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THE THIRD BOUNDARY VALUE PROBLEM IN POTENTIAL THEORY FOR DOMAINS WITH A PIECEWISE SMOOTH BOUNDARY

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Abstract. The paper investigates the third boundary value problem $\frac{\partial u}{\partial n} + \lambda u = \mu$ for the Laplace equation by the means of the potential theory. The solution is sought in the form of the Newtonian potential (1), (2), where ν is the unknown signed measure on the boundary. The boundary condition (4) is weakly characterized by a signed measure $\mathscr{T}\nu$. Denote by $\mathscr{T}\colon \nu \to \mathscr{T}\nu$ the corresponding operator on the space of signed measures on the boundary of the investigated domain G. If there is $\alpha \neq 0$ such that the essential spectral radius of $(\alpha I - \mathscr{T})$ is smaller than $|\alpha|$ (for example, if $G \subset R^3$ is a domain "with a piecewise smooth boundary" and the restriction of the Newtonian potential $\mathscr{U}\lambda$ on ∂G is a finite continuous functions) then the third problem is uniquely solvable in the form of a single layer potential (1) with the only exception which occurs if we study the Neumann problem for a bounded domain. In this case the problem is solvable for the boundary condition $\mu \in \mathscr{C}'$ for which $\mu(\partial G) = 0$.

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0. INTRODUCTION

Let G be a Borel set in the Euclidean m-space $\mathbb{R}^m$, $m \geqslant 2$, and suppose that the boundary B of G is compact and $B \neq \emptyset$. For every $\nu \in \mathscr{C}'$ (= the Banach space of all finite signed Borel measures with support in B), the corresponding Newtonian potential $\mathscr{U}\nu$ is defined by

$$\mathscr{U}\nu(x) = \int_B h_x(y)\,\mathrm{d}\nu(y), \quad x \in \mathbb{R}^m \tag{1}$$

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where

$$h_x(y) = \frac{1}{A}\log\frac{1}{|x-y|} \quad \text{for } m=2, \tag{2}$$
$$\frac{1}{A(m-2)}|x-y|^{2-m} \quad \text{for } m>0$$

and A is the area of the unit m-sphere.

Further, if there is a unit vector θ such that the symmetric difference of G and the half-space $\{x \in \mathbb{R}^m;\ (x-z)\cdot\theta > 0\}$ has m-dimensional density zero at z then $n^G(z) = \theta$ is termed the interior normal of G at z in Federer's sense. If there is no interior normal of G at z in this sense, we denote by $n^G(z)$ the zero vector in $\mathbb{R}^m$. The set $\{y \in \mathbb{R}^m;\ |n^G(y)| > 0\}$ is called the reduced boundary of G and will be denoted by $\hat{\partial} G$.

Denote for $z \in \mathbb{R}^m$, $r > 0$

$$v_r^G(z) = \int_{\hat{\partial} G \cap \mathcal{U}(z;r)} |n^G(y)\cdot \operatorname{grad} h_z(y)|\,\mathrm{d}\mathcal{H}_{m-1}(y),$$
$$V^G = \sup_{y\in B} v_\infty^G(y),$$
$$V_0^G = \lim_{r\to 0+} \sup_{y\in B} v_r^G(y).$$

Here $\mathcal{H}_k$ is the k-dimensional Hausdorff measure and $\mathcal{U}(z;r) = \{y \in \mathbb{R}^m; |z-y| < r\}$. Throughout this paper we shall assume that $V^G < \infty$. We may define for $x \in \mathbb{R}^m$, $f \in \mathcal{C}$, where $\mathcal{C}$ is the space of all bounded continuous functions on B equipped with the maximum norm,

$$W^G f(x) = \mathrm{d}_G(x) f(x) - \int_B f(y) n^G(y)\cdot \operatorname{grad} h_x(y)\,\mathrm{d}\mathcal{H}_{m-1}(y), \tag{3}$$

where

$$\mathrm{d}_G(x) = \lim_{r\to 0+} \frac{\mathcal{H}_m(\mathcal{U}(x;r)\cap G)}{\mathcal{H}_m(\mathcal{U}(x;r))}$$

is the m-dimensional density of G at the point x. The double layer potential $W^G f$ is a function harmonic on $\mathbb{R}^m - B$ and continuous on B. Besides that W^G is a bounded operator on $\mathcal{C}$. If $W^G f = g$ on B then $W^G f$ is a solution of the Dirichlet problem on $\mathbb{R}^m - \operatorname{cl} G$ with the boundary condition g. For $\nu \in \mathcal{C}'$ we define a signed measure $N^G \mathcal{U}\nu$

$$N^G\mathcal{U}\nu(M) = \int_B [d_G(x)\chi_M(x) - \int_{B\cap M} n^G(y)\cdot \operatorname{grad} h_x(y)\,\mathrm{d}\mathcal{H}_{m-1}(y)]\,\mathrm{d}\nu(x),$$

where χ_M is the characteristic function of the set M. If $N^G\mathcal{U}\nu = \mu$ then $\mathcal{U}\nu$ is a solution of the Neumann problem on $\operatorname{int} G$ with the boundary condition μ.

If W^G is a Fredholm operator on $\mathscr{C}$ then Fredholm's theorems hold for dual equations

$$W^G f = g,$$
$$N^G \mathscr{U} \nu = \mu.$$

If ∂G is Lipschitz, then W^G is a Fredholm operator in the space $L^2(\partial G)$. (For the L^p-theory of double layer potentials and its connection to boundary value problems see the papers [60], [13], [25], [26], [35], [38].) The operator W^G for a polyhedral boundary ∂G and certain Sobolev spaces is studied in [51]. If G is convex or if $V_0^G < \frac{1}{2}$ then W^G is Fredholm in the space $\mathscr{C}$ (see [28], [35]). If $G \subset \mathbb{R}^2$ and B is piecewise smooth without cusps then $V_0^G < \frac{1}{2}$ and W^G is a Fredholm operator. If $G \subset \mathbb{R}^3$ and B is piecewise smooth then it may happen that $V_0^G > \frac{1}{2}$ (see [33]). If $G \subset \mathbb{R}^3$ is a rectangular domain then W^G is a Fredholm operator with index 0 (cf. [33], [1]). The same holds for a polyhedral cone in $\mathbb{R}^3$ (cf. [50]).

The aim of the section 2 is to prove that W^G is a Fredholm operator with index 0 under assumption that $G \subset \mathbb{R}^3$ has a piecewise smooth boundary. We use a method which was proposed in [10], [40], [41] in connection with investigation of changes of the Fredholm radius of the Neumann operator $(2W^G - I)$ under a deformation. Here I is the identical operator.

In section 1 we study the third boundary value problem for open $G \subset \mathbb{R}^m$, where $m > 2$. Fix a nonnegative element λ of $\mathscr{C}'$ and suppose that $\mathscr{U}\lambda$ is bounded on B.

For each $\nu \in \mathscr{C}'$ we define the distribution $\mathscr{T}\nu$ by

$$\langle \varphi, \mathscr{T}\nu \rangle = \int_G \operatorname{grad} \varphi(x) \cdot \operatorname{grad} \mathscr{U}\nu(x) \, \mathrm{d}x + \int_B \varphi(x) \mathscr{U}\nu(x) \, \mathrm{d}\lambda(x),$$

$\varphi \in \mathscr{D}$, $\mathscr{D}$ being the class of all infinitely differentiable functions with compact support in $\mathbb{R}^m$ (see [44], [55]). The distribution $\mathscr{T}\nu$ is representable by a unique element of $\mathscr{C}'$ and the operator $\mathscr{T}\colon \nu \to \mathscr{T}\nu$ acting on $\mathscr{C}'$ is a bounded linear operator (see [44], theorem 5).

If B is a smooth hypersurface and λ is absolutely continuous with respect to the area measure H on B, then, under suitable conditions concerning $\mathscr{U}\nu$, $\langle \varphi, \mathscr{T}\nu \rangle$ transforms into

$$\int_B \varphi \Big(-\frac{\partial \mathscr{U}\nu}{\partial n} + q\mathscr{U}\nu \Big) \, \mathrm{d}H,$$

where $q = \frac{\mathrm{d}\lambda}{\mathrm{d}H}$, which shows that $\mathscr{T}\nu$ is a natural weak characterization of

$$-\frac{\partial \mathscr{U}}{\partial n} + q\mathscr{U}\nu. \tag{4}$$

The operator $\mathscr{T}$ is studied in [43], [44], [45], [46], [55]. In [46] the following theorem is proved:

Assume G to be a domain with $\mathrm{d}_G(y) \neq 0$ for every $y \in B$ and suppose that

(5) $$\inf_{\alpha \neq 0} \frac{\omega' \mathscr{T}_\alpha}{|\alpha|} < 1.$$

Then $\mathscr{T}(\mathscr{C}') = \mathscr{C}'$ with the only exception which occurs if G is bounded and $\lambda = 0$. In this case the range of $\mathscr{T}$ consists precisely of those $\nu \in \mathscr{C}'$ with $\nu(B) = 0$.

Here $\mathscr{T}_\alpha = \mathscr{T} - \alpha I$, I is the identity operator and

$$\omega' \mathscr{T}_\alpha = \inf_Q \|\mathscr{T}_\alpha - Q\|$$

Q ranging over the class of all operators acting on $\mathscr{C}'$ of the form

$$Q \ldots = \sum_{j=1}^{n} \langle f_j, \ldots \rangle \, m_j$$

where n is a positive integer, $m_j \in \mathscr{C}'$ and $f_j's$ are bounded Baire functions on B.

However in [33] an example is given of a rectangular domain G in $\mathbb{R}^3$ such that the condition (5) is not fulfilled even for $\lambda = 0$. We shall substitute the condition (5) by a weaker condition and then we shall prove the result of [46]. The technique of proofs remains the same as in [46].

If X is a Banach space we denote by $\mathscr{K}(X)$ the space of all compact linear operators on X. For each bounded linear operator Q on X we define

$$\|Q\|_{\mathrm{ess}} = \inf_{K \in \mathscr{K}(X)} \|Q + K\|,$$
$$r_{\mathrm{ess}} = \liminf_{n \to \infty} (\|Q^n\|_{\mathrm{ess}})^{1/n}.$$

We substitute the condition (5) by the condition

(6) $$a = \inf_{\alpha \neq 0} \frac{r_{\mathrm{ess}} \mathscr{T}_\alpha}{|\alpha|} < 1.$$

In the section 2 we will prove that the condition (6) is fulfilled for any domain $G \subset \mathbb{R}^3$ "with a piecewise smooth boundary" and $\lambda = 0$. According to the results in [45] the condition (6) is fulfilled even for each non-negative measure λ for which the restriction $\mathscr{U}\lambda$ on B is a finite continuous function.

1.1. Preliminaries. We shall suppose in this section that $G \subset \mathbb{R}^m$, $m > 2$, is an open set.

Let $\mathscr{B}$ denote the Banach space of all bounded Baire functions defined on B with the usual supremum norm. The symbol $\mathscr{B}'$ stands for the dual space of $\mathscr{B}$ and for $\mu \in \mathscr{C}'$ we shall denote by $|\mu|$ the indefinite variation of μ; of course, $\|\mu\| = |\mu|(B)$ is the norm of a μ in $\mathscr{C}'$.

According to [44], proposition 8 we may define on $\mathscr{B}$ the continuous operator V by

$$Vf(y) = \mathscr{U} f\lambda(y)\left[= \int_B f(x)h_y(x)\,\mathrm{d}\lambda(x)\right].$$

We define for $f \in \mathscr{B}$ and $y \in B$

$$\tilde{W}f(y) = \mathrm{d}_G(y)f(y) + \frac{1}{A}\int_B f(x)\frac{n(x)\cdot(x-y)}{|x-y|^m}\,\mathrm{d}\mathscr{H}_{m-1}(x).$$

Results in [29] (cf. also [4], [28], [36]) imply that $\tilde{W}$ is a bounded linear operator on $\mathscr{B}$ and

$$\int_G \operatorname{grad}\varphi(x)\cdot\operatorname{grad}\mathscr{U}\delta_y(x)\,\mathrm{d}\mathscr{H}_m(x) = \tilde{W}\varphi(y), \tag{7}$$

for each $\varphi \in \mathscr{D}$, $y \in B$. Here δ_y denotes the Dirac measure concentrated at y.

There is a close connection between the operator $T = V + \tilde{W}$ and the operator $\mathscr{T}$, namely, the restriction to $\mathscr{C}'$ of the dual operator T' of T coincides with the operator $\mathscr{T}$ (see [44], proposition 8), $T'/\mathscr{C}' = \mathscr{T}$.

Denoting by $\tilde{W}'$, V' the dual operator of $\tilde{W}$, V, respectively, we observe that $\tilde{W}'(\mathscr{C}') \subset \mathscr{C}'$, $V'(\mathscr{C}') \subset \mathscr{C}'$ (see [46], preliminaries 1).

1.2. Lemma. *Let X be a complex Banach space and Q be a bounded linear operator on X. Denote by X' the dual space of X and by Q' the dual operator of Q. Then*

$$r_{\mathrm{ess}}Q = r_{\mathrm{ess}}Q' = \inf\{r; r > 0, (\forall\alpha \in C, |\alpha| > r; (\alpha I - Q) \text{ is Fredholm})\}.$$

Put $\Omega = \{\alpha \in C; |\alpha| > r_{\mathrm{ess}}Q\}$. Then $\alpha I - Q$ is a Fredholm operator with index 0 for each $\alpha \in \Omega$. Denote by $\sigma(Q)$ the spectrum of the operator Q. The set $\Omega \cap \sigma(Q)$ is isolated in Ω.

P r o o f. Denote by Φ the set of all complex numbers α for which $\alpha I - Q$ is a Fredholm operator.

$$r_{\mathrm{ess}}Q = \sup\{|\alpha|;\ \alpha \notin \Phi\}$$

by [56], chapter IX, theorem 2.1 and theorem 1.3. According to [56], chapter VII, theorem 3.5 the operator $\alpha I - Q'$ is Fredholm if and only if $\alpha I - Q$ is Fredholm.

Hence

$$r_{\text{ess}} Q' = \sup\{|\alpha|;\ \alpha \notin \Phi\} = r_{\text{ess}} Q.$$

Since the index of $(\alpha I - Q)$ is constant on the domain Ω by [56], chapter VII, theorem 5.2 and $(\alpha I - Q)$ has index 0 for $|\alpha| > \|Q\|$, the index of $(\alpha I - Q)$ is null for $\alpha \in \Omega$.

Fix $d > r_{\text{ess}} Q$. Choose n such that $\|Q^n\|_{\text{ess}} < d^n$. The set $\sigma(Q^n) - \mathscr{U}(0; d^n)$ is finite by [39], lemma 2. Since $\sigma(Q^n) - \mathscr{U}(0; d^n) = \{\alpha^n;\ \alpha \in \sigma(Q) - \mathscr{U}(0; d)\}$ by [61], chapter VIII, 7 the set $\sigma(Q) \cap \Omega$ is isolated in Ω. □

1.3. Lemma. *Let X be a complex Banach space and Q be a bounded linear operator on X. Let Y be a closed subspace of X' such that $Q'(Y) \subset Y$ and denote by Q'/Y the restriction of Q' to Y. Then*

$$r_{\text{ess}}(Q'/Y) \leqslant r_{\text{ess}} Q.$$

P r o o f. Denote

$$\Omega = \{\alpha \in C; |\alpha| > r_{\text{ess}} Q\},$$
$$N = \sigma(Q) \cap \Omega.$$

The set N is isolated in Ω and $\alpha I - Q$ is Fredholm for all $\alpha \in \Omega$ by Lemma 1.2.

We shall prove that $(\alpha I - Q')/Y$ is Fredholm for all $\alpha \in \Omega$. Fix $\alpha \in \Omega$. Since $\alpha I - Q$ is Fredholm the operator $\alpha I - Q'$ is Fredholm too by [56], chapter V, theorem 4.1 and thus $\dim \operatorname{Ker}\big((\alpha I - Q')/Y\big) \leqslant \dim \operatorname{Ker}(\alpha I - Q') < \infty$, where $\operatorname{Ker}(\alpha I - Q')$ is the null space of $(\alpha I - Q')$.

Now we shall prove that $(\alpha I - Q')(Y)$ is a closed subspace of X'. According to [56], chapter V, theorem 1.4 there is a bounded operator F from $(\alpha I - Q')(X')$ to X' such that $(\alpha I - Q')F = I$ and X' is the direct sum of $Z = F(\alpha I - Q')(X')$ and $\operatorname{Ker}(\alpha I - Q')$. It is easy to see that Z is a closed subspace of X'. Put $Z_0 = Z \cap Y$. Now let $x_n \in Z_0$, $(\alpha I - Q')x_n \to y$. Then $x_n \to Fy$ and since Z_0 is closed we have $Fy \in Z_0$ and $y = (\alpha I - Q')Fy \in (\alpha I - Q')(Z_0)$. Hence $(\alpha I - Q')(Z_0)$ is closed. Now, we shall prove that the codimension of Z_0 in Y is finite. Denote $n = \dim \operatorname{Ker}(\alpha I - Q')$. Choose $y^1, \dots, y^{n+1} \in Y$. Denote by P the projection of X' onto $\operatorname{Ker}(\alpha I - Q')$ along Z. Then $Py^1, \dots, Py^{n+1}$ are linearly dependent. There are $c_1, \dots, c_{n+1}$ such that

$$\sum_{i=1}^{n+1} c_i P y^i = 0, \quad \sum_{i=1}^{n+1} |c_i|^2 > 0.$$

Therefore

$$\sum_{i=1}^{n+1} c_i y^i = \sum_{i=1}^{n+1} c_i (I-P) y^i \in Z_0.$$

So, there is a finite dimensional subspace Z_1 of Y such that Y is the direct sum of Z_0 and Z_1. Since

$$(\alpha I - Q')(Y) = (\alpha I - Q')(Z_0) + (\alpha I - Q')(Z_1),$$

$(\alpha I - Q')(Z_0)$ is closed and $(\alpha I - Q')(Z_1)$ has a finite dimension $(\alpha I - Q')(Y)$ is a closed subspace of X'.

Since $(\alpha I - Q')(Y)$ is a closed for all $\lambda \in \Omega$ we have $\dim \operatorname{Ker}\big((\alpha I - Q')/Y\big) > 0$ for all $\alpha \in \Omega \cap \partial\sigma(Q'/Y)$ by [56], chapter XII, theorem 10.1. But then necessarily $\Omega \cap \partial\sigma(Q'/Y) \subset N$ (see [56], chapter VII, theorem 3.2). Since $\Omega - \sigma(Q'/Y)$ is an open set we have $\Omega \cap \sigma(Q'/Y) \subset N$. Choose $\alpha \in \sigma(Q'/Y) \cap \Omega$. Then according to [56], chapter VI, theorem 4.5 there is a natural number k such that $\operatorname{Ker}\big((\alpha I - Q')^k\big) = \operatorname{Ker}\big((\alpha I - Q')^{k+m}\big)$ for all $m \geqslant 0$. Since $\operatorname{Ker}\big((\alpha I - Q')^m/Y\big) \subset \operatorname{Ker}\big((\alpha I - Q')^m\big)$ and $\operatorname{Ker}\big((\alpha I - Q')^k\big)$ is a finite dimensional space by [56], chapter V, theorem 2.3, there is a natural number n such that $\operatorname{Ker}\big((\alpha I - Q')^n/Y\big) = \operatorname{Ker}\big((\alpha I - Q')^{n+1}/Y\big)$. Since α is an isolated point of the spectrum of Q'/Y' and $(\alpha I - Q')(Y)$ is closed the operator $(\alpha I - Q')/Y$ is Fredholm by [56], chapter VI, theorem 4.2.

Since $(\alpha I - Q')/Y$ is a Fredholm operator for all $\alpha \in \Omega$ lemma 1.2 yields that $r_{\text{ess}}(Q'/Y) \leqslant r_{\text{ess}} Q$. □

1.4. Notation. Let C_0 stand for the class of all Borel subsets of $\mathbb{R}^m$ having the Newtonian capacity zero. It should be noted here that $\mathscr{H}_{m-1}(M) = 0$ for any $M \in C_0$ ([34], theorem 3.13) and $\lambda(M) = 0$ as well because λ has a bounded potential ([34], theorem 2.1). We shall say that a property holds quasi-everywhere in $Q \subset \mathbb{R}^m$ if it holds for all points in Q except possible those in a set $M \in C_0$.

Let us denote $\mathscr{C}'_*$ the set of all $\mu \in \mathscr{C}'$ with the following property. There are $M \in C_0$ and $c \in R_1$ such that the difference $\mathscr{U}\mu(x) = \mathscr{U}\mu^+(x) - \mathscr{U}\mu^-(x)$ is meaningful for each $x \in \mathbb{R}^m - M$ and $|\mathscr{U}\mu(x)| \leqslant c$ holds provided $x \in \mathbb{R}^m - M$ (as usual, $\mu = \mu^+ - \mu^-$ is the Jordan decomposition of μ). Clearly, $\mathscr{C}'_*$ is a linear subspace of $\mathscr{C}'$.

The function g is said to belong to the class $\tilde{\mathscr{B}}_0$, if it is defined quasi-everywhere in B and there is a function $h \in \mathscr{B}$ such that $g = h$ quasi-everywhere in B. For $g \in \tilde{\mathscr{B}}_0$ denote by $\tilde{g}$ the class of all $h \in \tilde{\mathscr{B}}_0$ that coincide with g quasi-everywhere in B. Let us denote by $\mathscr{B}_0$ the Banach space of such classes $\tilde{g}$ with the norm defined by

$$\|\tilde{g}\|_0 = \operatorname*{quasisup}_B |g|, \quad g \in \tilde{g},$$

where $\operatorname*{quasisup}_B |g|$ equals the infimum of all c's for which

$$\{x \in B; |g(x)| > c\} \in C_0$$

provided $B \notin C_0$; in the case that $B \in C_0$ we set $\operatorname*{quasisup}_B |g| = 0$.

An operator P acting on $\mathscr{B}$ is said to operate in $\mathscr{B}_0$ if $Pf = 0$ quasi-everywhere whenever $f = 0$ quasi-everywhere. Such an operator defines in an obvious manner an operator acting on $\mathscr{B}_0$ which will be denoted by $\tilde{P}$.

Let L be a linear space over the field of real numbers. We shall denote by $\hat{L}$ the set of all elements of the form $x + \mathrm{i}y$ where $x, y \in L$. If the sum of two elements of $\hat{L}$ and the multiplication of an element of $\hat{L}$ by a complex number are defined in an obvious way then $\hat{L}$ becomes a linear space over the field of complex numbers. Let Q be a linear operator acting on L. The same symbol will denote the extension of Q to $\hat{L}$ defined by

$$Q(x + \mathrm{i}y) = Q(x) + \mathrm{i}Q(y).$$

If an operator Q on L possesses an inverse operator Q^{-1}, then the extension of Q^{-1} to $\hat{L}$ is an inverse operator (on $\hat{L}$) of the extension of Q to $\hat{L}$.

For $f \in \hat{\mathscr{B}}$, $\tilde{g} \in \hat{\mathscr{B}}_0$ put

$$\|f\| = \sup_{x \in B} |f(x)|,$$
$$\|\tilde{g}\|_0 = \operatorname*{quasisup}_B |q|, \quad g \in \tilde{g}.$$

Note that $\hat{\mathscr{B}}$, $\hat{\mathscr{B}}_0$ with the above defined norms are Banach spaces and for any $\mu \in \hat{\mathscr{C}}'$

$$\|\mu\| = \sup \left| \int_B f \, \mathrm{d}\mu \right|$$

where the supremum is taken over all $f \in \hat{\mathscr{B}}$ with $\|f\| \leqslant 1$.

Similarly as above, an operator Q acting on $\hat{\mathscr{B}}$ is said to operate in $\hat{\mathscr{B}}_0$, if $Qf = 0$ quasi-everywhere whenever $f = 0$ quasi-everywhere. Such an operator defines an operator on $\hat{\mathscr{B}}_0$ that will be denoted by $\tilde{Q}$. The inequality $\|\tilde{Q}\|_0 \leqslant \|Q\|$ holds good. Note that if an operator P on $\mathscr{B}$ operates in $\mathscr{B}_0$, then its extension to $\hat{\mathscr{B}}$ operates in $\hat{\mathscr{B}}_0$.

For any $\mu \in \hat{\mathscr{C}}'_*$, $\mu = \mu^1 + \mathrm{i}\mu^2$, $\mathscr{U}\mu^j$ determines the only element of $\mathscr{B}_0$ which will be denoted by $\tilde{\mathscr{U}}\mu^j$ $(j = 1, 2)$. Defining

$$\tilde{\mathscr{U}}\mu = \tilde{\mathscr{U}}\mu^1 + \mathrm{i}\tilde{\mathscr{U}}\mu^2$$

we have $\tilde{\mathscr{U}}\mu \in \hat{\mathscr{B}}_0$ and the mapping

$$\tilde{\mathscr{U}} : \mu \to \tilde{\mathscr{U}}\mu$$

is a linear mapping of $\mathscr{C}'_*$ into $\hat{\mathscr{B}}_0$.

In what follows, fix $\gamma \in \mathbb{R}^1$ and put $T_\gamma = T - \gamma I$. According to our definitions, T, T_γ will also denote the above defined extension of T, T_γ to $\hat{\mathscr{B}}$, respectively.

Let Ω be the set of all complex numbers β with $|\beta| > r_{\text{ess}} T_\gamma$. Then $N = \Omega \cap \sigma(T_\gamma)$ is a countable set consisting of isolated points by lemma 1.2. For $\beta \in \Omega - N$ denote $I_{\beta\gamma} = (\beta I - T_\gamma)^{-1}$ the inverse operator of $(\beta I - T_\gamma)$.

An operator Q acting on $\hat{\mathscr{B}}$ is said to have the property (Φ), if it satisfies the following conditions:

Q operates in $\hat{\mathscr{B}}_0$,

$Q'(\mathscr{C}'_*) \subset \mathscr{C}'_*$,

$\tilde{\mathscr{U}} Q' \mu = \tilde{Q} \tilde{\mathscr{U}} \mu$ whenever $\mu \in \mathscr{C}'_*$.

We shall denote by Ω_0 the set of all $\beta \in \Omega - N$ for which $I_{\beta\gamma}$ has the property (Φ).

1.5. Lemma. $r_{\text{ess}}(T_\gamma) = r_{\text{ess}}(\mathscr{T}_\gamma)$.

Proof. Since $\mathscr{C}'$ is a closed subspace of $\hat{\mathscr{B}}'$ such that $T'_\gamma(\mathscr{C}') \subset \mathscr{C}'$ and $\mathscr{T}_\gamma = T'_\gamma \restriction \mathscr{C}'$ lemma 1.3 yields $r_{\text{ess}}(\mathscr{T}_\gamma) \leqslant r_{\text{ess}}(T_\gamma)$. Since $\hat{\mathscr{B}}$ is a closed subspace of $\mathscr{C}''$ and $\mathscr{T}'_\gamma \restriction \hat{\mathscr{B}} = T_\gamma$ we have $\mathscr{T}'_\gamma(\hat{\mathscr{B}}) \subset \hat{\mathscr{B}}$ and $r_{\text{ess}}(T_\gamma) \leqslant r_{\text{ess}}(\mathscr{T}_\gamma)$ by lemma 1.3. □

1.6. Lemma. *The sets Ω_0 and $\Omega - N$ coincide.*

Proof. See [46], proof of Lemma 9. □

1.7. Lemma Let $\alpha_0 \in \Omega$. *Let us denote*

$$N(\alpha_0) = \{y \in B; d_G(y) = \gamma + \alpha_0\}$$

and let p be any positive integer. Then each $f \in \hat{\mathscr{B}}$ satisfying

$$(\alpha_0 I - T_\gamma)^p f = 0, \tag{8}$$

$$\langle f, \mu \rangle = 0 \quad \text{for each } \mu \in \mathscr{C}'_* \tag{9}$$

has its support contained in $N(\alpha_0)$.

Proof. Denote by H the restriction of $\mathscr{H}_{m-1}$ to the reduced boundary $\hat{\partial} G$. Let (8) and (9) hold for an $f \in \hat{\mathscr{B}}$. By the argument from the proof of lemma 14 in [46] it follows that $f = 0$ λ-almost everywhere and H-almost everywhere as well. Now it is easily seen by the definition of T that

$$(\alpha_0 I - T_\gamma)^k f(y) = [\alpha_0 + \gamma - d_G(y)]^k f(y)$$

for each natural k. If $y \notin N(\alpha_0)$, then $f(y) = 0$ by (8). Consequently, the support of f is contained in $N(\alpha_0)$. □

Using this lemma and the reasoning from lemma 15 in [46] we obtain

1.8. Lemma. *Suppose that $\alpha_0 \in \Omega$, $N(\alpha_0) = \emptyset$ and p is a positive integer. Let $f_1, \ldots, f_q$ be linearly independent solutions of (8). Then there exist $\mu_1, \ldots, \mu_q \in \mathscr{C}'_*$ such that*

$$\langle f_i, \mu_j \rangle = \delta_{ij} \quad (\delta_{ij} = 0 \text{ for } i \neq j,\ \delta_{ii} = 1) \text{ for } 1 \leqslant i,\ j \leqslant q.$$

1.9. Lemma. *Let $\alpha_0 \in N$ and $r > 0$ such that the closed disc K centered at α_0 with radius r is contained in Ω and $K \cap N = \{\alpha_0\}$. Let C be the boundary of K. Let us define the operator A_{-1} acting on $\check{\mathscr{B}}$ by*

$$A_{-1} = (2\pi\mathrm{i})^{-1} \int_C I_{\alpha\gamma}\, \mathrm{d}\alpha \tag{10}$$

where the integral is taken over positively oriented circumference C. The operator A_{-1} has the property (Φ).

Proof. See [46], proof of lemma 11. □

1.10. Lemma. *Suppose that $\alpha_0 \in \Omega$ and $N(\alpha_0) = \emptyset$. If p is a positive integer and $\mu \in \check{\mathscr{B}}'$ satisfies*

$$(\alpha_0 I - T'_\gamma)^p \mu = 0 \tag{11}$$

then $\mu \in \check{\mathscr{C}}'_*$.

Proof. The assertion is trivial for $\alpha_0 \in \Omega - N$. Suppose that $\alpha_0 \in N$. Choose $r > 0$ small enough such that the closed disc K centered at α_0 with radius r is contained in Ω and $K \cap N = \{\alpha_0\}$. The operator A_{-1} from lemma 1.9 is a bounded projection on $\check{\mathscr{B}}$ and T_γ maps $A_{-1}(\check{\mathscr{B}})$ into $A_{-1}(\check{\mathscr{B}})$ (see [56], chapter 6). Denote by Q the restriction of the operator T_γ to the space $A_{-1}(\check{\mathscr{B}})$. Since the space $\check{\mathscr{B}}$ is the direct sum of the subspaces $A_{-1}(\check{\mathscr{B}})$ and $(I - A_{-1})(\check{\mathscr{B}})$, $(\alpha_0 I - T_\gamma)(\check{\mathscr{B}})$ is a subspace of the direct sum $(\alpha_0 I - Q)\big(A_{-1}(\check{\mathscr{B}})\big)$ and $(I - A_{-1})(\check{\mathscr{B}})$. Since $(\alpha_0 I - T_\gamma)$ is Fredholm by lemma 1.2, we have $\operatorname{codim}\, (\alpha_0 I - Q)\big(A_{-1}(\check{\mathscr{B}})\big) < \infty$. At the same time $(\alpha_0 I - Q)\big(A_{-1}(\check{\mathscr{B}})\big) = (\alpha_0 I - T_\gamma)(\check{\mathscr{B}}) \cap A_{-1}(\check{\mathscr{B}})$ is a closed subspace of $A_{-1}(\check{\mathscr{B}})$. Since the dimension of the null space of the operator $(\alpha_0 I - Q)$ is less than or equal to the dimension of the null space of the operator $(\alpha_0 I - T_\gamma)$, the operator $(\alpha_0 I - Q)$ is Fredholm. Since $\sigma(Q) = \{\alpha_0\}$ by [56], chapter 6, theorem 4.1, the operator $(\alpha I - Q)$ is Fredholm for each complex number α. According to [56], chapter 9, theorem 2.2 the space $A_{-1}(\check{\mathscr{B}})$ has a finite dimension. According to [61], chapter VIII, §8, theorem 4 the resolvent of the operator $(\alpha I - T_\gamma)$ has a pole at α_0. Similarly, the resolvent of the operator $(\alpha I - T'_\gamma)$ has a pole at α_0 too. These poles have the same order (compare [61], chap. VIII, 6, 8), say p_0. Clearly, we may assume that $p \geqslant p_0$.

Similarly as A_{-1}, define the operator $\mathscr{A}_{-1}$ on $\hat{\mathscr{B}}'$ by

$$\mathscr{A}_{-1} = (2\pi i)^{-1} \int_C I'_{\alpha\gamma}\, d\alpha$$

where C has the same meaning as in 1.9. Then the set Y of all solutions of the equation (11) coincides with $\mathscr{A}_{-1}(\hat{\mathscr{B}}')$ ([61], chap. VIII, 8). Since $\mathscr{A}_{-1} = A'_{-1}$ ([61], chap. VIII, 7), we have $Y = A'_{-1}(\hat{\mathscr{B}}')$. Similarly, denoting by X the set of all solutions of the equation (8), we get $X = A_{-1}(\hat{\mathscr{B}})$.

Let $f_1, \ldots, f_q$ be a basis of X. Then the operator A_{-1} possesses the form

$$A_{-1} \ldots = \sum_{k=1}^{q} \langle \ldots, \mu_k \rangle f_k \tag{12}$$

where $\mu_k \in \hat{\mathscr{B}}'$. Consequently,

$$A'_{-1} \ldots = \sum_{k=1}^{q} \langle f_k, \ldots \rangle \mu_k.$$

By virtue of lemma 1.8 we construct $\mu'_1, \ldots, \mu'_q \in \hat{\mathscr{C}}'_*$ such that $\langle f_j, \mu'_i \rangle = \delta_{ij}$, $1 \leqslant i, j \leqslant$. It follows from (12) that $A'_{-1}\mu'_k = \mu_k$ for $k = 1, \ldots, q$ and we conclude by lemma 1.9 that $\mu_k \in \hat{\mathscr{C}}'_*$. Since $Y = A'_{-1}(\hat{\mathscr{B}}')$, we have $Y \subset \hat{\mathscr{C}}'_*$ and the proof is complete. □

1.11. Theorem. *Suppose that $d_G(y) \neq 0$ for each $y \in B$ and (6) holds. Then*

$$T'\nu = 0$$

implies $\nu \in \mathscr{C}'_*$. In particular, if $\nu \in \mathscr{C}'$ satisfies

$$\mathscr{T}\nu = 0$$

then $\nu \in \mathscr{C}'_*$.

Proof. Let $T'\nu = 0$. Choose $\gamma \neq 0$ such that $r_{ess}(\mathscr{T}_\gamma) < |\gamma|$. Then $r_{ess}(T_\gamma) < |\gamma|$ by lemma 1.5. Since $N(-\gamma) = \emptyset$ lemma 1.10 yields that $\nu \in \mathscr{C}'_*$. □

Throughout the rest of the paragraph we shall assume that G has a finite number of components $G_1, \ldots, G_p$ such that $\operatorname{cl} G_i \cap \operatorname{cl} G_j = \emptyset$ for $i \neq j$.

1.12. Theorem. *Suppose that (6) holds, $d_G(y) \neq 0$ for each $y \in B$ and let $\nu \in \mathscr{B}'$ satisfy*

$$T'\nu = 0.$$

Then $\nu \in \mathscr{C}'$ and there are $c_1, \dots, c_p \in \mathbb{R}^1$ such that $\mathscr{U}\nu = c_i$ on G_i and $\sum_{i=1}^{p} c_i^2 \lambda(\partial G_i) = 0$. If $c_i = 0$ for $i = 1, \dots, p$ then $\nu = 0$.

Proof. Using theorem 1.11 we conclude $\nu \in \mathscr{C}'_* \subset \mathscr{C}'$ and $\mathscr{T}\nu = 0$. By the definition of $\mathscr{T}$

$$0 = \langle \varphi, \mathscr{T}\nu \rangle = \int_B \varphi(x) \mathscr{U}\nu(x) \, \mathrm{d}\lambda(x) + \int_G \operatorname{grad} \varphi(x) \cdot \operatorname{grad} \mathscr{U}\nu(x) \, \mathrm{d}\mathscr{H}_m(x)$$

for each $\varphi \in \mathscr{D}$. Since there exist functions $\varphi_n \in \mathscr{D}$ such that

$$\lim_{n\to\infty} \int_G \operatorname{grad} \varphi_n \cdot \operatorname{grad} \mathscr{U}\nu \, \mathrm{d}\mathscr{H}_m = \int_G |\operatorname{grad} \mathscr{U}\nu|^2 \, \mathrm{d}\mathscr{H}_m,$$

$$\lim_{n\to\infty} \int_B \varphi_n \mathscr{U}\nu \, \mathrm{d}\lambda = \int_B [\mathscr{U}\nu]^2 \, \mathrm{d}\lambda$$

according to [46] lemma 24 and lemma 25, we have

$$\int_G |\operatorname{grad} \mathscr{U}\nu(x)|^2 \, \mathrm{d}\mathscr{H}_m(x) + \int_B [\mathscr{U}\nu(x)]^2 \, \mathrm{d}\lambda(x) = 0. \tag{13}$$

Therefore there are $c_1, \dots, c_p$ such that $\mathscr{U}\nu = c_i$ on G_i. Let $\nu = \nu^+ - \nu^-$ be the Jordan decomposition of ν. We have $\mathscr{U}\nu^+(x) = \mathscr{U}\nu^-(x) + c_i$ for each $x \in G_i$. Since G_i has a positive m-dimensional density at any $z \in \partial G_i$, every fine neighbourhood of z (in the Cartan topology) meets G (see [3], chap. VII, §§2, 6) and we conclude from the Cartan Theorem ([3], chap. VII, §6) that $\mathscr{U}\nu^+(z) = c_i + \mathscr{U}\nu^-(z)$. Consequently, $\mathscr{U}\nu = c_i$ holds quasi-everywhere in ∂G_i. Noting that the same is true for λ-almost all points $x \in B$ we arrive at the equality $\sum_{i=1}^{p} c_i^2 \lambda(\partial G_i) = 0$ by (13).

Suppose that $c_i = 0$ for $i = 1, \dots, p$. Then $\mathscr{U}\nu^+ = \mathscr{U}\nu^-$ on G. Since $d_G(y) \neq 0$ for each $y \in B$, the set G is not thin at any $y \in B$ ([3], chap. VII, §2) and we have $\nu^+ = \nu^-$ (see [34], theorem 5.10 and chap. V, §1, section 2, 14). In this case $\nu = 0$. □

1.13. Lemma. *Let G_i is a bounded component of G such that $\lambda(\partial G_i) = 0$. If f_i is the characteristic function of ∂G_i then $Tf_i = 0$.*

Proof. Since $\operatorname{cl} G_i \cap \operatorname{cl} G_j = \emptyset$ for $i \neq j$ we can choose $\varphi \in \mathscr{D}$ such that $\varphi = 1$ on a nieghbourdhood of $\operatorname{cl} G_i$, $\varphi = 0$ on a neighourhood of $\operatorname{cl}(G - G_i)$. Then for

$y \in B$

$$\begin{aligned} Tf_i(y) &= Vf_i(y) + \tilde{W} f_i(y) = \int_{\partial G_i} h_y(x)\,\mathrm{d}\lambda(x) + \tilde{W}\varphi(y) \\ &= 0 + \int_G \operatorname{grad}\varphi(x)\cdot \operatorname{grad}\mathscr{U}\delta_y(x)\,\mathrm{d}\mathscr{H}_m(x) = 0 \end{aligned}$$

by (7). □

1.14. Theorem. *Suppose that $d_G(y) \neq 0$ for each $y \in B$ and (6) holds. Denote by $G_1, \dots, G_j$ all bounded components of G for which $\lambda(\partial G_i) = 0$. Then*

(14) $$\mathscr{T}(\mathscr{C}') = \{\nu \in \mathscr{C}';\ \nu(\partial G_i) = 0,\ i = 1, \dots, j\}.$$

Proof. According to lemma 1.5 and lemma 1.2 the operator T is Fredholm with index null. According to lemma 1.13 we have $\dim \operatorname{Ker} T \geqslant j$. If $T'\nu = 0$ then $\nu \in \mathscr{C}'$ by lemma 1.11 and according to theorem 1.12 there are $c_1, \dots, c_p \in \mathbb{R}^1$ such that $\mathscr{U}\nu = c_i$ on G_i. Since $\sum_{i=1}^{p} c_i^2 \lambda(\partial G_i) = 0$ by theorem 1.12 we have $c_i = 0$ for $\lambda(\partial G_i) > 0$. If G_i is unbounded then $c_i = \lim_{|x|\to\infty} \mathscr{U}\nu(x) = 0$. Hence $\dim \operatorname{Ker} T' \leqslant j$ by theorem 1.12. Since $\dim \operatorname{Ker} T = \dim \operatorname{Ker} T' = j$ because the index of T is equal to 0 (see [56], chapter VII, theorem 3.1) lemma 1.13 implies that $\operatorname{Ker} T = \{\sum_{i=1}^{j} \alpha_i f_i;\ \alpha_i \in \mathbb{R}^1\}$, where f_i is the characteristic function of ∂G_i. According to [56], chapter VII, theorem 3.1 we have $T'(\mathscr{B}') = \{\nu \in \mathscr{B}'; \langle f, \nu\rangle = 0\ \forall f \in \operatorname{Ker} T\} = \{\nu \in \mathscr{B}';\ \langle f_i, \nu\rangle = 0,\ i = 1, \dots, j\}$.

According to lemma 1.2 the operator $\mathscr{T}$ is Fredholm with index null. Since $\operatorname{Ker} \mathscr{T} = \operatorname{Ker} T'$ by theorem 1.11 we have $\operatorname{codim} \mathscr{T}(\mathscr{C}') = \dim \operatorname{Ker} \mathscr{T} = j$. Since $T(\mathscr{C}') \subset \mathscr{C}' \cap T'(\mathscr{B}') = \{\nu \in \mathscr{C}';\ \nu(\partial G_i) = 0,\ i = 1, \dots, j\}$ and $\operatorname{codim}\{\nu \in \mathscr{C}';\ \nu(\partial G_i) = 0,\ i = 1, \dots, j\} = j$, we have $\mathscr{T}(\mathscr{C}') = \{\nu \in \mathscr{C}';\ \nu(\partial G_i) = 0,\ i = 1, \dots, j\}$. □

1.15. Theorem. *Denote by $\mathscr{C}'_H$ the all elements of C' which are absolutely continuous with respect to $H = \mathscr{H}_{m-1}/\hat{\partial} G$. Suppose that $d_G(y) \neq 0$ for any $y \in B$, $\lambda \in \mathscr{C}'_H$ and (6) holds. Denote by $G_1, \dots, G_j$ all bounded components of G for which $\lambda(\partial G_i) = 0$. Then*

(15) $$\mathscr{T}(\mathscr{C}'_H) = \{\nu \in \mathscr{C}'_H;\ \nu(\partial G_i) = 0,\ i = 1, \dots, j\}.$$

Proof. It is known from proposition 12 in [44] that $\mathscr{T}(\mathscr{C}'_H) \subset \mathscr{C}'_H$ and $\mathscr{T}\nu \in \mathscr{C}'_H$ for a $\nu \in \mathscr{C}'$ implies $\nu \in \mathscr{C}'_H$. Theorem 1.14 yields

$$\mathscr{T}(\mathscr{C}'_H) \subset \{\nu \in \mathscr{C}'_H;\ \nu(\partial G_i) = 0,\ i = 1, \dots, j\}.$$

On the other hand if $\nu \in \mathscr{C}'_H$ and $\nu(\partial G_i) = 0$ for $i = 1, \dots, j$, then there is a $\mu \in \mathscr{C}'$ such that $\mathscr{T}\mu = \nu$ by theorem 1.14. Consequently, $\mu \in \mathscr{C}'_H$. □

2. The essential radius of the Neumann operator

In this section we shall study conditions under which the essential radius of the Neumann operator $(2W^G - I)$ is smaller than 1. Here $G \subset \mathbb{R}^m$, $m \geqslant 2$, is again a Borel set with a bounded boundary.

2.1. Lemma. *Let $D \subset \mathbb{R}^m$ be an open set, $\psi\colon D \to \mathbb{R}^m$ a diffeomorphism of class $C^{1+\alpha}$, where $0 < \alpha < 1$. Let G be bounded, $\operatorname{cl} G \subset D$.*

1) $\hat\partial\psi(G) = \psi(\hat\partial G)$ and $n^{\psi(G)}(\psi(x))$ is a normal vector to the hypersurface $\psi(\{z \in D;\ (z - x)\cdot n^G(x) = 0\})$ at $\psi(x)$ for each $x \in \hat\partial G$.

2) If $x \in B$, $D\psi(x) = I$, where $D\psi(x)$ is the differential of ψ at the point x then for every $\varepsilon > 0$ there is $r > 0$ such that for each $y \in B \cap \mathscr{U}(x;r)$ and for each Borel function f, $|f| \leqslant 1$

$$\left| \int_{B\cap\mathscr{U}(x;r)} f(z)\,\operatorname{grad} h_y(z)\cdot n^G(z)\,\mathrm{d}\mathscr{H}_{m-1}(z) \right.$$
$$\left. - \int_{\psi(B\cap\mathscr{U}(x,r))} f\left(\psi^{-1}(w)\right)\operatorname{grad} h_{\psi(y)}(w)\cdot n^{\psi(G)}(w)\,\mathrm{d}\mathscr{H}_{m-1}(w) \right| \leqslant \varepsilon.$$

Proof. For 1) see [40], lemma 7.

According to [41], lemma 3 for every $\delta > 0$ there is $R_1 > 0$ such that for every $z \in \hat\partial G$, $|z - x| < R_1$

$$|n^{\psi(G)}(\psi(z))\cdot D\psi(z) n^G(z) - 1| < \delta, \tag{16}$$

according to [41], lemma 4 there are positive constants R_2, K_1 such that for $r \in (0, R_2)$, $y \in B$, $z \in \hat\partial G$, $|y - x| < r$, $|x - z| < r$, $y \neq z$

$$\left|\frac{|z - y|^m}{|\psi(z) - \psi(y)|^m} - 1\right| \leqslant K_1 r^\alpha \tag{17}$$

and according to [41], lemma 6 there exist positive constants R_3, K_2 such that for every $y \in B$, $z \in \hat\partial G$, $0 < |y - z| < R_3$

$$\left|\operatorname{grad} h_{\psi(y)}(\psi(z))\cdot n^{\psi(G)}(\psi(z)) \right.$$
$$\left. - \frac{1}{A|\psi(z) - \psi(y)|^m}[(z - y)\cdot n^G(z)][n^{\psi(G)}(\psi(z))\cdot D\psi(z) n^G(z)]\right|$$
$$\leqslant K_2|y - z|^{1+\alpha-m}. \tag{18}$$

Since

$$\begin{aligned}
&\left|\operatorname{grad} h_{\psi(y)}(\psi(z)) \cdot n^{\psi(G)}(\psi(z)) - \operatorname{grad} h_y(z) \cdot n^G(z)\right| \\
&\leqslant \Big|\operatorname{grad} h_{\psi(y)}(\psi(z)) \cdot n^{\psi(G)}(\psi(z)) \\
&\quad - \frac{1}{A|\psi(z)-\psi(y)|^m}[(z-y)\cdot n^G(z)][n^{\psi(G)}(\psi(z)) \cdot D\psi(z) n^G(z)]\Big| \\
&\quad + \left|\operatorname{grad} h_y(z) \cdot n^G(z)[n^{\psi(G)}(\psi(z)) \cdot D\psi(z) n^G(z)]\left[\frac{|z-y|^m}{|\psi(z)-\psi(y)|^m} - 1\right]\right| \\
&\quad + \left|\operatorname{grad} h_y(z) \cdot n^G(z)[n^{\psi(G)}(\psi(z)) \cdot D\psi(z) n^G(z) - 1]\right|
\end{aligned}$$

there are positive constants c_1, r_1 such that for $y \in B \cap \mathscr{U}(x; r_1)$, $z \in \hat{\partial} G \cap \mathscr{U}(x; r_1)$ we have

$$\begin{aligned}
(19)\qquad &\left|\operatorname{grad} h_{\psi(y)}(\psi(z)) \cdot n^{\psi(G)}(\psi(z)) - \operatorname{grad} h_y(z) \cdot n^G(z)\right| \\
&\leqslant c_1 |y-z|^{\alpha+1-m} + \frac{\varepsilon}{6(V^G+\varepsilon)} \left|\operatorname{grad} h_y(z) \cdot n^G(z)\right|
\end{aligned}$$

(see (18), (17), (16)). Since $D\psi(x) = I$, we may choose r_1 small enough so that

$$\left(1 - \frac{\varepsilon}{6(V^G+\varepsilon)}\right)^{\frac{1}{m-1}} \leqslant \frac{|\psi(y)-\psi(z)|}{|y-z|} \leqslant \left(1 + \frac{\varepsilon}{6(V^G+\varepsilon)}\right)^{\frac{1}{m-1}}$$

for arbitrary $y, z \in B \cap \mathscr{U}(x; r_1)$. Thus for every non-negative Borel function g on B

$$\begin{aligned}
(20)\qquad &\left(1 - \frac{\varepsilon}{6(V^G+\varepsilon)}\right) \int_{B\cap\mathscr{U}(x;r_1)} g \, \mathrm{d}\mathscr{H}_{m-1} \leqslant \int_{\psi(B\cap\mathscr{U}(x;r_1))} g \circ \psi^{-1} \, \mathrm{d}\mathscr{H}_{m-1} \\
&\leqslant \left(1 + \frac{\varepsilon}{6(V^G+\varepsilon)}\right) \int_{B\cap\mathscr{U}(x;r_1)} g \, \mathrm{d}\mathscr{H}_{m-1}
\end{aligned}$$

and for every function g on B integrable with respect to $\mathscr{H}_{m-1}$

$$\begin{aligned}
(21)\qquad &\left|\int_{B\cap\mathscr{U}(x;r_1)} g \, \mathrm{d}\mathscr{H}_{m-1} - \int_{\psi(B\cap\mathscr{U}(x;r_1))} g \circ \psi^{-1} \, \mathrm{d}\mathscr{H}_{m-1}\right| \\
&\leqslant \frac{\varepsilon}{6(V^G+\varepsilon)} \int_{B\cap\mathscr{U}(x;r_1)} |g| \, \mathrm{d}\mathscr{H}_{m-1}.
\end{aligned}$$

According to [28], Corollary 2.17 and [40], lemma 9, there is a constant c_2 such that for each $y \in B$ and $r > 0$

$$(22)\qquad \int_{\hat{\partial} G\cap\mathscr{U}(y;r)} |y-z|^{\alpha+1-m} \, \mathrm{d}\mathscr{H}_{m-1}(z) \leqslant c_2 r^\alpha.$$

$\hat{\partial}\psi(G) = \psi(\hat{\partial}G)$ according to 1). If $r < \min\left(r_1, \frac{1}{2}(\varepsilon/4c_1c_2)^{1/\alpha}\right)$, $y \in B \cap \mathscr{U}(x;r)$, f is a Borel function on $B, |f| \leqslant 1$ then

$$\Big| \int_{\psi(B\cap\mathscr{U}(x;r))} f\big(\psi^{-1}(w)\big) \operatorname{grad} h_{\psi(y)}(w) \cdot n^{\psi(G)}(w) \,\mathrm{d}\mathscr{H}_{m-1}(w)$$

$$- \int_{B\cap\mathscr{U}(x;r)} f(z) \operatorname{grad} h_y(z) \cdot n^G(z) \,\mathrm{d}\mathscr{H}_{m-1}(z) \Big|$$

$$\leqslant \Big| \int_{\psi(B\cap\mathscr{U}(x;r))} f\big(\psi^{-1}(w)\big) [\operatorname{grad} h_{\psi(y)}(w) \cdot n^{\psi(G)}(w)$$

$$- \operatorname{grad} h_y\big(\psi^{-1}(w)\big) \cdot n^G\big(\psi^{-1}(w)\big)] \,\mathrm{d}\mathscr{H}_{m-1}(w) \Big|$$

$$+ \Big| \int_{\psi(B\cap\mathscr{U}(x;r))} f\big(\psi^{-1}(w)\big) \operatorname{grad} h_y\big(\psi^{-1}(w)\big) \cdot n^G\big(\psi^{-1}(w)\big) \,\mathrm{d}\mathscr{H}_{m-1}(w)$$

$$- \int_{B\cap\mathscr{U}(x;r)} f(z) \operatorname{grad} h_y(z) \cdot n^G(z) \,\mathrm{d}\mathscr{H}_{m-1}(z) \Big|$$

$$\leqslant \int_{\psi(\hat{\partial}G\cap\mathscr{U}(x;r))} \Big[c_1 |y - \psi^{-1}(w)|^{\alpha+1-m}$$

$$+ \frac{\varepsilon}{6(V^G+\varepsilon)} \big|\operatorname{grad} h_y\big(\psi^{-1}(w)\big) \cdot n^G\big(\psi^{-1}(w)\big)\big| \Big] \,\mathrm{d}\mathscr{H}_{m-1}(w)$$

$$+ \Big| \int_{\psi(B\cap\mathscr{U}(x;r))} f\big(\psi^{-1}(w)\big) \operatorname{grad} h_y\big(\psi^{-1}(w)\big) \cdot n^G\big(\psi^{-1}(w)\big) \,\mathrm{d}\mathscr{H}_{m-1}(w)$$

$$- \int_{B\cap\mathscr{U}(x;r)} f(z) \operatorname{grad} h_y(z) \cdot n^G(z) \,\mathrm{d}\mathscr{H}_{m-1}(z) \Big|$$

by (19). According to (20) and (21) we have

$$\Big| \int_{\psi(B\cap\mathscr{U}(x;r))} f\big(\psi^{-1}(w)\big) \operatorname{grad} h_{\psi(y)}(w) \cdot n^{\psi(G)}(w) \,\mathrm{d}\mathscr{H}_{m-1}(w)$$

$$- \int_{B\cap\mathscr{U}(x;r)} f(z) \operatorname{grad} h_y(z) \cdot n^G(z) \,\mathrm{d}\mathscr{H}_{m-1}(z) \Big|$$

$$\leqslant 2c_1 \int_{\hat{\partial}G\cap\mathscr{U}(x;r)} |y - z|^{\alpha+1-m} \,\mathrm{d}\mathscr{H}_{m-1}(z)$$

$$+ \frac{2\varepsilon}{6(V^G+\varepsilon)} \int_{\hat{\partial}G\cap\mathscr{U}(x;r)} \big|\operatorname{grad} h_y(z) \cdot n^G(z)\big| \,\mathrm{d}\mathscr{H}_{m-1}(z)$$

$$+ \frac{\varepsilon}{6(V^G+\varepsilon)} \int_{\hat{\partial}G\cap\mathscr{U}(x;r)} \big|\operatorname{grad} h_y(z) \cdot n^G(z)\big| \,\mathrm{d}\mathscr{H}_{m-1}(z) \leqslant \varepsilon$$

by (22). □

2.2. Lemma. *Suppose that for each $x \in B$ there are a natural number $n(x)$, a compact linear operator K_x on $\mathscr{C}$ and $\alpha_x \in \mathscr{C}$ such that $\alpha_x = 1$ in a neighbourhood* *of x and*

$$\|\alpha_x[(2W^G - I)^{n(x)} + K_x]\alpha_x f\| \leqslant q_x < 1 \tag{23}$$

for all $f \in \mathscr{C}$, $|f| \leqslant 1$. Then $r_{\text{ess}}(2W^G - I) < 1$.

P r o o f. For every $x \in B$ there is $\delta(x) > 0$ such that $\alpha_x = 1$ on $\mathscr{U}(x;\delta(x))$. Since B is compact there are $x^1, \ldots, x^k \in B$ such that

$$B \subset \bigcup_{i=1}^{k} \mathscr{U}(x^i;\delta(x^i)).$$

There exist $\beta_1, \ldots, \beta_k \in \mathscr{C}$, $0 \leqslant \beta_i \leqslant 1$, $\operatorname{spt}\beta_i \subset \mathscr{U}(x^i;\delta(x^i))$ such that

$$\sum_{i=1}^{k} \beta_i = 1$$

on B. Put

$$q = \max_{i=1,\ldots,k} q_{x_i}. \tag{24}$$

Choose a natural number w such that

$$kq^w < 1. \tag{25}$$

Put

$$n = w\prod_{i=1}^{k} n(x^i).$$

For $i \in \{1, \ldots, k\}$ put

$$n(i) = n(x^i), \quad m(i) = \frac{n}{n(i)}, \quad \alpha_0^i = \beta_i.$$

For $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, n+1\}$ choose a function $\alpha_j^i \in \mathscr{C}$ such that $0 \leqslant \alpha_j^i \leqslant 1$, $\alpha_j^i = 1$ on V_{j-1}^i a neighbourhood of $\operatorname{spt}\alpha_{j-1}^i$ and $\operatorname{spt}\alpha_j^i \subset \mathscr{U}(x^i;\delta(x^i))$. Denote A_j^i operator $A_j^i f = \alpha_j^i f$ on $\mathscr{C}$. The operator $A_j^i(2W^G - I)(I - A_{j+1}^i)$ is an integral operator on $\mathscr{C}$ with the kernel $-2\alpha_j^i(x)(1 - \alpha_{j+1}^i(y))n^G(y) \cdot \operatorname{grad} h_x(y)$ which is different from 0 only for $y \notin V_j^i \supset \operatorname{spt}\alpha_j^i$, $x \in \operatorname{spt}\alpha_j^i$ and thus this kernel is a bounded and equicontinuous function of the variable x. The operator $A^i_j(2W^G - I)(I - A^i_{j+1})$ is compact. Since

$$\begin{aligned}
&A^i_j(2W^G - I)^s(I - A^i_{j+s}) = A^i_j(2W^G - I)^{s-1}A^i_{j+s-1}(2W^G - I)(I - A^i_{j+s}) \\
&+ A^i_j(2W^G - I)^{s-2}A^i_{j+s-2}(2W^G - I)(I - A^i_{j+s-1})(2W^G - I)(I - A^i_{j+s}) \\
&\dots \\
&+ A^i_j(2W^G - I)A^i_{j+1}(2W^G - I)(I - A^i_{j+2})\dots(2W^G - I)(I - A^i_{j+s}) \\
&+ A^i_j(2W^G - I)(I - A^i_{j+1})(2W^G - I)(I - A^i_{j+2})\dots(2W^G - I)(I - A^i_{j+s})
\end{aligned}$$

the operator $A^i_j(2W^G - I)^s(I - A^i_{j+s})$ is compact, too. Since $\sum_{i=1}^{k} \beta_i = 1$ on B and $\alpha^i_1 = 1$ on $\operatorname{spt}\beta_i$ new have

$$\begin{aligned}
(26)\quad (2W^G - I)^n &= \sum_{i=1}^{k} \beta_i A^i_1(2W^G - I)^n \\
&= \sum_{i=1}^{k} \beta_i\{\{A^i_1(2W^G - I)^{n(i)}[A^i_{n(i)+1} + (I - A^i_{n(i)+1})]\} \\
&\circ \{[A^i_{n(i)+1} + (I - A^i_{n(i)+1})](2W^G - I)^{n(i)}[A^i_{2n(i)+1} + (I - A^i_{2n(i)+1})]\} \\
&\dots\{[A^i_{n(i)(m(i)-1)+1} \\
&\qquad + (I - A^i_{n(i)(m(i)-1)+1})](2W^G - I)^{n(i)}[A^i_{n+1} + (I - A^i_{n+1})]\}\}.
\end{aligned}$$

Calculate the right side of the equality. Since each member includes the term A^i_1, each member, which includes the term $(I - A^i_j)$, includes the term $A^i_r(2W^G - I)^{n(i)}(I - A^i_{r+n(i)})$ for some integer r. Since the operator $A^i_r(2W^G - I)^{n(i)}(I - A^i_{r+n(i)})$ is compact we have by (26)

$$\begin{aligned}
&r_{\text{ess}}(2W^G - I) \leqslant [\|(2W^G - I)^n\|_{\text{ess}}]^{1/n} \\
&= \Big\{\Big\|\sum_{i=1}^{k} \beta_i[A^i_1(2W^G - I)^{n(i)}A^i_{n(i)+1}][A^i_{n(i)+1}(2W^G - I)^{n(i)}A^i_{2n(i)+1}] \\
&\quad \dots[A_{n(i)(m(i)-1)+1}(2W^G - I)^{n(i)}A^i_{n+1}]\Big\|_{\text{ess}}\Big\}^{1/n} \\
&\leqslant \Big\{\Big\|\sum_{i=1}^{k} \beta_i\{A^i_1[(2W^G - I)^{n(i)} + K_{x^i}]A^i_{n(i)+1}\} \\
&\quad \dots\{A^i_{n(i)(m(i)-1)+1}[(2W^G - I)^{n(i)} + K_{x^i}]A^i_{n+1}\}\Big\|\Big\}^{1/n}
\end{aligned}$$

$$\leqslant \left[\sum_{i=1}^{k} \prod_{j=1}^{m(i)} \|A^i_{(j-1)n(i)+1}[(2W^G - I)^{n(i)} + K_{x^i}]A^i_{jn(i)+1}\|\right]^{1/n}$$

$$\leqslant \left[\sum_{i=1}^{k} q^{m(i)}\right]^{1/n} \leqslant [kq^w]^{1/n} < 1$$

by (23), (24), (24), because $\alpha_{x^i} = 1$ on spt α^i_j. □

2.3. Theorem. *Suppose that for each $x \in B$ there are $r(x) > 0$, an open set D_x with a compact boundary and diffeomorphism $\psi_x : \mathscr{U}(x; r(x)) \to \mathbb{R}^m$ of class $C^{1+\alpha}$, where $\alpha > 0$, such that*

$$\psi_x(G \cap \mathscr{U}(x; r(x))) = D_x \cap \psi_x(\mathscr{U}(x; r(x))), \quad V^{D_x} < \infty,$$

$r_{\mathrm{ess}}(2W^{D_x} - I) < 1$ and $D\psi_x(x) = I$. Then $r_{\mathrm{ess}}(2W^G - I) < 1$.

Proof. Fix $x \in B$. Put $D \equiv D_x$, $\psi \equiv \psi_x$. Denote

$$S = 2W^G - I,$$
$$\tilde{S} = 2W^D - I.$$

According to the assumption there is a natural number n and a compact operator K on $\mathscr{C}(\partial D)$ (the space of the continuous functions on ∂D) such that

$$\|(\tilde{S})^n + K\| < \frac{1}{4}. \tag{27}$$

Denote

$$L = \max(\|S\|, \|\tilde{S}\|). \tag{28}$$

Since $D\psi(x) = I$ according to lemma 2.1 there is a $\delta_0 > 0$ such that for $y \in B \cap \mathscr{U}(x; \delta_0)$, $f \in \mathscr{C}$, $|f| \leqslant 1$

$$\begin{aligned} &\left| \int_{B \cap \mathscr{U}(x;\delta_0)} f(z) \operatorname{grad} h_y(z) \cdot n^G(z) \, \mathrm{d}\mathscr{H}_{m-1}(z) \right. \\ &\quad \left. - \int_{\psi(B \cap \mathscr{U}(x;\delta_0))} f(\psi^{-1}(w)) \operatorname{grad} h_{\psi(y)}(w) \cdot n^{\psi(G)}(w) \, \mathrm{d}\mathscr{H}_{m-1}(w) \right| \\ &< \frac{1}{8(4L+1)^n}. \end{aligned} \tag{29}$$

Choose $\delta_1, \ldots, \delta_n$ such that $\delta_j < \delta_{\frac{j-1}{2}}$, for $y \in B - \mathscr{U}(x; \delta_{j-1}/2)$

$$\int_{B \cap \mathscr{U}(x;\delta_j)} |\operatorname{grad} h_y \cdot n^G| \, \mathrm{d}\mathscr{H}_{m-1} < \frac{1}{8(4L+1)^n} \tag{30}$$

and for $y \in \partial D - \psi\big(\mathscr{U}(x;\delta_{j-1}/2)\big)$

$$\text{(31)} \qquad \int_{\partial D \cap \psi(\mathscr{U}(x;\delta_j))} \left|\operatorname{grad} h_y \cdot n^D\right| \, \mathrm{d}\mathscr{H}_{m-1} < \frac{1}{8(4L+1)^n}.$$

Put

$$\alpha(t) = \begin{cases} 1 & \text{for } t \in \langle 0, \frac{1}{2} \rangle, \\ 3-4t & \text{for } t \in (\frac{1}{2}, \frac{3}{4}), \\ 0 & \text{for } t \geqslant \frac{3}{4} \end{cases}$$

and denote

$$\alpha_j(y) = \alpha(|x-y|/\delta_j).$$

For function f defined on $\mathscr{U}(x;\delta_0)$ put

$$\begin{aligned} (\tilde{P}f)(y) = & f\big(\psi^{-1}(y)\big) && \text{for } y \in \psi\big(\mathscr{U}(x;\delta_0)\big), \\ & 0 && \text{for the remaining } y \in \mathbb{R}^m. \end{aligned}$$

Similarly, for function f defined on $\psi\big(\mathscr{U}(x;\delta_0)\big)$ put

$$\begin{aligned} (Pf)(y) = & f\big(\psi(y)\big) && \text{for } y \in \mathscr{U}(x;\delta_0), \\ & 0 && \text{for the remaining } y \in \mathbb{R}^m. \end{aligned}$$

We will prove that for $j = 1, \ldots, n$ and $f \in \mathscr{C}$, $|f| \leqslant 1$

$$\text{(32)} \qquad \|\alpha_{j-1} S^j \alpha_j f - \alpha_{j-1} P[(\tilde{S})^j \tilde{P}(\alpha_j f)]\| \leqslant \frac{1}{4(4L+1)^{n-j+1}}.$$

If $y \in \hat{\partial} G \cap \mathscr{U}(x;\delta_0)$ then $d_G(y) = d_D\big(\psi(y)\big) = \frac{1}{2}$ by lemma 2.1. If $y \in B \cap \mathscr{U}(x;\delta_0)$ and there is a $\varrho > 0$ such that $\mathscr{H}_m\big(G \cap \mathscr{U}(y;\varrho)\big) = 0$ then $d_G(y) = d_D\big(\psi(y)\big) = 0$. If $y \in B \cap \mathscr{U}(x;\delta_0)$ and there is a $\varrho > 0$ such that $\mathscr{H}_m\big(\mathscr{U}(y;\varrho) - G\big) = 0$ then $d_G(y) = d_D\big(\psi(y)\big) = 1$. If $y \in B_1 = \hat{\partial} G \cup \{y \in B;\ \exists \varrho > 0,\ \mathscr{H}_m\big(G \cap \mathscr{U}(y;\varrho)\big) = 0\} \cup \{y \in B;\ \exists \varrho > 0,\ \mathscr{H}_m\big(\mathscr{U}(y;\varrho) - G\big) = 0\}$ then according to (29) and (3)

$$\text{(33)} \qquad |\alpha_{j-1}(y) S(\alpha_j f)(y) - \alpha_{j-1}(y)\big[P\big(\tilde{S}\tilde{P}(\alpha_j f)\big)\big](y)| \leqslant \frac{1}{4(4L+1)^n}.$$

Since B_1 is dense in B by the Isoperimetric Lemma (see [28], p. 50) the continuity of $\alpha_{j-1}\{S(\alpha_j f) - P[\tilde{S}\tilde{P}(\alpha_j f)]\}$ yields (33) for all $y \in B$. Thus the relation (32) holds for $j = 1$.

Now, let the relation (32) holds for $j = r$. According to (30) and (3)

$$\text{(34)} \qquad \|(1-\alpha_r) S \alpha_{r+1} f\| \leqslant \frac{1}{4(4L+1)^n}.$$

According to (31)

$$(35) \qquad \|(1-\tilde{P}\alpha_r)\tilde{S}\tilde{P}(\alpha_{r+1}f)\| \leqslant \frac{1}{4(4L+1)^n}.$$

We have

$$\begin{aligned}
(36)\quad & \|\alpha_r S^{r+1}\alpha_{r+1}f - \alpha_r P[(\tilde{S})^{r+1}\tilde{P}(\alpha_{r+1}f)]\| \\
& \leqslant \|\alpha_r S^r(1-\alpha_r)S\alpha_{r+1}f\| + \|\alpha_r S^r\alpha_r S\alpha_{r+1}f - \alpha_r P[(\tilde{S})^r\tilde{P}(\alpha_r S\alpha_{r+1}f)]\| \\
& \quad + \|\alpha_r P[(\tilde{S})^r\tilde{P}(\alpha_r S\alpha_{r+1}f)] - \alpha_r P[(\tilde{S})^r(\tilde{P}\alpha_r)\tilde{S}\tilde{P}(\alpha_{r+1}f)]\| \\
& \quad + \|\alpha_r P[(\tilde{S})^r(\tilde{P}\alpha_r)\tilde{S}\tilde{P}(\alpha_{r+1}f)] - \alpha_r P[(\tilde{S})^{r+1}\tilde{P}(\alpha_{r+1}f)]\|.
\end{aligned}$$

Now we estimate the terms in the right side of (36).

$$\|\alpha_r S^r(1-\alpha_r)S\alpha_{r+1}f\| \leqslant \|S^r\|\,\|(1-\alpha_r)S\alpha_{r+1}f\| \leqslant L^r\frac{1}{4(4L+1)^n}$$

by (28) and (34). Since $\|S\alpha_{r+1}f\| \leqslant L$ by (28) and $0 \leqslant \alpha_r \leqslant \alpha_{r-1}$ we obtain

$$\begin{aligned}
& \|\alpha_r S^r\alpha_r(S\alpha_{r+1}f) - \alpha_r(P(\tilde{S})^r\tilde{P}(\alpha_r S\alpha_{r+1}f)]\| \\
& \leqslant \|\alpha_{r-1}S^r\alpha_r(S\alpha_{r+1}f) - \alpha_{r-1}P[(\tilde{S})^r\tilde{P}(\alpha_r S\alpha_{r+1}f)]\| \\
& \leqslant L\frac{1}{4(4L+1)^{n-r+1}}
\end{aligned}$$

using that the relation (32) holds for $j=r$ and the function $\frac{1}{L}S\alpha_{r+1}f$.

$$\begin{aligned}
& \|\alpha_r P[(\tilde{S})^r\tilde{P}(\alpha_r S\alpha_{r+1}f)] - \alpha_r P[(\tilde{S})^r(\tilde{P}\alpha_r)\tilde{S}\tilde{P}(\alpha_{r+1}f)]\| \\
& = \|(\tilde{P}\alpha_r)(\tilde{S})^r[\tilde{P}(\alpha_r S\alpha_{r+1}f) - (\tilde{P}\alpha_r)\tilde{S}\tilde{P}(\alpha_{r+1}f)]\| \\
& \leqslant \|\tilde{S}\|^r\|\alpha_r S\alpha_{r+1}f - \alpha_r P[\tilde{S}\tilde{P}(\alpha_{r+1}f)]\| \leqslant L^r\frac{1}{4(4L+1)^n}
\end{aligned}$$

by (28) and (33).

$$\begin{aligned}
& \|\alpha_r P[(\tilde{S})^r(\tilde{P}\alpha_r)\tilde{S}\tilde{P}(\alpha_{r+1}f)] - \alpha_r P[(\tilde{S})^{r+1}\tilde{P}(\alpha_{r+1}f)]\| \\
& = \|(\tilde{P}\alpha_r)(\tilde{S})^r(\tilde{P}\alpha_r - 1)\tilde{S}\tilde{P}(\alpha_{r+1}f)\| \leqslant \|\tilde{S}\|^r\|(1-\tilde{P}\alpha_r)\tilde{S}\tilde{P}(\alpha_{r+1}f)\| \\
& \leqslant L^r\frac{1}{4(4L+1)^n}
\end{aligned}$$

by (28) and (35). Using these estimates and (36) we obtain

$$\begin{aligned}
& \|\alpha_r S^{r+1}\alpha_{r+1}f - \alpha_r P[(\tilde{S})^{r+1}\tilde{P}(\alpha_{r+1}f)]\| \\
& \leqslant 3\frac{L^r}{4(4L+1)^n} + \frac{L}{4(4L+1)^{n-r+1}} \leqslant \frac{1}{4(4L+1)^{n-r}}
\end{aligned}$$

which is the relation (32) for $j = r + 1$. So we have proved the relation (32) by the induction.

Using (32) for $j = n$ and (27) we obtain

$$\begin{aligned}
&\|\alpha_{n-1} S^n \alpha_n f + \alpha_{n-1} P[K\tilde{P}(\alpha_n f)]\| \\
&\leqslant \|\alpha_{n-1} S^n \alpha_n f - \alpha_{n-1} P[(\tilde{S})^n \tilde{P}(\alpha_n f)]\| \\
&\quad + \|\alpha_{n-1} P[(\tilde{S})^n \tilde{P}(\alpha_n f) + K\tilde{P}(\alpha_n f)]\| \\
&\leqslant \frac{1}{4(4L+1)} + \frac{1}{4} \leqslant \frac{1}{2}.
\end{aligned}$$

Hence, the assumptions of the lemma 2.2 are fulfilled and $r_{\mathrm{ess}}(2W^G - I) < 1$. □

2.4. Remark. It is well-known that if G is a set with sufficiently smooth boundary, a convex set or a complement of a convex set then $r_{\mathrm{ess}}(2W^G - I) < 1$. (See for example [28].)

2.5. Definition. Let $\Omega \subset \mathbb{R}^m$ be an open set. We call Ω an open polyhedral set if its boundary $\partial\Omega$ is locally a hypersurface (i.e. every point of $\partial\Omega$ has a neighbourhood in $\partial\Omega$ which is homeomorphic to $\mathbb{R}^{m-1}$) and $\partial\Omega$ is formed by a finite number of $(m-1)$-dimensional polyhedrons.

2.6. Proposition. *If $G \subset \mathbb{R}^3$ is a polyhedral set then $r_{\mathrm{ess}}(2W^G - I) < 1$.*

Proof. At first we define W^M for a polyhedral cone $M \subset \mathbb{R}^3$. We denote by $C(\partial M)$ the space of bounded continuous functions on ∂M having a finite limit at infinity equipped by the maximum norm. We define a bounded linear operator W^M on $C(\partial M)$

$$W^M f(x) = d_M(x) - \int_{\partial M} f(y) n^M(y) \cdot \operatorname{grad} h_x(y) \, \mathrm{d}\mathcal{H}_2(y)$$

for $f \in C(\partial M)$, $x \in \partial M$. The spectral radius of $(2W^M - I)$ is less than 1 (see [50], cf. [19]).

Fix $x \in B$. Then there are a polyhedral cone M and $\delta > 0$ such that $G \cap \mathcal{U}(x;\delta) = M \cap \mathcal{U}(x;\delta)$. Further there is a natural number n such that

$$\|(2W^M - I)^n\| < \tfrac{1}{4}.$$

Put $\psi = I$ and repeat the conclusion from the proof of theorem 2.3. We obtain that for each $x \in B$ there are $\delta(x) > 0$ and a natural number $n(x)$ such that

$$\|\alpha_x (2W^G - I)^{n(x)} \alpha_x f\| \leqslant \tfrac{1}{2}$$

for all $f \in \mathscr{C}$, $|f| \leqslant 1$, where

$$\alpha_x(y) = \begin{cases} 1 & \text{for } |x-y| \leqslant \delta(x)/2, \\ 3 - 4|x-y|/\delta(x) & \text{for } \delta(x)/2 < |x-y| < \frac{3}{4}\delta(x), \\ 0 & \text{for } |x-y| \geqslant \frac{3}{4}\delta(x). \end{cases}$$

According to lemma 2.2 we have $r_{\text{ess}}(2W^G - I) < 1$. □

2.7. Remark. If $G \subset \mathbb{R}^2$ is a domain with a piecewise smooth boundary and $\inf_{y \in B} |d_G(y) - \frac{1}{2}| \neq \frac{1}{2}$ then $r_{\text{ess}}(2W^G - I) < 1$. (See [2], [7], [29], [49].)

3. Domains with a piecewise-smooth boundary

In this paragraph we shall suppose that $G \subset \mathbb{R}^3$ is an open set with a compact boundary. Suppose that for each $x \in B$ there are $r(x) > 0$, a domain D_x which is polyhedral, convex or a complement of a convex domain and a diffeomorphism $\psi_x \colon \mathscr{U}(x; r(x)) \to \mathbb{R}^3$ of class $C^{1+\alpha}$, where $\alpha > 0$, such that $\psi_x\big(G \cap \mathscr{U}(x; r(x))\big) = D_x \cap \psi_x\big(\mathscr{U}(x; r(x))\big)$. Since the assumptions of theorem 2.3 are fulfilled with sets $[D\psi_x(x)]^{-1}(D_x)$ and diffeomorphisms $[D\psi_x(x)]^{-1}\psi_x$ (see remark 2.4 and proposition 2.6) we have $r_{\text{ess}}(2W^G - I) < 1$.

3.1. Theorem on the third boundary value problem. *Let λ be a nonnegative element of $\mathscr{C}'$ and suppose that $\mathscr{U}\lambda$ is bounded and continuous on B. Let $\mu \in \mathscr{C}'$. Then there is a solution of the third problem*

$$-\frac{\partial u}{\partial n} + \lambda u = \mu$$

in the form $\mathscr{U}\nu$ with $\nu \in \mathscr{C}'$ if and only if $\mu(\partial\Omega) = 0$ for each bounded component Ω of G for which $\lambda(\partial\Omega) = 0$. The measure ν is uniquely determined if and only if G has no bounded component Ω for which $\lambda(\partial\Omega) = 0$. If $\lambda, \mu \in \mathscr{C}'_H$ then $\nu \in \mathscr{C}'_H$, too. If $(2\mathscr{T}_{\frac{1}{2}})^k \mu \to 0$ for $k \to \infty$ then we may put

$$\nu = \sum_{k=0}^{\infty} (-2\mathscr{T}_{\frac{1}{2}})^k 2\mu.$$

Proof. According to [45], proposition 9 the operator V is compact and $V(\mathscr{C}) \subset \mathscr{C}$. Since $r_{\text{ess}}(2W^G - I) < 1$ we have $r_{\text{ess}}\big(2W^G + 2(V/\mathscr{C}) - I\big) < 1$, where $V/\mathscr{C}$ is the restriction of V to $\mathscr{C}$. Since

$$\mathscr{T}_{\frac{1}{2}} = \tfrac{1}{2}\big(2W^G + 2(V/\mathscr{C}) - I\big)',$$

lemma 1.2 yields

$$r_{\text{ess}}(\mathcal{T}_{\frac{1}{2}}) = \tfrac{1}{2} r_{\text{ess}}\big(2W^G + 2(V/C) - I\big) < \tfrac{1}{2}.$$

According to theorem 1.14 there is $\nu \in \mathscr{C}'$ such that $\mathcal{T}\nu = \mu$ if and only if $\mu(\partial\Omega) = 0$ for each bounded component Ω of G for which $\lambda(\partial\Omega) = 0$. Since $N^G\mathscr{U}$ is a dual operator to W^G we have for $f \in \mathscr{C}$

$$\begin{aligned}\langle f, (\mathscr{U}\nu)\lambda + N^G\mathscr{U}\nu\rangle &= \int_B\int_B f(x) h_y(x)\,\mathrm{d}\lambda(x)\,\mathrm{d}\nu(y) + \langle f, N^G\mathscr{U}\nu\rangle \\ &= \langle Vf,\nu\rangle + \langle W^G f,\nu\rangle = \langle Tf,\nu\rangle = \langle f, \mathcal{T}\nu\rangle\,.\end{aligned}$$

Thus $\mathscr{U}\nu$ is a solution of the third problem

$$-\frac{\partial u}{\partial n} + \lambda u = \mu$$

if and only if $\mathcal{T}\nu = \mu$. Since $\mathcal{T}$ is a Fredholm operator with index 0, because $r_{\text{ess}}(\mathcal{T}_{\frac{1}{2}}) < \frac{1}{2}$, the measure ν is uniquely determined iff $\mathcal{T}(\mathscr{C}') = \mathscr{C}'$, what happens if and only if G has no bounded component Ω for which $\lambda(\partial\Omega) = 0$. If $\lambda, \mu \in \mathscr{C}'_H$ then proposition 12 in [44] implies $\nu \in \mathscr{C}'_H$.

Suppose now that $(2\mathcal{T}_{\frac{1}{2}})^k\mu \to 0$ for $k \to \infty$. Since $r_{\text{ess}}(2\mathcal{T}_{\frac{1}{2}}) < 1$ there are a natural number n and a compact linear operator K on $\mathscr{C}'$ such that $\|(2\mathcal{T}_{\frac{1}{2}})^n + K\| < 1$. According to [39] the series

$$\sum_{j=0}^{\infty}(-2\mathcal{T}_{\frac{1}{2}})^{nj}\mu$$

converges. For given $\varepsilon > 0$ there is a natural number k such that for $m_2 \geqslant m_1 \geqslant k$ we have

$$\Big\|\sum_{j=m_1}^{m_2}(-2\mathcal{T}_{\frac{1}{2}})^{nj}\mu\Big\| < \varepsilon\Big[\sum_{i=0}^{n-1}\|(-2\mathcal{T}_{\frac{1}{2}})^i\|\Big]^{-1}.$$

If $m_2 \geqslant m_1 \geqslant nk$ we have

$$\Big\|\sum_{p=m_1}^{m_2}(-2\mathcal{T}_{\frac{1}{2}})^p\mu\Big\| \leqslant \sum_{i=0}^{n-1}\|(-2\mathcal{T}_{\frac{1}{2}})^i\|\Big\|\sum_{\substack{j\\ m_1\leqslant nj\leqslant m_2-i}}(-2\mathcal{T}_{\frac{1}{2}})^{nj}\mu\Big\| < \varepsilon.$$

The series

$$\nu = \sum_{j=0}^{\infty}(-2\mathcal{T}_{\frac{1}{2}})^j 2\mu$$

converges and $\mathcal{T}\nu = \frac{1}{2}[I + 2\mathcal{T}_{\frac{1}{2}}]\nu = \mu$. □

3.2. Theorem on the Dirichlet problem. *Denote by $G_1, \dots, G_p$ bounded components of G. Fix $x_j \in \operatorname{int} G_j$ $(j = 1, \dots, p)$. Given $g \in \mathscr{C}$, then there are constant $c_1, \dots, c_p$ and an $f \in \mathscr{C}$ such that*

$$W^G f + \sum_{j=1}^{p} c_j h_{x_j}$$

represents a solution of the Dirichlet problem for $C = \mathbb{R}^3 - \operatorname{cl} G$ and the boundary condition g. The constants $c_1, \dots, c_p$ are uniquely determined. The function f is uniquely determined iff G is unbounded and connected. If $(I - 2W^G)^j f \to 0$ for $j \to \infty$ then we may put

$$f = \sum_{k=0}^{\infty} (I - 2W^G)^k 2g$$

and $c_1 = \ldots = c_p = 0$.

Proof. Since $r_{\mathrm{ess}}(2W^G - I) < 1$ the operator W^G is Fredholm with index 0 by lemma 1.2. Since $N^G \mathscr{U}$ is a dual operator to W^G (see [28], proposition 2.20) we have $\dim \operatorname{Ker} N^G \mathscr{U} = \operatorname{codim} N^G \mathscr{U}(\mathscr{C}') = p$ by theorem 3.1 and [56], chapter V, theorem 4.1.

Now, we will prove that we can choose $\mu_1, \dots, \mu_p \in \operatorname{Ker} N^G \mathscr{U}$ such that

$$\langle h_{x_i}, \mu_j \rangle = \delta_{ij} \quad \text{for } i, j = 1, \dots, p. \tag{37}$$

If $\nu \in \operatorname{Ker} N^G \mathscr{U}$ then there are $\psi_n \in \mathscr{D}$ such that

$$\lim_{n \to \infty} \int_G \operatorname{grad} \psi_n(x) \cdot \operatorname{grad} \mathscr{U}\nu(x) \, \mathrm{d}\mathscr{H}_m(x) = \int_G |\operatorname{grad} U\nu(x)|^2 \, \mathrm{d}\mathscr{H}_m(x)$$

(see [46], lemma 24 and lemma 25). Since

$$\int_G \operatorname{grad} \psi_n(x) \cdot \operatorname{grad} \mathscr{U}\nu(x) \, \mathrm{d}\mathscr{H}_m(x) = \langle \psi_n, N^G \mathscr{U}\nu \rangle = 0$$

we have $\operatorname{grad} \mathscr{U}\nu = 0$ in G. The function $\mathscr{U}\nu$ is constant in each component of G. If $\mathscr{U}\nu = 0$ in $G_1 \cup \ldots \cup G_p$ then $\mathscr{U}\nu \equiv 0$ in G. Let $\nu = \nu^+ - \nu^-$ be the Jordan decomposition of ν. Since $d_G(y) \neq 0$ for each $y \in B$, the set G is not thin at any $y \in B$ (see [3], chap. VII, §2) and we have $\nu^+ = \nu^-$ (see [34], theorem 5.10 and chap. V, §1, section 2, 14). In this case $\nu = 0$.

Since $\dim \operatorname{Ker} N^G\mathscr{U} = p$ there are $\mu_1, \dots, \mu_p$ which form a base of $\operatorname{Ker} N^G\mathscr{U}$ such that (37) holds. The function

$$\tilde{g} = 2g - \sum_{j=1}^{p} c_j h_{x_j}$$

will belong to $W^G(\mathscr{C})$ iff

$$\langle \tilde{g}, \mu_j \rangle = 0, \quad 1 \leqslant j \leqslant p.$$

We put $c_j = \langle 2g, \mu_j \rangle$.

The rest of the proof is the same as in the proof of theorem 3.1. □

3.3. Note. The attentive reader will note that the restriction to $\mathbb{R}^3$ is dectated by using the fact that the spectral radius of $(2W^G - I)$ is less than 1 for a polyhedral cone in $\mathbb{R}^3$ (cf. [50]). It would be very interesting to know whether similar result holds in higher dimensions.

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