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A SUPERSOLUTION-SUBSOLUTION METHOD FOR NONLINEAR BIHARMONIC EQUATIONS IN $\mathbb{R}^N$

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Dedicated to Professor Masuo Hukuhara on the occasion of his ninetieth birthday

0. INTRODUCTION

The method of supersolutions and subsolutions, or the method of barriers, has proved to be a powerful tool for establishing the existence of solutions of second order elliptic partial differential equations in unbounded domains, particularly, in the whole Euclidean space $\mathbb{R}^N$ ([1, 10, 12, 13]). Since this method depends heavily on the maximum principle for second order elliptic operators and since no such maximum principle holds for higher order elliptic operators, there is no supersolution-subsolution method of general nature which enables to construct solutions of elliptic equations of higher order. It would not be so unnatural, however, to expect that one might develop, in a spirit similar to the second order case, an existence principle, which could be called a higher order supersolution-subsolution method, for solving higher order elliptic problems in unbounded domains, provided one considers a severely restricted class of elliptic equations. The truth of this expectation has recently been observed by Furusho and Kusano [7] and Kusano and Swanson [11] who have proposed new principles of super- and subsolutions which can be effectively used to construct bounded solutions for a special class of equations including perturbed biharmonic equations of the form $\Delta^2 u = f(x,u)$, $x \in \mathbb{R}^N$. A close look at the papers [7, 11] suggests that the basic principles obtained therein can possibly be extended or refined so as to be able to cover a wider class of higher order elliptic equations.

The purpose of this paper is to provide a refined version of the supersolution-subsolution method [7, 11] which is applicable to the nonlinear biharmonic equation

$$\text{(A)} \qquad \Delta(|\Delta u|^{p-2}\Delta u) = f(x,u), \quad x \in \mathbb{R}^N,$$

where $p > 1$, $N \geqslant 3$, and f is a function of class $C^{\alpha}_{\mathrm{loc}}(\mathbb{R}^N \times \mathbb{R}_+)$ for some $\alpha \in (0,1)$, $\mathbb{R}_+ = (0,\infty)$. Our attention will be restricted to bounded positive entire solutions $u \in C^{2+\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ such that $|\Delta u|^{p-2}\Delta u \in C^{2+\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ and satisfy the equation throughout $\mathbb{R}^N$. Such a solution is said to be symmetric if it depends only on $|x| = \left(\sum_{i=1}^{N} x_i^2\right)^{1/2}$. The main existence principle given in Section 1 states that if suitable "supersolutions" and "subsolutions" are shown to exist for the equation (A), then (A) does possess entire solutions lying between them. Establishing the existence of entire solutions of (A), therefore, depends on the possibility of constructing supersolutions and subsolutions with the required properties. In Sections 3 and 4 we prove four theorems ensuring the existence of bounded positive entire solutions with different asymptotic behaviors of (A) by constructing supersolutions and subsolutions with the corresponding asymptotic behaviors as symmetric entire solutions of symmetric equations of the type

$$\text{(B)} \qquad \Delta(|\Delta u|^{p-2}\Delta u) = g(|x|,u), \quad x \in \mathbb{R}^N.$$

The problem is thus reduced to solving an ordinary differential equation which is the one-dimensional polar form of (B). The desired solutions of this ODE are obtained as solutions of suitable integral equations which are formed in terms of "inverse", denoted by Ψ_p (see (2.4)), of the polar form of the "p-biharmonic" operator $\Delta(|\Delta u|^{p-2}\Delta u)$. For this purpose a crucial role is played by the estimates for the asymptotic behavior of Ψ_p stated in Section 2. An example illustrating our main results is presented in Section 5.

Theoretical importance of p-biharmonic operator has been recognized by several authors including Drábek, Fučík and Kufner; see e.g. [2, 3, 4]. The existing literature on the p-biharmonic operator seems to be concerned exclusively with boundary value problems in bounded domains, and this observation motivated our attempt at studying perturbed p-biharmonic equations in unbounded domains.

1. Supersolution-subsolution principle

We begin by stating a supersolution-subsolution principle on which our construction of entire solutions of the equation (A) is based.

Theorem 1.1. *Let f be a function of class $C^{\alpha}_{\text{loc}}(\mathbb{R}^N \times \mathbb{R})$. Suppose that there exist functions $v, w \in C^{2+\alpha}_{\text{loc}}(\mathbb{R}^N)$ such that*

$$|\Delta v|^{p-2}\Delta v, |\Delta w|^{p-2}\Delta w \in C^{2+\alpha}_{\text{loc}}(\mathbb{R}^N), \tag{1.1}$$

$$v(x) \geqslant w(x), \quad -\Delta v(x) \geqslant -\Delta w(x), \quad x \in \mathbb{R}^N, \tag{1.2}$$

and, for every fixed $x \in \mathbb{R}^N$, the inequalities

$$\Delta(|\Delta v(x)|^{p-2}\Delta v(x)) \geqslant f(x,\sigma) \geqslant \Delta(|\Delta w(x)|^{p-2}\Delta w(x)) \tag{1.3}$$

hold for all σ satisfying $v(x) \geqslant \sigma \geqslant w(x)$. Then, the equation (A) *has an entire solution u such that*

$$v(x) \geqslant u(x) \geqslant w(x), \quad -\Delta v(x) \geqslant -\Delta u(x) \geqslant -\Delta w(x), \quad x \in \mathbb{R}^N. \tag{1.4}$$

The functions $v(x)$ and $w(x)$ satisfying the conditions (1.1)–(1.3) are called, respectively, a supersolution and a subsolution of the equation (A).

As is easily seen, when $f(x,u)$ is monotone in u the condition (1.3) in the above theorem takes a simpler form as described in the following corollaries.

Corollary 1.1. *Suppose that $f(x,u)$ is nondecreasing in u for each fixed $x \in \mathbb{R}^N$. If there exist functions $v, w \in C^{2+\alpha}_{\text{loc}}(\mathbb{R}^N)$ satisfying* (1.1), (1.2) *and*

$$\Delta(|\Delta v|^{p-2}\Delta v) \geqslant f(x,v), \quad \Delta(|\Delta w|^{p-2}\Delta w) \leqslant f(x,w), \quad x \in \mathbb{R}^N, \tag{1.5}$$

then (A) *has an entire solution u satisfying* (1.4).

Corollary 1.2. *Suppose that $f(x,u)$ is nonincreasing in u for each fixed $x \in \mathbb{R}^N$. If there exist functions $v, w \in C^{2+\alpha}_{\text{loc}}(\mathbb{R}^N)$ satisfying* (1.1), (1.2) *and*

$$\Delta(|\Delta v|^{p-2}\Delta v) \geqslant f(x,w), \quad \Delta(|\Delta w|^{p-2}\Delta w) \leqslant f(x,v), \quad x \in \mathbb{R}^N, \tag{1.6}$$

then (A) *has an entire solution u satisfying* (1.4).

Recently Furusho [6] has established a general existence principle for second order semilinear elliptic systems, of which the following theorem is a special case.

Theorem 0. *Let $\varphi(t,u)$ and $\psi(t,u)$ be functions of class $C^{\alpha}_{\text{loc}}(\mathbb{R}^N \times \mathbb{R})$. Suppose that there exist functions v_1, v_2, w_1, w_2 of class $C^{2+\alpha}_{\text{loc}}(\mathbb{R}^N)$ such that*

$$v_1(x) \geqslant w_1(x), \quad v_2(x) \geqslant w_2(x), \quad x \in \mathbb{R}^N, \tag{1.7}$$

and, for each fixed $x \in \mathbb{R}^N$, the inequalities

$$-\Delta v_1(x) \geqslant \varphi(x,\tau) \geqslant -\Delta w_1(x), \quad -\Delta v_2(x) \geqslant \psi(x,\sigma) \geqslant -\Delta w_2(x), \tag{1.8}$$

hold for all σ and τ satisfying

$$v_1(x) \geqslant \sigma \geqslant w_1(x), \quad v_2(x) \geqslant \tau \geqslant w_2(x). \tag{1.9}$$

Then there exists a solution $(u_1, u_2) \in C^{2+\alpha}_{\text{loc}}(\mathbb{R}^N) \times C^{2+\alpha}_{\text{loc}}(\mathbb{R}^N)$ of the system

$$-\Delta u_1 = \varphi(x, u_2), \quad -\Delta u_2 = \psi(x, u_1), \quad x \in \mathbb{R}^N, \tag{1.10}$$

such that

$$v_1(x) \geqslant u_1(x) \geqslant w_1(x), \quad v_2(x) \geqslant u_2(x) \geqslant w_2(x), \; x \in \mathbb{R}^N. \tag{1.11}$$

We will show that Theorem 1.1 follows from Theorem 0. For simplicity we introduce the notation

$$\xi^{a*} = |\xi|^{a-1}\xi = |\xi|^a \operatorname{sgn} \xi, \quad \xi \in \mathbb{R}, \quad a > 0, \tag{1.12}$$

in terms of which the p-biharmonic operator is written as $\Delta(\Delta u)^{(p-1)*}$.

Suppose that v and w satisfy the conditions of Theorem 1.1. We put

$$v_1 = v, \quad w_1 = w, \quad v_2 = -(\Delta v)^{(p-1)*}, \quad w_2 = -(\Delta w)^{(p-1)*}. \tag{1.13}$$

Then, (1.2) implies that $v_1 \geqslant w_1$ and $v_2 \geqslant w_2$ in $\mathbb{R}^N$. Let $x \in \mathbb{R}^N$ be fixed and let σ and τ satisfy the inequalities $v_1(x) \geqslant \sigma \geqslant w_1(w)$ and $v_2(x) \geqslant \tau \geqslant w_2(x)$. From (1.3) it follows that

$$-\Delta v_1(x) = (v_2(x))^{\frac{1}{p-1}*} \geqslant \tau^{\frac{1}{p-1}*} \geqslant (w_2(x))^{\frac{1}{p-1}*} = -\Delta w_1(x)$$

and

$$-\Delta v_2(x) \geqslant f(x,\sigma) \geqslant -\Delta w_2(x).$$

Theorem 0 then ensures the existence of functions u_1, u_2 of class $C^{2+\alpha}_{\text{loc}}(\mathbb{R}^N)$ such that

$$-\Delta u_1 = u_2^{\frac{1}{p-1}*}, \quad -\Delta u_2 = f(x, u_1)$$

and

$$v_1 \geqslant u_1 \geqslant w_1, \quad v_2 \geqslant u_2 \geqslant w_2$$

in $\mathbb{R}^N$, which, in view of (1.13), implies that the function $u = u_1$ is an entire solution of (A) satisfying (1.4).

2. One-dimensional inverse of the p-biharmonic operator

In order to prove the existence of an entire solution of the equation (A) on the basis of Theorem 1.1 or its corollaries we have to construct its supersolutions and subsolutions with the required properties. The actual construction of such super- and subsolutions of (A) will be the subject of the next section. The purpose of this section of preparatory nature is to summarized the results concerning a type of "one-dimensional inverse" of the p-biharmonic operator $\Delta((\Delta u)^{(p-1)*}) \equiv \Delta(|\Delta u|^{p-2}\Delta u)$ which will play a crucial role in the development of Section 3.

We denote by $L^1_\lambda (\lambda > 0)$ the set of all real-valued measurable functions $h(t)$ on $\mathbb{R}_+$ such that

$$\int_0^\infty s^\lambda |h(s)| \, \mathrm{d}s < \infty.$$

We define the integral operator Ψ acting on $C(\overline{\mathbb{R}}_+) \cap L^1_1$, $\overline{\mathbb{R}}_+ = [0, \infty)$, by

$$\begin{aligned}
(\Psi h)(t) &= t^{2-N} \int_0^t \left(s^{N-3} \int_s^\infty r h(r) \, \mathrm{d}r \right) \mathrm{d}s \\
&= \frac{1}{N-2} \left[\int_0^t \left(\frac{s}{t}\right)^{N-2} s h(s) \, \mathrm{d}s + \int_t^\infty s h(s) \, \mathrm{d}s \right], \quad t > 0, \qquad (2.1) \\
(\Psi h)(0) &= \lim_{t \to +0} (\Psi h)(t).
\end{aligned}$$

It is easy to verify that Ψ maps $C(\overline{\mathbb{R}}_+) \cap L^1_1$ into $C^2(\overline{\mathbb{R}}_+)$ and that

$$\lim_{t \to \infty} (\Psi h)(t) = 0, \tag{2.2}$$

$$\Delta(\Psi h)(|x|) = -h(|x|), \quad x \in \mathbb{R}^N \tag{2.3}$$

for every $h \in C(\overline{\mathbb{R}}_+) \cap L^1_1$. This shows that $-\Psi$ is a type of inverse of one-dimensional polar form of the Laplace operator $\Delta \cdot = t^{1-N} D_t (t^{N-1} D_t \cdot)$, $D_t = \mathrm{d}/\mathrm{d}t$. More detailed information about (2.2) is given in the following lemma.

Lemma 2.1. (i) *If $h \in C(\overline{\mathbb{R}}_+) \cap L^1_1$ and $h(t) \geqslant 0$ for $t \geqslant 0$, then*

$$\frac{1}{N-2}\int_0^\infty \min\{s, s^{N-1}\} h(s)\,\mathrm{d}s \cdot t_*^{2-N} \leqslant (\Psi h)(t) \leqslant \frac{1}{N-2}\int_0^\infty sh(s)\,\mathrm{d}s, \quad t \geqslant 0,$$

where $t_* = \max\{1, t\}$.

(ii) *If $h \in C(\overline{\mathbb{R}}_+) \cap L^1_{N-1}$ and $h(t) \geqslant 0$ for $t \geqslant 0$, then*

$$\frac{1}{N-2}\int_0^\infty \min\{s, s^{N-1}\} h(s)\,\mathrm{d}s \cdot t_*^{2-N} \leqslant (\Psi h)(t)$$
$$\leqslant \frac{1}{N-2}\int_0^\infty \max\{s, s^{N-1}\} h(s)\,\mathrm{d}s \cdot t_*^{2-N}, \quad t \geqslant 0.$$

For the definition and the properties of Ψ see Fukagai [5] and Kusano et al [9].

Let us consider the operator Ψ_p defined by

$$\Psi_p \cdot = \Psi(\Psi(\cdot)^{\frac{1}{p-1}*}). \tag{2.4}$$

For the meaning of $*$ see (1.12). If $h \in C(\overline{\mathbb{R}}_+)$ is in the domain of this operator, denoted by $\mathrm{dom}(\Psi_p)$, application of (2.3) shows that the function $u(x) = (\Psi_p h)(|x|)$ satisfies the equation

$$\Delta((\Delta u(x))^{(p-1)*}) = h(|x|), \quad x \in \mathbb{R}^N,$$

so that Ψ_p is a kind of inverse of the one-dimensional polar form of the p-biharmonic operator

$$\Delta((\Delta\cdot)^{(p-1)*}) \equiv t^{1-N} D_t(t^{N-1} D_t(t^{1-N} D_t(t^{N-1} D_t \cdot))^{(p-1)*}).$$

In what follows we restrict our attention to the case where $N > 2p$ and obtain useful results on the domain and the asymptotic decay of Ψ_p by using the decaying function $q_{N,p}(t)$ defined by

$$q_{N,p}(t) = \begin{cases} t_*^{2-N} & \text{if } 1 < p < 2 - \frac{2}{N}, \\ t_*^{2-N}\log(et_*) & \text{if } p = 2 - \frac{2}{N}, \\ t_*^{\frac{2p-N}{p-1}} & \text{if } 2 - \frac{2}{N} < p < \frac{N}{2}. \end{cases} \tag{2.5}$$

Lemma 2.2. *Suppose that $p > 1$ and $N > 2p$.*

(i) *If* $1 < p \leqslant 2$, *then* $C(\overline{\mathbb{R}}_+) \cap L^1_{2p-1} \subset dom(\Psi_p)$. *If* $h \in C(\overline{\mathbb{R}}_+) \cap L^1_{2p-1}$ and $h(t) \geqslant 0$ *for* $t \geqslant 0$, *then*

$$(2.6) \qquad C_1(N,p)\left(\int_0^\infty \min\{s, s^{N-1}\}h(s)\,\mathrm{d}s\right)^{\frac{1}{p-1}} q_{N,p}(t) \leqslant (\Psi_p h)(t) \leqslant C_2(N,p)\left(\int_0^\infty s_*^{2p-1}h(s)\,\mathrm{d}s\right)^{\frac{1}{p-1}}, \quad t \geqslant 0,$$

for some positive constants $C_1(N,p)$ *and* $C_2(N,p)$ *depending only on* N *and* p.

(ii) *If* $p > 2$, *then* $C(\overline{\mathbb{R}}_+) \cap L^1_{2p'-1} \subset dom(\Psi_p)$ *for any* p' *with* $p' > p$ *and* $N > 2p'$. *If* $h \in C(\overline{\mathbb{R}}_+) \cap L^1_{2p'-1}$ *for such a* p' *and* $h(t) \geqslant 0$ *for* $t \geqslant 0$, *then*

$$(2.7) \qquad C_1(N,p)\left(\int_0^\infty \min\{s, s^{N-1}\}h(s)\,\mathrm{d}s\right)^{\frac{1}{p-1}} q_{N,p}(t) \leqslant (\Psi_p h)(t) \leqslant C_2'(N,p,p')\left(\int_0^\infty s_*^{2p'-1}h(s)\,\mathrm{d}s\right)^{\frac{1}{p-1}}, \quad t \geqslant 0,$$

where $C_1(N,p)$ *is as in* (i) *and* $C_2'(N,p,p')$ *is a constant depending only on* N, p *and* p'.

Lemma 2.3. *Suppose that* $p > 1$ *and* $N > 2p$. *If* $h \in C(\overline{\mathbb{R}}_+) \cap L^1_{N-1}$ and $h(t) \geqslant 0$ *for* $t \geqslant 0$, *then*

$$(2.8) \quad C_1(N,p)\left(\int_0^\infty \min\{s, s^{N-1}\}h(s)\,\mathrm{d}s\right)^{\frac{1}{p-1}} q_{N,p}(t) \leqslant (\Psi_p h)(t) \leqslant C_3(N,p)\left(\int_0^\infty \max\{s, s^{N-1}\}h(s)\,\mathrm{d}s\right)^{\frac{1}{p-1}} q_{N,p}(t), \quad t \geqslant 0,$$

where $C_1(N,p)$ *is as in* (i) *of Lemma 2.2 and* $C_3(N,p)$ *is a constant depending only on* N *and* p.

To prove Lemma 2.2 it suffices to combine Lemma 2.1 with Lemma 1.3 of Furusho and Kusano [8]. We omit the details. The proof of Lemma 2.3 is given in [8; Lemma 1.6].

3. Existence of uniformly positive entire solutions

Existence theorems for the equation (A) will be obtained in this and the next sections under the standing hypothesis

(F_1) $f \in C^{\alpha}_{\mathrm{loc}}(\mathbb{R}^N \times \mathbb{R}_+)$ for some $\alpha \in (0,1)$, and there is a function $F \in C^{\alpha}_{\mathrm{loc}}(\overline{\mathbb{R}}_+ \times \mathbb{R}_+)$ such that

$$|f(x,u)| \leqslant F(|x|,u) \quad \text{for } (x,u) \in \mathbb{R}^N \times \mathbb{R}_+.$$

This section concerns positive entire solutions which are asymptotic to positive constants as x tends to infinity. An additional hypothesis for $F(t,u)$ is selected from the list below.

(F_2) $F(t,u)$ is nondecreasing in u for any fixed $t \geqslant 0$. For any fixed $(t,u) \in \overline{\mathbb{R}}_+ \times \mathbb{R}_+$, $F(t,\lambda u)/\lambda^{p-1}$ is nondecreasing in $\lambda > 0$ and

$$\lim_{\lambda \to +0} \frac{F(t,\lambda u)}{\lambda^{p-1}} = 0.$$

(F_3) $F(t,u)$ is nondecreasing in u for any fixed $t \geqslant 0$. For any fixed $(t,u) \in \overline{\mathbb{R}}_+ \times \mathbb{R}_+$, $F(t,\lambda u)/\lambda^{p-1}$ is nonincreasing in $\lambda > 0$ and

$$\lim_{\lambda \to \infty} \frac{F(t,\lambda u)}{\lambda^{p-1}} = 0.$$

(F_4) $F(t,u)$ is nonincreasing in u for any fixed $t \geqslant 0$.

Our first main theorem is stated below.

Theorem 3.1. *Let $p > 1$ and $N > 2p$ and let (F_1) and one of (F_2), (F_3), and (F_4) be satisfied. Suppose that if $1 < p \leqslant 2$ then there exists a positive constant $c > 0$ such that*

$$\int_0^{\infty} t^{2p-1} F(t,c)\,\mathrm{d}t < \infty \tag{3.1}$$

and that if $p > 2$ then there exist positive constants c and p' such that $p' > p$, $N > 2p'$ and

$$\int_0^{\infty} t^{2p'-1} F(t,c)\,\mathrm{d}t < \infty. \tag{3.2}$$

Then the equation (A) possesses infinitely many positive entire solutions $u(x)$ such that

$$\lim_{|x| \to \infty} u(x) = \textit{constant} > 0 \quad \textit{and} \quad \lim_{|x| \to \infty} \Delta u(x) = 0. \tag{3.3}$$

Proof. The proof is given only for the case where $1 < p \leqslant 2$, since an almost parallel argument holds for the case where $p > 2$. Suppose first that either (F_2) or (F_3) is satisfied. In view of (3.1) we then see that

$$\lim_{k \to *} k^{1-p} \int_0^\infty t^{2p-1} F(t, 2k)\,\mathrm{d}t = 0,$$

where $* = 0$ or ∞ according as (F_2) or (F_3) holds, and hence there exists an interval $I \subset \mathbb{R}_+$ with the property that

$$C_2(N,p)\left(\int_0^\infty s_*^{2p-1} F(s, 2k)\,\mathrm{d}s\right)^{\frac{1}{p-1}} \leqslant k \quad \text{for all } k \in I, \tag{3.4}$$

where $C_2(N,p)$ is the constant which appeared in Lemma 2.2.

Let $k \in I$ be fixed arbitrarily and consider the set $Y \subset C(\overline{\mathbb{R}}_+)$ and the mapping $\mathscr{F}_1 \colon Y \to C(\overline{\mathbb{R}}_+)$ defined by

$$Y = \{y \in C(\overline{\mathbb{R}}_+) \colon k \leqslant y(t) \leqslant 2k, \quad t \geqslant 0\},$$

and

$$(\mathscr{F}_1 y)(t) = k + [\Psi_p F(\cdot, y)](t) \equiv k + [\Psi(\Psi F(\cdot, y))^{\frac{1}{p-1}}](t), \quad t \geqslant 0.$$

It can be shown that $\mathscr{F}_1$ maps Y, which is a closed convex subset of the Fréchet space $C(\overline{\mathbb{R}}_+)$ with the usual metric topology, continuously into a relatively compact subset of Y.

(i) Let $y \in Y$. Then, clearly $(\mathscr{F}_1 y)(t) \geqslant k$ for $t \geqslant 0$. Using Lemma 2.2 and (3.4) we have

$$\begin{aligned}
[\Psi_p F(\cdot, y)](t) &\leqslant C_2(N,p)\left(\int_0^\infty s_*^{2p-1} F(s, y(s))\,\mathrm{d}s\right)^{\frac{1}{p-1}} \\
&\leqslant C_2(N,p)\left(\int_0^\infty s_*^{2p-1} F(s, 2k)\,\mathrm{d}s\right)^{\frac{1}{p-1}} \leqslant k, \quad t \geqslant 0,
\end{aligned}$$

which shows that $(\mathscr{F}_1 y)(t) \leqslant 2k$ for $t \geqslant 0$. It follows that $\mathscr{F}_1(Y) \subset Y$.

(ii) To prove the continuity of $\mathscr{F}_1$ let $\{y_\nu\}$ be a sequence in Y converging to $y \in Y$ in $C(\overline{\mathbb{R}}_+)$. First note that

$$\begin{aligned}
&|[\Psi F(\cdot, y_\nu)](t) - [\Psi F(\cdot, y)](t)| \\
&\quad = \frac{1}{N-2}\left|\int_0^t (s/t)^{N-2} s(F(s, y_\nu(s)) - F(s, y(s)))\,\mathrm{d}s\right. \\
&\qquad \left. + \int_t^\infty s(F(s, y_\nu(s)) - F(s, y(s)))\,\mathrm{d}s\right| \\
&\quad \leqslant \frac{1}{N-2}\int_0^\infty s|F(s, y_\nu(s)) - F(s, y(s))|\,\mathrm{d}s, \quad t \geqslant 0.
\end{aligned} \tag{3.5}$$

Since $s|F(s,y_\nu(s)) - F(s,y(s))| \leqslant 2sF(s,2k)$, $s \geqslant 0$, and $sF(s,2k)$ is integrable over $\overline{\mathbb{R}}_+$ by (3.1), the Lebesgue dominated convergence theorem implies that

$$\lim_{\nu\to\infty} \int_0^\infty s|F(s,y_\nu(s)) - F(s,y(s))|\,\mathrm{d}s = 0. \tag{3.6}$$

From (3.5) and (3.6) it follows that $[\Psi F(\cdot,y_\nu)](t)$ converges to $\Psi F(\cdot,y)(t)$ uniformly on $\overline{\mathbb{R}}_+$. Using this fact, the definition of Ψ_p (cf. (2.4)) and (i) of Lemma 2.1 we conclude that $[\Psi_p F(\cdot,y_\nu)](t)$ converges to $[\Psi_p F(\cdot,y)](t)$ uniformly on $\overline{\mathbb{R}}_+$. This establishes the continuity of $\mathscr{F}_1$ in the topology of $C(\overline{\mathbb{R}}_+)$.

(iii) The set $\mathscr{F}_1(Y)$ is uniformly bounded on $\overline{\mathbb{R}}_+$ since $\mathscr{F}_1(Y) \subset Y$. Let $y \in Y$. Note that $F(s,y(s)) \leqslant F(s,2k), s \geqslant 0$. Since the condition $F(\cdot,2k) \in L^1_{2p-1}$ implies that $(\Psi F(\cdot,2k))^{\frac{1}{p-1}} \in L^1_1$ (cf. Lemma 1.3 of [8]), we obtain

$$\begin{aligned}|(\mathscr{F}_1 y)'(t)| &= \int_0^t (s/t)^{N-1}[\Psi F(\cdot,y)]^{\frac{1}{p-1}}(s)\,\mathrm{d}s \\ &\leqslant \int_0^t [\Psi F(\cdot,2k)]^{\frac{1}{p-1}}(s)\,\mathrm{d}s \leqslant \int_0^\infty [\Psi F(\cdot,2k)]^{\frac{1}{p-1}}(s)\,\mathrm{d}s, \quad t \geqslant 0,\end{aligned}$$

which implies that $\mathscr{F}_1(Y)$ is equicontinuous on $\overline{\mathbb{R}}_+$. This proves the relative compactness of $\mathscr{F}_1(Y)$ in $C(\overline{\mathbb{R}}_+)$.

Thus all the conditions of the Schauder-Tychonoff fixed point theorem are satisfied for $\mathscr{F}_1$, and so there exists $y \in Y$ such that $y = \mathscr{F}_1 y$, that is,

$$y(t) = k + [\Psi_p F(\cdot,y)](t) \equiv k + [\Psi(\Psi F(\cdot,y))^{\frac{1}{p-1}}](t), \quad t \geqslant 0. \tag{3.7}$$

Define the functions v and w by

$$v(x) = y(|x|), \quad w(x) = 2k - y(|x|), \quad x \in \mathbb{R}^N. \tag{3.8}$$

We then easily see that

$$\begin{aligned}v(x) - w(x) = 2(y(|x|) - k) \geqslant 0, \quad x \in \mathbb{R}^N,\\ -\Delta v(x) = [\Psi F(\cdot,y)]^{\frac{1}{p-1}}(|x|) \geqslant -[\Psi F(\cdot,y)]^{\frac{1}{p-1}}(|x|) = -\Delta w(x), \quad x \in \mathbb{R}^N.\end{aligned}$$

Furthermore, for any fixed $x \in \mathbb{R}^N$, we have

$$\begin{aligned}\Delta(|\Delta v(x)|^{p-2}\Delta v(x)) &= F(|x|,v(x)) \geqslant F(|x|,\sigma) \geqslant f(x,\sigma) \\ &\geqslant -F(|x|,\sigma) \geqslant -F(|x|,v(x)) = \Delta(|\Delta w(x)|^{p-2}\Delta w(x))\end{aligned}$$

provided $v(x) \geqslant \sigma \geqslant w(x)$. This shows that $v(x)$ and $w(x)$ are a supersolution and a subsolution of (A) satisfying the conditions of Theorem 1.1. Consequently, the equation (A) has an entire solution $u(x)$ such that

$$v(x) \geqslant u(x) \geqslant w(x), \quad -\Delta v(x) \geqslant -\Delta u(x) \geqslant -\Delta w(x), \quad x \in \mathbb{R}^N. \tag{3.9}$$

Since by (3.7) and (3.8)

$$\lim_{|x|\to\infty} v(x) = \lim_{t\to\infty} y(t) = k, \quad \lim_{|x|\to\infty} w(x) = k,$$
$$\lim_{|x|\to\infty} \Delta v(x) = -\lim_{t\to\infty} [\Psi F(\cdot, y)](t) = 0, \quad \lim_{|x|\to\infty} \Delta w(x) = 0,$$

from (3.9) it follows that

$$\lim_{|x|\to\infty} u(x) = k, \quad \lim_{|x|\to\infty} \Delta u(x) = 0. \tag{3.10}$$

This implies that $u(x)$ is an entire solution of (A) satisfying (3.3). Since k is an arbitrary constant chosen from the interval I (cf. (3.4)), the above construction shows that (A) has an infinitude of positive entire solutions with the required asymptotic property.

Next suppose that the hypothesis (F_4) is satisfied. In this case we have $\lim_{k\to\infty} k^{1-p} \cdot \int_0^\infty t^{2p-1} F(t,k)\,dt = 0$ by (3.1), and so there is an interval $I \subset \mathbb{R}_+$ such that

$$C_2(N,p)\left(\int_0^\infty t^{2p-1} F(t,k)\,dt\right)^{\frac{1}{p-1}} \leqslant \tfrac{1}{2}k \quad \text{for all } k \in I.$$

For any fixed $k \in I$ define

$$Z = \{z \in C(\overline{\mathbb{R}}_+) \colon \tfrac{1}{2}k \leqslant z(t) \leqslant k,\ t \geqslant 0\},$$
$$(\mathscr{F}_2 z)(t) = k - [\Psi_p F(\cdot, z)](t), \quad t \geqslant 0.$$

By the Schauder-Tychonoff fixed point theorem, there exists a function z in Z such that

$$z(t) = k - [\Psi_p F(\cdot, z)](t), \quad t \geqslant 0.$$

It is easy to show that the functions $v(x)$ and $w(x)$ defined by

$$v(x) = 2k - z(|x|), \quad w(x) = z(|x|), \quad x \in \mathbb{R}^N$$

are a supersolution and a subsolution of (A) satisfying the conditions of Theorem 1.1. Therefore (A) has an entire solution $u(x)$ satisfying (3.9). Since $\lim_{|x|\to\infty} w(x) = \lim_{t\to\infty} z(t) = k$ and $\lim_{|x|\to\infty} \Delta w(x) = \lim_{t\to\infty} [\Psi F(\cdot, z(t))]^{\frac{1}{p-1}}(t) = 0$, it follows that the solution $u(x)$ satisfies (3.10). Since $k \in I$ is arbitrary, there exist infinitely many such entire solutions of (A). The proof of Theorem 3.1 is thus complete. □

Remark. A question arises: When are the solutions obtained in Theorem 3.1 in $C^4(\mathbb{R}^N)$? A partial answer to this question now follows.

Let u be such an entire solution of (A). Then the functions $u_1 = u$, $u_2 = -(\Delta u)^{(p-1)*}$ satisfy

$$-\Delta u_1 = u_2^{\frac{1}{p-1}*}, \quad -\Delta u_2 = f(x, u_1), \quad x \in \mathbb{R}^N. \tag{3.11}$$

Since $f(x, u_1) \in C^\alpha_{\mathrm{loc}}(\mathbb{R}^N)$, from the second equation in (3.11) and the Schauder estimates we see that $u_2 \in C^{2+\beta}_{\mathrm{loc}}(\mathbb{R}^N)$ for some $\beta \in (0, \alpha)$.

Suppose that $f(x,u) > 0$ [or < 0] for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}_+$. Since $-\Delta u_2 > 0$ [or < 0] in $\mathbb{R}^N$ and $\lim\limits_{|x|\to\infty} u_2(x) = 0$, the maximum principle implies that $u_2(x) > 0$ [or < 0] throughout $\mathbb{R}^N$. Hence $u_2^{\frac{1}{p-1}*} \in C^{2+\gamma}_{\mathrm{loc}}(\mathbb{R}^N)$ for some $\gamma \in (0, \beta)$, and using this fact we conclude from the first equation in (3.11) that $u = u_1 \in C^{4+\gamma}_{\mathrm{loc}}(\mathbb{R}^N)$. Note that p may be any constant greater that 1.

In case $f(x,u)$ is not of constant sign, u_2 may have a zero in $\mathbb{R}^N$, in which case we have to restrict the value of p within the interval $1 < p < \frac{3}{2}$ in order to have that $u_2^{\frac{1}{p-1}*} \in C^{2+\delta}_{\mathrm{loc}}(\mathbb{R}^N)$ for some $\delta \in (0, \beta)$ and hence $u \in C^{4+\delta}_{\mathrm{loc}}(\mathbb{R}^N)$.

4. Existence of decaying positive entire solutions

Our purpose here is to construct, by means of Theorem 1.1, positive entire solutions which decay to zero as $|x| \to \infty$. The following structure hypotheses for $f(x,u)$ are needed for this purpose.

(F_5) $f \in C^\alpha_{\mathrm{loc}}(\mathbb{R}^N \times \mathbb{R}_+), \alpha \in (0,1)$; f satisfies

$$0 < \varphi(|x|, u) \leqslant f(x,u) \leqslant F(|x|, u), \quad (x,u) \in \mathbb{R}^N \times \mathbb{R}_+, \tag{4.1}$$

where φ and F are functions a of class $C^\alpha_{\mathrm{loc}}(\overline{\mathbb{R}}_+ \times \mathbb{R}_+)$ such that $\varphi(t,u)$ and $F(t,u)$ are nondecreasing in u for each fixed $t \geqslant 0$. In addition F satisfies (F_3) and φ satisfies the following conditions:

$\varphi(t, \lambda u)/\lambda^{p-1}$ is nonincreasing in $\lambda > 0$ and $t \geqslant 0$ and $u > 0$;

$\lim\limits_{\lambda\to+0} \varphi(t, \lambda u)/\lambda^{p-1} = \infty$ for any t in some interval $J \subset \overline{\mathbb{R}}_+$ and for any $u > 0$.

(F_6) $f \in C^\alpha_{\mathrm{loc}}(\mathbb{R}^N \times \mathbb{R}_+), \alpha \in (0,1)$; f satisfies (4.1) for some functions $\varphi(t,u)$ and $F(t,u)$ of class $C^\alpha_{\mathrm{loc}}(\overline{\mathbb{R}}_+ \times \mathbb{R}_+)$ which are nonincreasing in u for any $t \geqslant 0$. In addition there exists a positive constant ϱ such that

$$F(t, \lambda u) \leqslant \lambda^{-\varrho} F(t,u) \quad \text{for any } \lambda > 0 \quad \text{and} \quad (t,u) \in \overline{\mathbb{R}}_+ \times \mathbb{R}_+, \tag{4.2}$$

$$\lim_{\lambda\to+0} \frac{\varphi(t, \lambda^{-\frac{\varrho}{p-1}})}{\lambda^{p-1}} = \infty \quad \text{uniformly for } t \in [0,1]. \tag{4.3}$$

Theorem 4.1. *Let $p > 1$ and $N > 2p$ and suppose that* (F_5) *is satisfied. Suppose in addition that* (3.1) *holds if $1 < p \leqslant 2$ and* (3.2) *holds if $p > 2$. Then the equation* (A) *possesses a positive entire solution $u \in C^4(\mathbb{R}^N)$ such that*

$$\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} \Delta u(x) = 0. \tag{4.4}$$

Theorem 4.2. *Let $p > 1$ and $N > 2p$ and suppose that* (F_6) *is satisfied. Suppose in addition that if $1 < p \leqslant 2$ then there exists a positive constant c such that*

$$\int_0^\infty t^{2p-1} F(t, cq_{N,p}(t))\,\mathrm{d}t < \infty \tag{4.5}$$

and if $p > 2$ then there exist positive constants c and p' such that $p' > p$, $N > 2p'$ and

$$\int_0^\infty t^{2p'-1} F(t, cq_{N,p}(t))\,\mathrm{d}t < \infty. \tag{4.6}$$

Then the equation (A) *has a positive entire solution $u \in C^4(\mathbb{R}^N)$ satisfying* (4.4).

Proof of Theorem 4.1. We give a proof only for the case where $1 < p \leqslant 2$. Using (3.1), (F_5) and the Lebesgue convergence theorem, we can choose $k > 1$ so that

$$\begin{aligned} &C_1(N,p)\Big(\int_0^\infty \min\{s, s^{N-1}\}\varphi(s, k^{-1}q_{N,p}(s))\,\mathrm{d}s\Big)^{\frac{1}{p-1}} \geqslant k^{-1},\\ &C_2(N,p)\Big(\int_0^\infty s_*^{2p-1} F(s,k)\,\mathrm{d}s\Big)^{\frac{1}{p-1}} \leqslant k, \end{aligned} \tag{4.7}$$

where $C_1(N,p)$ and $C_2(N,p)$ are the constants appearing in Lemma 2.2. Consider the mapping

$$(\mathscr{G}z)(t) = [\Psi_p F(\cdot, z)](t), \quad t \geqslant 0, \tag{4.8}$$

on the set

$$Z_1 = \{z \in C(\overline{\mathbb{R}}_+)\colon k^{-1}q_{N,p}(t) \leqslant z(t) \leqslant k,\ t \geqslant 0\}.$$

If $z \in Z_1$, then, using (i) of Lemma 2.2 and (4.7), we obtain for $t \geqslant 0$

$$
\begin{aligned}
(\mathscr{G}z)(t) &\geqslant C_1(N,p)\left(\int_0^\infty \min\{s, s^{N-1}\}F(s,z(s))\,\mathrm{d}s\right)^{\frac{1}{p-1}} q_{N,p}(t)\\
&\geqslant C_1(N,p)\left(\int_0^\infty \min\{s, s^{N-1}\}\varphi(s,k^{-1}q_{N,p}(s))\,\mathrm{d}s\right)^{\frac{1}{p-1}} q_{N,p}(t) \geqslant k^{-1}q_{N,p}(t),\\
(\mathscr{G}z)(t) &\leqslant C_2(N,p)\left(\int_0^\infty s^{2p-1}F(s,z(s))\,\mathrm{d}s\right)^{\frac{1}{p-1}}\\
&\leqslant C_2(N,p)\left(\int_0^\infty s^{2p-1}F(s,k)\,\mathrm{d}s\right)^{\frac{1}{p-1}} \leqslant k.
\end{aligned}
$$

This shows that $\mathscr{G}$ maps Z_1 into itself. We can show the continuity of $\mathscr{G}$ and the relative compactness of $\mathscr{G}(Z_1)$ in the topology of $C(\overline{\mathbb{R}}_+)$ as in the proof of Theorem 3.1. Therefore there exists $z \in Z_1$ such that $z(t) = [\Psi_p F(\cdot,z)](t)$, $t \geqslant 0$, by the Schauder-Tychonoff theorem. With this $z(t)$ define

$$
v(x) = z(|x|), \quad w(x) = [\Psi_p\varphi(\cdot,k^{-1}q_{N,p})](|x|), \quad x \in \mathbb{R}^N.
$$

We then have

$$
v(x) = [\Psi_p F(\cdot,z)](|x|) \geqslant [\Psi_p\varphi(\cdot,k^{-1}q_{N,p})](|x|) = w(x),
$$

and

$$
-\Delta v(x) = (\Psi F(\cdot,z))^{\frac{1}{p-1}}(|x|) \geqslant (\Psi\varphi(\cdot,k^{-1}q_{N,p}))^{\frac{1}{p-1}}(|x|) = -\Delta w(x)
$$

for $x \in \mathbb{R}^N$. Furthermore if, for any fixed $x \in \mathbb{R}^N$, σ satisfies $v(x) \geqslant \sigma \geqslant w(x)$, then

$$
\begin{aligned}
\Delta((\Delta v(x))^{(p-1)*}) &= -\Delta(-\Delta v(x))^{p-1} = F(x,v(x))\\
&\geqslant F(x,\sigma) \geqslant f(x,\sigma) \geqslant \varphi(|x|,\sigma) \geqslant \varphi(|x|,w(x))\\
&\geqslant \varphi(|x|,k^{-1}q_{N,p}(|x|)) = -\Delta(-\Delta w(x))^{p-1} = \Delta((\Delta w(x))^{(p-1)*}).
\end{aligned}
$$

From Theorem 1.1 it follows that (A) has an entire solution u such that $v(x) \geqslant u(x) \geqslant w(x)$ and $-\Delta v(x) \geqslant -\Delta u(x) \geqslant -\Delta w(x)$ for $x \in \mathbb{R}^N$. The decaying property (4.4) of u follows from the fact that $v(x)$, $w(x)$, $\Delta v(x)$ and $\Delta w(x)$ tend to zero as $|x| \to \infty$, which is easily verified. That $u \in C^4(\mathbb{R}^N)$ has already been observed in the remark following Theorem 3.1. $\square$

Proof of Theorem 4.2. We restrict our attention to the case where $1 < p \leqslant 2$. We define

$$\varphi_0(t,u) = \zeta_0(t)\varphi(t,u), \quad (t,u) \in \overline{\mathbb{R}}_+ \times \mathbb{R}_+,$$

where ζ_0 is a function of class $C^\alpha_{\mathrm{loc}}(\overline{\mathbb{R}}_+)$ such that $0 \leqslant \zeta_0(t) \leqslant 1$ for $0 \leqslant t \leqslant 1$ and $\zeta_0(t) = 0$ for $t \geqslant 1$, and consider the function

$$z_0(t) = [\Psi_p\varphi_0(\cdot,1)](t), \quad t \geqslant 0.$$

Since $\varphi_0(\cdot,1) \in L^1_{N-1}$ (note that $\varphi_0(t,1) = 0$ for $t \geqslant 1$), from Lemma 2.3 we have

$$\alpha_1 q_{N,p}(t) \leqslant z_0(t) \leqslant \alpha_2 q_{N,p}(t), \quad t \geqslant 0, \tag{4.9}$$

for some positive constants α_1 and α_2 depending only on N and p. We claim that there exists a constant $k > 0$ such that

$$F(t, k\alpha_2 q_{N,p}(t)) \geqslant k^{p-1}\varphi_0(t,1), \tag{4.10}$$

$$\varphi(t, (k\alpha_1)^{-\frac{\varrho}{p-1}}M) \geqslant k^{p-1}\varphi_0(t,1) \tag{4.11}$$

for $t \geqslant 0$, where $M = \sup\limits_{t\geqslant 0}[\Psi_p F(\cdot, q_{N,p})](t)$. These inequalities hold trivially for $t \geqslant 1$. So, we suppose that $0 \leqslant t \leqslant 1$. Let $K > 0$ be any constant. Then form (4.3) it follows that for all small $\lambda > 0$

$$\varphi(t, \lambda^{-\frac{\varrho}{p-1}}) \geqslant K\lambda^{p-1}. \tag{4.12}$$

From this with $K = \max\limits_{0\leqslant t\leqslant 1}\varphi_0(t,1)$, we can take a constant $k > 0$ (sufficiently small) such that $k\alpha_2 \leqslant k^{-\frac{\varrho}{p-1}}$ and $\varphi(t, k^{-\frac{\varrho}{p-1}}) \geqslant k^{p-1}\varphi_0(t,1)$. Using the decreasing property of F and φ in u, we then obtain

$$F(t, k\alpha_2 q_{N,p}(t)) = F(t, k\alpha_2) \geqslant F(t, k^{-\frac{\varrho}{p-1}}) \geqslant \varphi(t, k^{-\frac{\varrho}{p-1}}) \geqslant k^{p-1}\varphi_0(t,1),$$

which proves (4.10). Substituting $\lambda = k\alpha_1 M^{-\frac{p-1}{\varrho}}$ and $K = \alpha_1^{1-p}M^{\frac{(p-1)^2}{\varrho}} \max\limits_{0\leqslant t\leqslant 1}\varphi_0(t,1)$ into (4.12), we have $\varphi(t, (k\alpha_1)^{-\frac{\varrho}{p-1}}M) \geqslant k^{p-1}\max\limits_{0\leqslant t\leqslant 1}\varphi_0(t,1)$, which proves (4.11). Thus a constant $k > 0$ can be chosen so that (4.10) and (4.11) hold for all $t \geqslant 0$.

We now define the functions $v(x)$ and $w(x)$ by

$$v(x) = [\Psi_p F(\cdot, kz_0)](|x|), \quad w(x) = kz_0(|x|), \quad x \in \mathbb{R}^N. \tag{4.13}$$

That $v(x)$ is well defined follows from (3.1) and (4.9). These functions become a supersolution and a subsolution of (A) satisfying the conditions of Theorem 1.1. In fact, using (4.9) and (4.10) we have

$$\begin{aligned} v(x) &\geqslant [\Psi_p F(\cdot, k\alpha_2 q_{N,p})](|x|) \geqslant k[\Psi_p \varphi_0(\cdot, 1)](|x|) = kz_0(|x|) = w(x), \\ -\Delta v(x) &= [\Psi F(\cdot, kz_0)]^{\frac{1}{p-1}}(|x|) \geqslant [\Psi F(\cdot, k\alpha_2 q_{N,p})]^{\frac{1}{p-1}}(|x|) \\ &\geqslant k[\Psi \varphi_0(\cdot, 1)]^{\frac{1}{p-1}}(|x|) = -\Delta w(x), \quad x \in \mathbb{R}^N. \end{aligned}$$

On the other hand, using the decreasing property of φ in u combined with (4.11) and the inequalities

$$F(t, kz_0(t)) \leqslant F(t, k\alpha_1 q_{N,p}(t)) \leqslant (k\alpha_1)^{-\varrho} F(t, q_{N,p}(t)), \quad t \geqslant 0,$$

following from (4.2), we see that

$$\begin{aligned} \varphi(|x|, v(x)) &= \varphi(|x|, [\Psi_p F(\cdot, kz_0)](|x|)) \\ &\geqslant \varphi(|x|, (k\alpha_1)^{-\frac{\varrho}{p-1}} [\Psi_p F(\cdot, q_{N,p})](|x|)) \\ &\geqslant \varphi(|x|, (k\alpha_1)^{-\frac{\varrho}{p-1}} M) \geqslant k^{p-1} \varphi_0(|x|, 1), \quad x \in \mathbb{R}^N. \end{aligned} \tag{4.14}$$

Let $x \in \mathbb{R}^N$ be fixed and let σ be such that $v(x) \geqslant \sigma \geqslant w(x)$. It follows from (4.13) and (4.14) that

$$\begin{aligned} \Delta((\Delta v(x))^{(p-1)*}) &= F(|x|, w(x)) \geqslant F(|x|, \sigma) \geqslant f(x, \sigma) \\ &\geqslant \varphi(|x|, \sigma) \geqslant \varphi(|x|, v(x)) \geqslant k^{p-1} \varphi_0(|x|, 1) = \Delta((\Delta w(x))^{(p-1)*}). \end{aligned}$$

Therefore, by Theorem 1.1, there exists an entire solution $u \in C^4(\mathbb{R}^N)$ of (A) with the property that $v \geqslant u \geqslant w$ and $-\Delta v \geqslant -\Delta u \geqslant -\Delta w$ in $\mathbb{R}^N$, which implies that u and Δu tend to zero as $|x| \to \infty$. This completes the proof of Theorem 4.2. □

No information about the exact order of decay of the entire solutions is available from the proofs of Theorems 4.1 and 4.2. It is possible to give conditions under which (A) possesses a positive entire solution having the same order of decay as the function $q_{N,p}(t)$ defined by (2.5).

Theorem 4.3. *Let $p > 1$ and $N > 2p$ and suppose that either* (F_5) *or* (F_6) *is satisfied. If there is a constant $c > 0$ such that*

$$\int_0^\infty t^{N-1} F(t, cq_{N,p}(t))\, dt < \infty, \tag{4.15}$$

then (A) *possesses a positive entire solution* $u \in C^4(\mathbb{R}^N)$ *such that*

$$(4.16) \qquad \begin{aligned} k_1^{-1} q_{N,p}(|x|) &\leqslant u(x) \leqslant k_1 q_{N,p}(|x|), \\ k_2^{-1} |x|_*^{\frac{2-N}{p-1}} &\leqslant -\Delta u(x) \leqslant k_2 |x|_*^{\frac{2-N}{p-1}}, \quad x \in \mathbb{R}^N, \end{aligned}$$

for some positive constants k_1 *and* k_2.

Proof. Suppose that (F_5) holds. Take a constant $k > 0$ such that

$$C_1(N,p)\left(\int_0^\infty \min\{s, s^{N-1}\}\varphi(s, k^{-1}q_{N,p}(s))\,\mathrm{d}s\right)^{\frac{1}{p-1}} \geqslant k^{-1},$$
$$C_3(N,p)\left(\int_0^\infty \max\{s, s^{N-1}\}F(s, kq_{N,p}(s))\,\mathrm{d}s\right)^{\frac{1}{p-1}} \leqslant k,$$

where $C_1(N,p)$ and $C_3(N,p)$ are the constants indicated in Lemma 2.3. This is possible because of (F_5) and (4.15). Let the set Z_2 be defined by

$$Z_2 = \{z \in C(\overline{\mathbb{R}}_+)\colon k^{-1}q_{N,p}(t) \leqslant z(t) \leqslant kq_{N,p}(t), t \geqslant 0\}.$$

Consider again the mapping $\mathscr{G}$ used in the proof of Theorem 4.1 (cf. (4.8)). Then, it can be shown without difficulty that $\mathscr{G}$ send Z_2 continuously into a relatively compact subset of Z_2. So, there exists a fixed element z of $\mathscr{G}$ in Z_2. With this z define

$$v(x) = z(|x|), \quad w(x) = [\Psi_p\varphi(\cdot, k^{-1}q_{N,p})](|x|), \quad x \in \mathbb{R}^N.$$

Then, proceeding as in the proof of Theorem 4.1, we can prove that $v(x)$ and $w(x)$ are a supersolution and a subsolution of (A) satisfying the requirements of Theorem 1.1, so that there exists an entire solution $u \in C^4(\mathbb{R}^N)$ such that $v \geqslant u \geqslant w$ and $-\Delta v \geqslant -\Delta u \geqslant -\Delta w$ in $\mathbb{R}^N$. That u satisfies (4.16) is a consequence of the fact that both v and w have the same type of asymptotic behavior as $|x| \to \infty$.

Suppose that (F_6) holds. Since the condition (4.15) is stronger than (4.5) and (4.6), we can proceed exactly as in the proof of Theorem 4.2 to establish the existence of a positive entire solution u of (A) lying between the functions $v(x)$ and $w(x)$ defined by (4.13). Lemma 2.3 shows that under (4.15) these $v(x)$ and $w(x)$ behave like positive constant multiples of $q_{N,p}(|x|)$, and so the solution u obtained is shown to have the desired asymptotic property (4.16). This sketches the proof. The verification of the details is left to the reader. □

5. Example

An example illustrating the above theorems is given below.

Example. Consider the equation

$$\Delta(|\Delta u|^{p-2}\Delta u) = a(x)|u|^{r-2}u \tag{5.1}$$

in $\mathbb{R}^N$, where $p > 1$ and $r \in \mathbb{R}$ are constants, and $a(x)$ is a function of class $C^{\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$. This is a special case of (A) in which $f(x,u) = a(x)|u|^{r-2}u$. The condition (F_1) is satisfied with the choice of $F(t,u) = a^*(t)u^{r-1}$, where $a^*(t) = \max\limits_{|x|=t}|a(x)|$. It is easy to see that

$$(\mathrm{F}_2) \iff r > p, \quad (\mathrm{F}_3) \iff 1 \leqslant r < p \quad \text{and} \quad (\mathrm{F}_4) \iff r < 1.$$

In what follows we suppose that $N > 2p$.

Theorem 3.1 applied to (5.1) yields the following statement concerning the existence of uniformly positive entire solutions.

I. Let $r \neq p$. Suppose that if $1 < p \leqslant 2$ then

$$\int_0^{\infty} t^{2p-1}a^*(t)\,\mathrm{d}t < \infty \tag{5.2}$$

and that if $p > 2$, then there is p' such that $p' > p$, $N > 2p'$ and

$$\int_0^{\infty} t^{2p'-1}a^*(t)\,\mathrm{d}t < \infty. \tag{5.3}$$

Then (5.1) has infinitely many positive entire solutions u satisfying

$$\lim_{|x|\to\infty} u(x) = \text{constant} > 0, \quad \lim_{|x|\to\infty} \Delta u(x) = 0. \tag{5.4}$$

In order to discuss the existence of decaying entire solutions it is necessary to assume that $a(x) > 0$ throughout $\mathbb{R}^N$. We note that if $1 \leqslant r < p$ the condition (F_5) is satisfied with the choice $\varphi(t,u) = a_*(t)u^{r-1}$ and $F(t,u) = a^*(t)u^{r-1}$, where $a_*(t) = \min\limits_{|x|=t} a(x)$ and $a^*(t) = \max\limits_{|x|=t} a(x)$. On the other hand, the condition (F_6) is shown to hold with the same choice of φ and F if r satisfies $2 - p < r < 1$; for example, it suffices to take $\varrho = 1 - r$ in (4.2) and (4.3). The results which follow from Theorems 4.1–4.3 applied to (5.1) with positive $a(x)$ are listed below.

II. (i) Let $1 \leqslant r < p$. If (5.2) or (5.3) holds according to whether $1 < p \leqslant 2$ or $p > 2$, then (5.1) has a positive entire solution u such that

$$\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} \Delta u(x) = 0. \tag{5.5}$$

(ii) Let $2 - p < r < 1$. Suppose that if $1 < p \leqslant 2$, then

$$\int_0^\infty t^{2p-1} a^*(t)(q_{N,p}(t))^{r-1}\,\mathrm{d}t < \infty \tag{5.6}$$

and that if $p > 2$, then there is p' such that $p' > p$, $N > 2p'$ and

$$\int_0^\infty t^{2p'-1} a^*(t)(q_{N,p}(t))^{r-1}\,\mathrm{d}t < \infty. \tag{5.7}$$

Then (5.1) has a positive entire solution u satisfying (5.5).

III. Suppose that either $1 \leqslant r < p$ or $2 - p < r < 1$.

(i) If $1 < p < 2 - \frac{2}{N}$ and

$$\int_0^\infty t_*^{N-1-(N-2)(r-1)} a^*(t)\,\mathrm{d}t < \infty, \tag{5.8}$$

then (5.1) has a positive entire solution u of class C^4 satisfying

$$k_1^{-1}|x|_*^{2-N} \leqslant u(x) \leqslant k_1 |x|_*^{2-N}, \quad x \in \mathbb{R}^N, \tag{5.9}$$

$$k_2^{-1}|x|_*^{\frac{2-N}{p-1}} \leqslant -\Delta u(x) \leqslant k_2 |x|_*^{\frac{2-N}{p-1}}, \quad x \in \mathbb{R}^N, \tag{5.10}$$

for some positive constants k_1 and k_2.

(ii) If $p = 2 - \frac{2}{N}$ and

$$\int_0^\infty t_*^{N-1-(N-2)(r-1)} (\log t_*)^{r-1} a^*(t)\,\mathrm{d}t < \infty,$$

then (5.1) has a positive entire solution u of class C^4 satisfying

$$k_1^{-1}|x|_*^{2-N}\log(e|x|_*) \leqslant u(x) \leqslant k_1|x|_*^{2-N}\log(e|x|_*), \quad x \in \mathbb{R}^N, \tag{5.11}$$

and (5.10) for some positive constants k_1 and k_2.

(ii) If $2 - \frac{2}{N} < p < \frac{2}{N}$ and

$$\int_0^\infty t_*^{N-1-(N-2p)\frac{r-1}{p-1}} a^*(t)\,\mathrm{d}t < \infty, \tag{5.12}$$

then (5.1) has a positive entire solution u of class C^4 satisfying

$$k_1^{-1}|x|_*^{\frac{2p-N}{p-1}} \leqslant u(x) \leqslant k_1|x|_*^{\frac{2p-N}{p-1}}, \quad x \in \mathbb{R}^N, \tag{5.13}$$

and (5.10) for some positive constants k_1 and k_2.

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