mathbb{N}$ that

$$\left\|x_k'(t) - x'(t)\right\| \leqslant L_1 \left\|x_k(t) - x(t)\right\| + \left\|h(t, x_k(t)) - h(t, x(t))\right\|, \quad t \in [0, T].$$

From Proposition 3 we get

$$\left\|x_k(t) - x(t)\right\| \leqslant \mathrm{e}^{TL_1} \left\|x_k(0) - x_0\right\| + \mathrm{e}^{TL_1} L_2 \int_0^t \left\|x_k(s) - x(s)\right\| \, \mathrm{d}s$$

and

$$\left\|x_k(t) - x(t)\right\| \leqslant \mathrm{e}^{TL_1} \left\|x_k(0) - x_0\right\| + 2T\mathrm{e}^{TL_1} b, \qquad t \in [0, T].$$

Therefore,

$$|||x_k - x||| \leqslant \left(\sum_{j=0}^{m} \frac{T^j (\mathrm{e}^{TL_1} L_2)^j}{j!} \right) \mathrm{e}^{TL_1} \left\|x_k(0) - x_0\right\| + \frac{2T^{m+1}}{m!} \left(\mathrm{e}^{TL_1} L_2\right)^m \mathrm{e}^{TL_1} b,$$

$m \in \mathbb{N}_0$.

Now let $y \in [0, \infty)_{\mathbb{N}}$. It holds that

$$\limsup_{k \to \infty} \left\langle |||x_k - x|||, y \right\rangle \leqslant \left\langle \frac{2T^{m+1}}{m!} (\mathrm{e}^{TL_1} L_2)^m \mathrm{e}^{TL_1} b, y \right\rangle, \quad m \in \mathbb{N}_0.$$

Condition (4) implies the convergence of the right-hand side of this inequality to zero as $m \to \infty$. Therefore, $\lim\limits_{k \to \infty} x_k = x$ in $\left(C([0, T], F), ||| \cdot |||\right)$. □

4. Examples

1) We consider $\left(\mathbb{R}^{\mathbb{N}}, \|\cdot\|\right)$, $\|x\| = \left(|x_n|\right)_{n=1}^{\infty}$, $f(t, x) = g(t, x) + h(t, x)$ with

$$g(t, x) = \left(t^n x_n \arctan(x_n)\right)_{n=1}^{\infty}, \qquad h(t, x) = \left(\alpha_n \arctan(t^n x_{n+1})\right)_{n=1}^{\infty},$$

$(t, x) \in [0, T] \times \mathbb{R}^{\mathbb{N}}$, where $\alpha = (\alpha_n)_{n=1}^{\infty} \in (0, \infty)^{\mathbb{N}}$. We can choose

$$L_1 = \operatorname{diag}\left(\frac{\pi + 1}{2} T^n\right)$$

and

$$L_2 = \begin{pmatrix} 0 & \alpha_1 T & 0 & 0 & \dots \\ 0 & 0 & \alpha_2 T^2 & 0 & \dots \\ 0 & 0 & 0 & \alpha_3 T^3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

$\sigma(L_1)$ is at most countable and $\sigma(L_2)$ is uncountable (see e. g. [7]).

Now,

$$L_2 \mathrm{e}^{TL_1} = L_2 \operatorname{diag}\left(\mathrm{e}^{\frac{\pi+1}{2}\,T^{n+1}}\right)$$
$$= \begin{pmatrix} 0\alpha_1 T\mathrm{e}^{\frac{\pi+1}{2}\,T^3}00\ldots \\ 00\alpha_2 T^2\mathrm{e}^{\frac{\pi+1}{2}\,T^4}0\ldots \\ 000\alpha_3 T^3\mathrm{e}^{\frac{\pi+1}{2}\,T^5}\ \ldots \\ \vdots\vdots\vdots\vdots \end{pmatrix}.$$

Furthermore, $\|h(t,x)\| \leqslant b := \dfrac{\pi}{2}\,\alpha$.

Now assume $\alpha_{n+1}T^n\mathrm{e}^{\frac{\pi+1}{2}\,T^{n+2}} \leqslant c$, $n \in \mathbb{N}$, for some $c > 0$. Then

$$L_2\mathrm{e}^{TL_1}b = \left(\frac{\pi}{2}\alpha_n\alpha_{n+1}T^n\mathrm{e}^{\frac{\pi+1}{2}\,T^{n+2}}\right)_{n=1}^{\infty} \leqslant c\left(\frac{\pi}{2}\,\alpha_n\right)_{n=1}^{\infty} = cb.$$

Then (5) holds and, according to Theorem 3, (1) is uniquely solvable for every $x_0 \in \mathbb{R}^{\mathbb{N}}$.

Remark that $L = L_1 + L_2$ is a Lipschitz matrix for f in (2) and that $\sigma(L)$ is uncountable. Hence Theorem 1 is not applicable. Since f is not bounded in $\mathbb{R}^{\mathbb{N}}$, also Theorem 2 is not applicable.

2) We consider $\big(C([1,\infty),\mathbb{R}), \|\cdot\|\big)$, $\|x\| = \left(\max\limits_{s\in[n,n+1]}|x(s)|\right)_{n=1}^{\infty}$, $f(x) = g(x)+h(x)$ with

$$\big(g(x)\big)(s) = x(s+1)\max\{\sin(\pi s), 0\},$$
$$\big(h(x)\big)(s) = \arctan\big(x(s+1)\big)\max\{\sin(\pi(s+1)), 0\},$$

$s \in [1,\infty)$, $(t,x) \in [0,T] \times C([1,\infty),\mathbb{R})$.

We can choose

$$L_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

and

$$L_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

In this example, $\sigma(L_1) = \sigma(L_2) = \{0\}$, but $\sigma(L_1+L_2) = \mathbb{C}$ (cf. [7]), and $L_1^2 = L_2^2 = 0$.

We have

$$L_2 e^{TL_1} = L_2(I + TL_1) = L_2 + TL_2L_1 = \begin{pmatrix} 0 & 1 & T & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & T & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & T & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix},$$

and it holds that $\|h(x)\| \leqslant b := \Big(\frac{\pi}{4}\big(1 + (-1)^{n+1}\big)\Big)_{n=1}^{\infty}$.

Therefore $L_2 e^{TL_1} b = Tb$. Hence (5) is satisfied and, using Theorem 3, the initial value problem (1) is uniquely solvable for every $x_0 \in C([1,\infty), \mathbb{R})$.

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