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DESCRIPTIONS OF EXCEPTIONAL SETS IN THE CIRCLES FOR FUNCTIONS FROM THE BERGMAN SPACE

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Abstract. Let D be a domain in $\mathbb{C}^2$. For $w \in \mathbb{C}$, let $D_w = \{z \in \mathbb{C} \mid (z, w) \in D\}$. If f is a holomorphic and square-integrable function in D, then the set $E(D, f)$ of all w such that $f(.,w)$ is not square-integrable in D_w is of measure zero. We call this set the exceptional set for f. In this note we prove that for every $0 < r < 1$,and every G_δ-subset E of the circle $C(0,r) = \{z \in \mathbb{C} \mid |z| = r\}$,there exists a holomorphic square-integrable function f in the unit ball B in $\mathbb{C}^2$ such that $E(B, f) = E$.

1. INTRODUCTION

In [1] the following question was investigated: Let D be an open set in $\mathbb{C}^{n+m}$. Let $L^2H(D)$ be the space of all functions from the space $L^2(D)$ (with respect to the Lebesgue measure) which are holomorphic in D. Given $w \in C^m$, set

$$D_w = D \cap (\mathbb{C}^n \times \{w\}),$$

and let $p(D_w)$ be the projection of D_w onto the first coordinate space, i.e.

$$p(D_w) = \{z \in \mathbb{C}^n \mid (z, w) \in D\}.$$

For $f \in L^2H(D)$, the function $f|_{D_w}$ can be considered as a function holomorphic on the open set $p(D_w)$ in $\mathbb{C}^n$ (this set, as well as the set D_w, can be empty). Denote by $E(D, f)$ the set of all $w \in \mathbb{C}^m$ with $p(D_w) \neq \emptyset$ and such that $f|_{D_w}$ is not in $L^2(p(D_w))$. We call the set $E(D, f)$ the exceptional set for the function f. By Fubini theorem, $E(D, f)$ has Lebesgue measure zero in $\mathbb{C}^m$. What further properties have

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the exceptional sets? (The investigation of the properties of functions from various spaces on lower-dimensional slices was carried out by several authors; see e.g. [3].) It was shown in [1] that if D is a Hartogs domain in $\mathbb{C}^2$ (i.e. $n = m = 1$) then $E(D, f)$ is a G_δ-set; conversely, for every G_δ-set E in $\mathbb{C}$ of Lebesgue measure zero there exists a Hartogs domain $D \subset \mathbb{C}^2$ and a function $f \in L^2H(D)$ such that $E = E(D, f)$. It was conjectured that if D is a domain of holomorphy and $n = 1$ (i.e. every $p(D_w)$ is an open subset in the complex plane), then the exceptional set cannot be a closed curve. However, in [2] we have proved

Theorem A ([2], Thm. 1). *Given r with $0 < r < 1$, there exists a function $f \in L^2H(B)$ such that $E(B, f)$ is the circle $C(0, r) = \{z \in \mathbb{C} \mid |z| = r\}$.*

In this note we strengthen this result by showing the following theorem:

Theorem 1. *Let B be the unit ball in $\mathbb{C}^2$. Fix r with $0 < r < 1$, and let E be an arbitrary G_δ-subset of $C(0, r)$. Then there exists a function $f \in L^2H(B)$ such that $E(B, f) = E$.*

2. The exceptional sets for L^2H-functions in the unit ball

In this section we prove Theorem 1. The construction of a function f satisfying the assertion of Theorem A proceeds as follows (see [2]): Let $g(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ be holomorphic in the unit disc U in the plane, and set $\tilde{g}(z, w) = g(z)$, $(z, w) \in U \times \mathbb{C}$. For a given r with $0 < r < 1$, let

$$\Phi(z, w) = ((1 - r^2)^{1/2} z - rw, rz + (1 - r^2)^{1/2} w).$$

Suppose that $\tilde{g} \in L^2H(B)$. Then $\tilde{g} \circ \Phi^{-1}$ is also in $L^2H(B)$. Moreover, if

$$E(g) = \{p \in \partial U \mid g \notin L^2(D(r^2 p, 1 - r^2))\} \tag{1}$$

(where $D(z_0, \varrho)$ denotes the disc in the complex plane with center at z_0 and of radius ϱ), then (see [2], p. 80)

$$E(\tilde{g} \circ \Phi^{-1}, B) = \{rp \mid p \in E(g)\}. \tag{2}$$

Also, if $h(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ is holomorphic in U, and $\tilde{h}(z, w) = h(z)$, then $\tilde{h} \in L^2H(B)$ if and only if (see e.g. [4])

$$\sum_{n=0}^{\infty} (n + 1)^{-2} |a_n|^2 < +\infty, \tag{3}$$

Fix $0 < r < 1$, and let E be a given G_δ-subset of $C(0,r)$. It follows from (1), (2), and (3) that in order to obtain the function $f = \tilde{g} \circ \Phi^{-1}$ satisfying the conditions of Theorem 1, it is enough to construct a function g holomorphic in U, $g(z) = \sum_{n=0}^{\infty} a_n z^n$, such that

(i) $$\sum_{n=0}^{\infty} (n+1)^{-2} |a_n|^2 < +\infty$$

and

(ii) $$\text{for every } p \in \partial U \text{ with } rp \in E, \quad \int_{D(r^2 p, 1-r^2)} |g|^2 \, \mathrm{d}m = +\infty,$$

and for all other $p \in \partial U$ this integral is finite. (Here and in the sequel, m denotes the Lebesgue measure in $\mathbb{C}$.) For simplicity set $\varrho = r^2$, and let $G = \{p \mid rp \in E\}$. Then the condition (ii) above can be rewritten in the form:

(ii′) $$\text{for every } p \in \partial U, \text{ the integral } \int_{D(\varrho p, 1-\varrho)} |g|^2 \, \mathrm{d}m = +\infty,$$

precisely when $p \in G$.

We have $G = \bigcap_{k=1}^{\infty} G_k$, where each G_k is an open subset of ∂U. The construction of the function g is the elementary, though a bit technical. The idea is the following: g should be the sum of a series $g = \sum_{k=0}^{\infty} g_k$, where each g_k is holomorphic in some neighborhood of $\overline{U}$, the coefficients of the Taylor expansion of g at 0 satisfy (i), for every k and for every $p \in \partial U \setminus G_k$ we have

$$\int_{D(\varrho p, 1-\varrho)} |g_k|^2 \, \mathrm{d}m < 2^{-k},$$

and for every $p \in G$ and every (arbitrarily large) $M > 0$ one can find a non-negative integer $n(p)$ and an open non-empty subset W of $D(\varrho p, 1-\varrho)$ such that

$$\int_W |g_{n(p)}|^2 \, \mathrm{d}m > M,$$

whereas the sum

$$\sum_{n \neq n(p)} \int_{D(\varrho p, 1-\varrho)} |g_n|^2 \, dm$$

is small; e.g. this sum could be less than one. It follows from the above that for $p \in G$, the integral

$$\int_{D(\varrho p, 1-\varrho)} |g_n|^2 \, dm$$

is finite precisely when $p \in \partial U \setminus G$.

We first need a preliminary construction. Let F_1, F_2 be two open arcs in ∂U such that $\overline{F_1} \cap \overline{F_2} = \emptyset$. Denote by L the open arc in ∂U consisting of F_1, F_2, and one of the connected components of $\partial U \setminus (F_1 \cup F_2)$. Let K be a closed subset of $\overline{U}$ such that $K \cap (L \setminus (F_1 \cup F_2)) = \emptyset$. Assume moreover that for every $p \in \partial U \setminus L$,

$$D(\varrho p, 1-\varrho) \subset K. \tag{4}$$

Fix positive numbers ε, η and M and a number τ with $0 < \tau < 1$. In [2] we have constructed a function $h(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ holomorphic in U and such that

$$\sum_{n=0}^{\infty} (n+1)^{-2} |a_n|^2 < \eta, \tag{5}$$

$$\text{for every } p \in \partial U\colon \int_{D(\tau\varrho, 1-\tau)} |h(z)|^2 \, dm(z) = +\infty. \tag{6}$$

Let V be a simply connected smoothly bounded domain in $\mathbb{C}$ such that $U \cup (\partial U \setminus L) \subset V$ and $L \setminus (F_1 \cup F_2) \subset \partial V$. Let $\psi : V \longrightarrow U$ be a conformal mapping of V onto U. For $0 < t < t' < 1$, consider the annulus $A(0; t, t') = \{z \in \mathbb{C} \mid t < |z| < t'\}$. If t is sufficiently close to 1, then the set $\psi^{-1}(\overline{A(0; t, t')}) \subset V \setminus K$. Consider the function $h \circ \psi$. For $\delta > 0$ sufficiently small, we have

$$\int_K |\delta(h \circ \psi)|^2 \, dm < \varepsilon. \tag{7}$$

In particular, because of (4), for every $p \in \partial U \setminus L$ we have

$$\int_{D(\varrho p, 1-\varrho)} |\delta(h \circ \psi)|^2 \, dm < \varepsilon. \tag{8}$$

Consider once more the construction of the function h from [2]. It follows from it that the function h not only satisfies (6), but also that

$$\int_{D(\tau p,1-\tau)\cap A(0,t,t')} |h|^2 \, dm \longrightarrow +\infty \tag{9}$$

uniformly with respect to $p \in \partial U$ as $t' \nearrow 1$. Since the real jacobian of ψ is uniformly bounded and uniformly bounded away from zero, it follows from (9) with τ conveniently chosen with respect to ϱ that if t' is sufficiently close to 1, then for every $p \in L \setminus (F_1 \cup F_2)$ we have

$$\int_{D(\varrho p,1-\varrho)\cap\psi^{-1}(\overline{A(0;t,t')})} |\delta(h \circ \psi)|^2 \, dm > M. \tag{10}$$

Moreover, if $\delta(h \circ \psi)(z) = \sum_{n=0}^{\infty} b_n z^n$ in U, then (5) is still preserved with a_n replaced by b_n.

Now shift the domain V and the function $\delta(h \circ \psi)$ in the direction of the outward normal vector to ∂U at a point p_0, where p_0 is e.g. the middle point of the arc $L \setminus (F_1 \cup F_2)$. If the domain V was chosen sufficiently close to U (so that the function ψ is not far from the identity) and if this shifting is sufficiently small, then the resulting shifted domain V' contains $\overline{U}$, and the shifted function k, holomorphic in V', is also not far from $\delta(h \circ \psi)$. Hence, if $k(z) = \sum_{n=0}^{\infty} c_n z^n$ in U, then the condition

$$\sum_{n=0}^{\infty} (n+1)^{-2} |c_n|^2 < \eta$$

still holds. Also, the conditions (7) and (10) hold with $\delta(h \circ \psi)$ replaced by k.

Consider the set $\overline{U} \cap \psi^{-1}(\overline{A(0;t,t')})$. We can find two curves γ_1 and γ_2 such that γ_1 and γ_2 are contained in $\overline{U}$, $\gamma_1 \cap \gamma_2 = \emptyset$, both γ_1 and γ_2 have their initial points on F_1 and their end points on F_2, the interior points of γ_1 and γ_2 are in $U \setminus K$, and if we denote by H the closed subset of $\overline{U}$ bounded by γ_1, γ_2 and the parts of the arcs F_1 and F_2 joining the initial points and the end points of γ_1 and γ_2, then $H \cap K = \emptyset$, and $\overline{U} \cap \psi^{-1}(\overline{A(0;t,t')}) \subset H$. Then, because of (10), we have

$$\int_{D(\varrho p,1-\varrho)\cap H} |h|^2 \, dm > M.$$

For convenience, let us call the set H, as constructed above, a band supported by the arcs F_1 and F_2.

We have thus proved the following:

Let ϱ, F_1, F_2, L, K, ε, η, and M be as above. Then there exists a function h holomorphic in a neighborhood of $\overline{U}$, $h(z) = \sum_{n=0}^{\infty} c_n z^n$, and a band H supported by F_1 and F_2 such that $H \cap K = \emptyset$, and

$$\sum_{n=0}^{\infty} (n+1)^{-2} |c_n|^2 < \eta,$$

$$\int_K |h|^2 \, \mathrm{d}m < \varepsilon$$

(and hence for every $p \in \partial U \setminus L$:

$$\int_{D(\varrho p, 1-\varrho)} |h|^2 \, \mathrm{d}m < \varepsilon,$$

since $D(\varrho p, 1-\varrho) \subset K$ by assumption), and for every $p \in L \setminus (F_1 \cup F_2)$:

$$\int_{D(\varrho p, 1-\varrho) \cap H} |h|^2 \, \mathrm{d}m > M.$$

We now pass to the construction of the function g. Let π denote the radial projection of $\mathbb{C} \setminus \{0\}$ onto ∂U, $\pi(z) = z/|z|$. Choose a strictly increasing sequence $\{s_n\}_{n=1}^{\infty}$ of positive numbers such that $s_n \nearrow 1$ as $n \longrightarrow \infty$. We can assume that G is a proper subset of ∂U, since the case $G = \partial U$ was already considered in [2]. Then we can assume that also G_1 is a proper subset of ∂U. The set G_1 is a sum (finite or countable) of open arcs in $\partial U, G_1 = \bigcup_k G_k^{(1)}$. Fix k and consider the arc $G_k^{(1)}$. Suppose first that $G_k^{(1)} \cap G$ is relatively compact in $G_k^{(1)}$. We can then shrink the set G_2 in such a way that $G_k^{(1)} \cap G_2$ is also relatively compact in $G_k^{(1)}$, but the set $G = \bigcap_l G_l$ remains the same (of course, the sets $G_k^{(1)} \cap G_l, l = 3, 4, \ldots$, need to be changed in general). Choose two open arcs $F_{k,0}^{(1)}$ and $F_{k,1}^{(1)}$ in $G_1^{(1)} \setminus \overline{G_2}$ such that $\overline{F_{k,0}^{(1)}} \cup \overline{F_{k,1}^{(1)}} \subset G_k^{(1)} \setminus \overline{G_2}$, and such that $F_{k,0}^{(1)}$ ($F_{k,1}^{(1)}$) lies on the left (on the right) of $G_k^{(1)} \cap \overline{G_2}$. (We can choose e.g. a negative, i.e. clockwise, orientation on ∂U; then the notions "on the left" or "on the right" have precise and natural meaning; in the sequel we will use both these notions only in this context.) Let $L_{k,0}^{(1)}$ be the open arc in $G_k^{(1)}$ which consists of $F_{k,0}^{(1)}, F_{k,1}^{(1)}$ and of those points $p \in G_k^{(1)}$ which lie between $F_{k,0}^{(1)}$ and $F_{k,1}^{(1)}$. Denote $J_{k,0}^{(1)} = L_{k,0}^{(1)} \setminus (\overline{F_{k,0}^{(1)}} \cup \overline{F_{k,1}^{(1)}})$. The set

$$(11)_1 \qquad K_{k,0}^{(1)} = \bigcup \{\overline{D(\varrho p, 1-\varrho)} \mid p \in \partial U \setminus L_{k,0}^{(1)}\} \cup \overline{D(0, s_1)}$$

is a compact subset of $\overline{U}$, and $K^{(1)}_{k,0} \cap \overline{J^{(1)}_{k,0}} = \emptyset$. By the preliminary construction there exists a band $H^{(1)}_{k,0}$, supported by $F^{(1)}_{k,0}$ and $F^{(1)}_{k,1}$, such that

$$(12)_1 \; H^{(1)}_{k,0} \cap K^{(1)} = \emptyset,\ \pi(H^{(1)}_{k,0}) \subset L^{(1)}_{k,0},\ \text{and } \overline{\bigcup\{\overline{D(\varrho p, 1-\varrho)} \cap H^{(1)}_{k,0} \mid p \in J^{(1)}_{k,0}\}}$$
$$\text{is a compact subset of } U.$$

There also exists a function $g^{(1)}_{k,0} = \sum_{n=0}^{\infty} a^{(1)}_{k,0,n} z^n$, holomorphic in a neighborhood of $\overline{U}$, such that

$$\sum_{n=0}^{\infty} (n+1)^{-2} \left|a^{(1)}_{k,0,n}\right|^2 < 2^{-2(k+1)},$$

and for every $p \in J^{(1)}_{k,0}$ we have

$$(13)_1 \qquad \int\limits_{D(\varrho,1-\varrho)\cap H^{(1)}_{k,0}} \left|g^{(1)}_{k,0}\right|^2 \mathrm{d}m > 3$$

and

$$(14)_1 \qquad \int\limits_{K^{(1)}_{k,0}} \left|g^{(1)}_{k,0}\right|^2 \mathrm{d}m < 2^{-2(k+1)}.$$

Suppose now that $G^{(1)}_k \cap G$ is not a relatively compact subset of $G^{(1)}_k$. Then $G^{(1)}_k \cap G_l$ is not relatively compact in $G^{(1)}_k$ either for $l = 2, 3, \ldots$. Suppose that the points of $G^{(1)}_k \cap G$ are arbitrarily close to both ends of the arc $G^{(1)}_k$. Take a two-sided sequence $\{F^{(1)}_{k,m}\}_{m=-\infty}^{\infty}$ of open subarcs of $G^{(1)}_k$ such that each $\overline{F^{(1)}_{k,m}}$ is contained in G_3, the sets $\{\overline{F^{(1)}_{k,m}}\}_{m=-\infty}^{\infty}$ are pairwise disjoint, each $F^{(1)}_{k,m+1}$ lies on the right of $F^{(1)}_{k,m}$, the sets $\{F^{(1)}_{k,m}\}_{m=-\infty}^{\infty}$ cluster only at the end points of $G^{(1)}_k$, and

$$(15)_1 \qquad \begin{array}{c}\text{between each two sets } \overline{F^{(1)}_{k,m}} \text{ and } \overline{F^{(1)}_{k,m+1}} \text{ there exists in} \\ G^{(1)}_k \text{ an open and non-void subset of } G_4.\end{array}$$

Denote by $L^{(1)}_{k,m}$ the open arc in $G^{(1)}_k$, which consists of $F^{(1)}_{k,m}$, $F^{(1)}_{k,m+1}$, and of those points $p \in G^{(1)}_k$ which lie between $F^{(1)}_{k,m}$ and $F^{(1)}_{k,m+1}$. Denote $J^{(1)}_{k,m} = L^{(1)}_{k,m} \setminus (\overline{F^{(1)}_{k,m}} \cup \overline{F^{(1)}_{k,m+1}})$. The set

$$(16) \qquad K^{(1)}_{k,m} = \bigcup\{\overline{D(\varrho p, 1-\varrho)} \mid p \in \partial U \setminus L^{(1)}_{k,m}\} \cup \overline{D(0, s_1)}$$

is a compact subset of $\overline{U}$, and $K_{k,m}^{(1)} \cap \overline{J_{k,m}^{(1)}} = \emptyset$. By the above preliminary construction, for each integer m there exists a band $H_{k,m}^{(1)}$ supported by $F_{k,m}^{(1)}$ and $F_{k,m+1}^{(1)}$, such that

$$(17)_1 \qquad H_{k,m}^{(1)} \cap K_{k,m}^{(1)} = \emptyset, \pi(H_{k,m}^{(1)}) \subset L_{k,m}^{(1)}, G_k^{(1)} = \bigcup\{\pi(H_{k,m}^{(1)}) \mid m \in \mathbb{Z}\},$$
$$\text{and } \overline{\bigcup\{\overline{D(\varrho p, 1-\varrho)} \cap H_{k,m}^{(1)} \mid p \in J_{k,m}^{(1)}\}} \text{ is a compact subset of } U.$$

There also exists a function $g_{k,m}^{(1)}(z) = \sum_{n=0}^{\infty} a_{k,m,n}^{(1)} z^n$, holomorphic in a neighborhood of $\overline{U}$, such that

$$(18)_1 \qquad \sum_{n=0}^{\infty} (n+1)^{-2} |a_{k,m,n}^{(1)}|^2 < 2^{-2(k+|m|+2)},$$

and for every $p \in J_{k,m}^{(1)}$ we have

$$(19)_1 \qquad \int_{D(\varrho p, 1-\varrho) \cap H_{k,m}^{(1)}} |g_{k,m}^{(1)}|^2 \, dm \geqslant 3$$

and

$$(20)_1 \qquad \int_{K_{k,m}^{(1)}} |g_{k,m}^{(1)}|^2 \, dm < 2^{-(k+1+|m|)}.$$

If $G_k^{(1)} \cap G$ is not relatively compact in $G_k^{(1)}$, but is separated from the left (right) end point of $G_k^{(1)} \cap G$, then we can shrink G_2 if necessary in such a way that the set G remains unchanged, but $G_k^{(1)} \cap G_2$ is now separated from the left (right) end point of $G_k^{(1)}$. Then we choose a one-sided sequence $\{F_{k,m}^{(1)}\}_{m=1}^{\infty}$ ($\{F_{k,m}^{(1)}\}_{m=-\infty}^{-1}$) of open subarcs of $G_k^{(1)}$, and an arc $F_{k,0}^{(1)} \subset G_k^{(1)} \setminus \overline{G_2}$ such that the sequences $\{F_{k,m}^{(1)}\}_{m=0}^{\infty}$ or $\{F_{k,m}^{(1)}\}_{m=-\infty}^{0}$ (i.e. with $F_{k,0}^{(1)}$ added), the resulting sets $K_{k,m}^{(1)}$, and the bands $H_{k,m}^{(1)}$ have similar properties as above. Also, we obtain the functions $g_{k,m}^{(1)}$ with the above properties.

Then denote $H_1 = \bigcup_{k,m} H_{k,m}^{(1)}$, $F_1 = \bigcup_{k,m} F_{k,m}^{(1)}$, and $J_1 = \bigcup_{k,m} J_{k,m}^{(1)}$, where we sum over all admissible k and m. One can easily see that the sets $F_{k,m}^{(1)}$, $K_{k,m}^{(1)}$, and $H_{k,m}^{(1)}$

can be chosen in such a way that, moreover,

$$(21)_1 \qquad \begin{gathered}\text{for every } p \in \overline{U},\\ p \text{ belongs to all sets } K_{k,m}^{(1)} \text{ except at most two,}\end{gathered}$$

$$(22)_1 \qquad \begin{gathered}\text{for every admissible pairs } (k_1,m_1) \text{ and } (k_2,m_2), \text{ if}\\ (k_1,m_1) \neq (k_2,m_2), \text{ we have}\\ \overline{\bigcup\{\overline{D(\varrho p,1-\varrho)} \cap H_{k_1,m_1}^{(1)} \mid p \in J_{k_1,m_1}^{(1)}\}} \cap \\ \overline{\bigcup\{\overline{D(\varrho p,1-\varrho)} \cap H_{k_2,m_2}^{(1)} \mid p \in J_{k_2,m_2}^{(1)}\}} = \emptyset,\end{gathered}$$

and for every $p \in J_{k_1,m_1}^{(1)}$ we have

$$(23)_1 \qquad \overline{D(\varrho p,1-\varrho)} \cap H_{k_2,m_2}^{(1)} = \emptyset.$$

Set

$$(24)_1 \qquad g_1(z) = \sum_{k,m} g_{k,m}^{(1)}(z) =: \sum_{n=0}^{\infty} a_n^{(1)} z^n,$$

where the summation ranges over all admissible k and m. Then g_1 is well-defined and holomorphic in U, and

$$(25)_1 \qquad \sum_{n=0}^{\infty} (n+1)^{-2} |a_n^{(1)}|^2 \leqslant 2^{-2},$$

for every $p \in \partial U \setminus G_1$ we have

$$(26)_1 \qquad \int_{D(\varrho p,1-\varrho)} |g_1|^2 \, \mathrm{d}m < 2^{-1},$$

and for every $p \in J_1$ we have

$$(27)_1 \qquad \int_{D(\varrho p,1-\varrho)\cap H_1} |g_1|^2 \, \mathrm{d}m \geqslant 1 + 1.$$

(The last inequality follows from $(19)_1, (22)_1, (20)_1$, the definition of $K_{k,m}^{(1)}$, and the facts that $J_1 = \bigcup_{k,m} J_{k,m}^{(1)}$ and the sets $J_{k,m}^{(1)}$ are pairwise disjoint.)

The construction of H_2 and the function g_2 differs in some details from the previous one. As before, $G_2 = \bigcup_k G_k^{(2)}$ (the finite or countable sum), where each $G_k^{(2)}$ is an open arc in ∂U. Fix k and consider the arc $G_k^{(2)}$. Suppose first that $G_k^{(2)} \cap G$ is relatively compact in $G_k^{(2)}$. It can also happen that some of the arcs $F_{\kappa,\mu}^{(1)}$ are in $G_k^{(2)}$; but it follows from the previous step that they form a relatively compact subset $\Phi_k^{(2)}$ of $G_k^{(2)}$. Therefore we can shrink G_3 if necessary, without changing G and $\Phi_k^{(2)}$, so that $G_k^{(2)} \cap G_3$ is also relatively compact in $G_k^{(2)}$. We then choose two open arcs $F_{k,0}^{(2)}$ and $F_{k,1}^{(2)}$ in $G_k^{(2)} \setminus \overline{G_3}$, lying on the left (on the right) of $G_k^{(2)} \cap \overline{G_3}$, and such that also $\overline{F_{k,0}^{(2)}} \cup \overline{F_{k,1}^{(2)}} \subset G_k^{(2)} \setminus \overline{G_3}$. As above, let $L_{k,0}^{(2)}$ be the open arc in $G_k^{(2)}$ which consists of $F_{k,0}^{(2)}, F_{k,1}^{(2)}$, and of those points $p \in G_k^{(2)}$ which lie between $F_{k,0}^{(2)}$ and $F_{k,1}^{(2)}$. Set $J_{k,0}^{(2)} = L_{k,0}^{(2)} \setminus (\overline{F_{k,0}^{(2)}} \cup \overline{F_{k,1}^{(2)}})$. Note that all arcs $\overline{F_{\kappa,\mu}^{(1)}}$ which are in $G_k^{(2)} \cap G_3$, are also in $J_{k,0}^{(2)}$. The set

$$(11)_2 \qquad K_{k,0}^{(2)} = \bigcup \{\overline{D(\varrho p, 1-\varrho)} \mid p \in \partial U \setminus L_{k,0}^{(2)}\} \cup \overline{D(0, s_2)}$$

is a compact subset of $\overline{U}$, and $K_{k,0}^{(2)} \cap \overline{J_{k,0}^{(2)}} = \emptyset$. By the preliminary construction there exists a band $H_{k,0}^{(2)}$, supported by $F_{k,0}^{(2)}$ and $F_{k,1}^{(2)}$, such that

$$(12)_2 \qquad H_{k,0}^{(2)} \cap K_{k,0}^{(2)} = \emptyset, \ \pi(H_{k,0}^{(2)}) \subset L_{k,0}^{(2)},$$
$$\text{and } \overline{\bigcup \{\overline{D(\varrho p, 1-\varrho)} \cap H_{k,0}^{(2)} \mid p \in J_{k,0}^{(2)}\}} \text{ is a compact subset of } U.$$

We may choose $H_{k,0}^{(2)}$ in such a way that in addition,

$$H_{k,0}^{(2)} \cap \overline{\bigcup \{H_1 \cap D(\varrho p, 1-\varrho) \mid p \in J_1\}} = \emptyset.$$

Then we replace the set $K_{k,0}^{(2)}$ by a larger compact subset of $\overline{U}$ (called also $K_{k,0}^{(2)}$) such that

$$(28)_2 \qquad \overline{\bigcup \{H_1 \cap D(\varrho p, 1-\varrho) \mid p \in J_1\}} \subset K_{k,0}^{(2)},$$
$$\text{and still } H_{k,0}^{(2)} \cap K_{k,0}^{(2)} = \emptyset.$$

By the original construction, there exists a function $g_{k,0}^{(2)} = \sum_{n=0}^{\infty} a_{k,0,n}^{(2)} z^n$, holomorphic in a neighborhood of $\overline{U}$, such that

$$\sum_{n=0}^{\infty} (n+1)^{-2} |a_{k,0,n}^{(2)}|^2 < 2^{-2(k+2)},$$

for every $p \in J_{k,0}^{(2)}$ we have

$$(13)_2 \qquad \int_{D(\varrho p, 1-\varrho) \cap H_{k,0}^{(2)}} |g_{k,0}^{(2)}|^2 \, dm > 4 + \int_{D(\varrho p, 1-\varrho) \cap H_{k,0}^{(2)}} |g_1|^2 \, dm,$$

and

$$(14)_2 \qquad \int\limits_{K^{(2)}_{k,0}} |g^{(2)}_{k,0}|^2 \,\mathrm{d}m < 2^{-(k+2)}.$$

(The condition $(13)_2$ can be fulfilled since the set

$$\overline{\bigcup\{\overline{D(\varrho p, 1-\varrho)} \cap H^{(2)}_{k,0} \mid p \in J^{(2)}_{k,0}\}}$$

is compact in U by $(12)_2$, and so the values of $\int\limits_{D(\varrho p,1-\varrho)} |g_1|^2 \,\mathrm{d}m$ are uniformly bounded for $p \in J^{(2)}_{k,0}$.)

Suppose now that $G^{(2)}_k \cap G$ is not a relatively compact subset of $G^{(2)}_k$. Then $G^{(2)}_k \cap G_l$ is not relatively compact in $G^{(2)}_k$ either for $l = 3, 4, \ldots$. Suppose that the points of $G^{(2)}_k \cap G$ are arbitrarily close to both ends of the arc $G^{(2)}_k$. Take a two-sided sequence $\{F^{(2)}_{k,m}\}_{m=-\infty}^{\infty}$ of open subarcs of $G^{(2)}_k$ such that each $\overline{F^{(2)}_{k,m}}$ is contained in G_4, the sets $\{\overline{F^{(2)}_{k,m}}\}_{m=-\infty}^{\infty}$ are pairwise disjoint, each $F^{(2)}_{k,m+1}$ lies on the right of $F^{(2)}_{k,m}$, the sets $\{F^{(2)}_{k,m}\}_{m=-\infty}^{\infty}$ cluster only at the end points of $G^{(2)}_k$, and

$(15)_1$ between each two sets $\overline{F^{(2)}_{k,m}}$ and $\overline{F^{(2)}_{k,m+1}}$ there exists in $G^{(2)}_k$ an open and non-void subset of G_5.

Let $L^{(2)}_{k,m}$ and $J^{(2)}_{k,m}$ have similar meaning as in the first step. The set

$$(29) \qquad K^{(2)}_{k,m} = \bigcup\{\overline{D(\varrho p, 1-\varrho)} \mid p \in \partial U \setminus L^{(2)}_{k,m}\} \cup \overline{D(0, s_2)}$$

is a compact subset of $\overline{U}$, and $K^{(2)}_{k,m} \cap \overline{J^{(2)}_{k,m}} = \emptyset$. By the original construction, for each integer m there exists a band $H^{(2)}_{k,m}$ supported by $F^{(2)}_{k,m}$ and $F^{(2)}_{k,m+1}$, such that

$(17)_1$ $H^{(2)}_{k,m} \cap K^{(2)}_{k,m} = \emptyset, \pi(H^{(2)}_{k,m}) \subset L^{(2)}_{k,m}, G^{(2)}_k = \bigcup\{\pi(H^{(2)}_{k,m}) \mid m \in \mathbb{Z}\}$, and $\overline{\bigcup\{\overline{D(\varrho, 1-\varrho)} \cap H^{(2)}_{k,m} \mid p \in J^{(2)}_{k,m}\}}$ is a compact subset of U.

We may choose $H^{(2)}_{k,m}$ in such a way that in addition

$$(30)_2 \qquad H^{(2)}_{k,m} \cap \overline{\bigcup\{H_1 \cap D(\varrho p, 1-\varrho) \mid p \in J_1\}} \subset K^{(2)}_{k,m},$$
and still $H^{(2)}_{k,m} \cap K^{(2)}_{k,m} = \emptyset$.

By the second part of the preliminary construction there exists a function $g^{(2)}_{k,m}(z) = \sum_{n=0}^{\infty} a^{(2)}_{k,m,n} z^n$, holomorphic in a neighborhood of $\bar{U}$, such that

$$(18)_2 \qquad \sum_{n=0}^{\infty} (n+1)^{-2} \left|a^{(2)}_{k,m,n}\right|^2 < 2^{-2(k+|m|+3)},$$

for every $p \in J^{(2)}_{k,m}$ we have

$$(19)_2 \qquad \int\limits_{D(\varrho p, 1-\varrho) \cap H^{(1)}_{k,m}} \left|g^{(2)}_{k,m}\right|^2 dm \geqslant 4 + \int\limits_{D(\varrho p, 1-\varrho) \cap H^{(2)}_{k,m}} |g_1|^2 \, dm,$$

and

$$(20)_2 \qquad \int\limits_{K^{(2)}_{k,m}} \left|g^{(2)}_{k,m}\right|^2 dm < 2^{-(k+|m|+3)}.$$

(Since the set

$$\overline{\bigcup \{\overline{D(\varrho p, 1-\varrho)} \cap H^{(2)}_{k,m} \mid p \in J^{(2)}_{k,m}\}}$$

is compact in U, the values of integrals on the right-hand side of $(19)_2$ are uniformly bounded for $p \in J^{(2)}_{k,m}$; so the condition $(19)_2$ can be fulfilled by some function $g^{(2)}_{k,m}$.)

If $G^{(2)}_k \cap G$ is not relatively compact in $G^{(2)}_k$, but is separated from the left (right) end point of $G^{(2)}_k$, then we can shrink G_3 if necessary in such a way that the sets G and $\Phi^{(2)}_k$ (where $\Phi^{(2)}_k$ is the sum of the arcs $F^{(1)}_{\kappa,\mu}$) remain unchanged, but $G^{(2)}_k \cap G_3$ is now separated from the left (right) end point of $G^{(2)}_k$. Then we choose a one-sided sequence $\{F^{(2)}_{k,m}\}_{m=1}^{\infty}$ ($\{F^{(2)}_{k,m}\}_{m=-\infty}^{-1}$) of open subarcs of $G^{(2)}_k$, and a supplementary arc $F^{(2)}_{k,0} \subset G^{(2)}_k \setminus \overline{G_3}$ such that the sequences $\{F^{(2)}_{k,m}\}_{m=0}^{\infty}$ or $\{F^{(2)}_{k,m}\}_{m=-\infty}^{0}$ (with $F^{(2)}_{k,0}$ added), the resulting sets $K^{(2)}_{k,m}$, the bands $H^{(2)}_{k,m}$, and the functions $g^{(2)}_{k,m}$ have similar properties as above.

Moreover, by $(15)_1$, we can assume that for each admissible m, κ, and μ we have

$$\overline{F^{(2)}_{k,m}} \cap \overline{F^{(1)}_{\kappa,\mu}} = \emptyset.$$

It follows from this in particular that all arcs $\overline{F^{(1)}_{\kappa,\mu}}$ which are in $G^{(2)}_k \cap G_3$, are actually in $\bigcup_m J^{(2)}_{k,m}$.

Then denote $H_2 = \bigcup_{k,m} H^{(2)}_{k,m}$, $F_2 = \bigcup_{k,m} F^{(2)}_{k,m}$, and $J_2 = \bigcup_{k,m} J^{(2)}_{k,m}$, where we sum over all admissible k and m. One can check that the sets $F^{(2)}_{k,m}$, $K^{(2)}_{k,m}$ and $H^{(2)}_{k,m}$ can be chosen in such a way that, moreover,

$(21)_2$ for every $p \in \overline{U}$,

$$p \text{ belongs to all sets } K^{(2)}_{k,m} \text{ except at most two },$$

$(22)_2$ For every admissible pairs (k_1, m_1) and (k_2, m_2), if $(k_1, m_1) \neq (k_2, m_2)$, then

$$\overline{\bigcup\{\overline{D(\varrho p, 1-\varrho)} \cap H^{(2)}_{k_1,m_1} \mid p \in J^{(2)}_{k_1,m_1}\}} \cap \overline{\bigcup\{\overline{D(\varrho p, 1-\varrho)} \cap H^{(2)}_{k_2,m_2} \mid p \in J^{(2)}_{k_2,m_2}\}} = \emptyset,$$

and for every $p \in J^{(2)}_{k_1,m_1}$ we have

$$\overline{D(\varrho p, 1-\varrho)} \cap H^{(2)}_{k_2,m_2} = \emptyset. \tag{23$_2$}$$

Set

$$g_2(z) = \sum_{k,m} g^{(2)}_{k,m}(z) =: \sum_{n=0}^{\infty} a^{(2)}_n z^n, \tag{24$_2$}$$

where we sum over all admissible k and m. Then g_2 is a well-defined holomorphic function in U, and

$$\sum_{n=0}^{\infty} (n+1)^{-2} |a^{(2)}_n|^2 \leqslant 2^{-4}, \tag{25$_2$}$$

for every $p \in \partial U \setminus G_2$ we have

$$\int_{D(\varrho p, 1-\varrho)} |g_2|^2 \,\mathrm{d}m < 2^{-2}, \tag{26$_2$}$$

and for every $p \in J_2$ we have

$$\int_{D(\varrho p, 1-\varrho) \cap H_2} |g_2|^2 \,\mathrm{d}m \geqslant 3 + \int_{D(\varrho p, 1-\varrho) \cap H_2} |g_1|^2 \,\mathrm{d}m. \tag{27$_2$}$$

(The last inequality follows from $(19)_2$, $(22)_2$, $(20)_2$, the definition of $K^{(2)}_{k,m}$, and the facts that $J_2 = \bigcup\limits_{k,m} J^{(2)}_{k,m}$ and the sets $J^{(2)}_{k,m}$ are pairwise disjoint.)

Now it is clear how to construct inductively the function g_t having assumed that the functions $g_1, \ldots, g_{t-1}$ were already constructed. We have, as before, $G_t = \bigcup\limits_{k} G^{(t)}_k$

where each $G_k^{(t)}$ is an open arc in ∂U. We fix k and suppose first that $G_k^{(t)} \cap G$ is relatively compact in $G_k^{(t)}$. After shrinking G_{t+1} if necessary, without changing G and $\Phi_k^{(t)}$ (the sum of arcs $F_{\kappa,\mu}^{(t-1)}$ in $G_k^{(t)}$), we can assume that $G_k^{(t)} \cap G_{t+1}$ is also relatively compact in $G_k^{(t)}$, and then choose $F_{k,0}^{(t)}$ and $F_{k,1}^{(t)}$ in $G_k^{(t)} \setminus \overline{G_{t+1}}$ as before. It follows that

$$(31)_t \qquad \text{all arcs } \overline{F_{\kappa,\mu}^{(t-1)}} \text{ which are in } G_k^{(t)} \text{ are, by the induction hypothesis, in } G_k^{(t)} \cap G_{t+1}, \text{ so they are also in } J_{k,0}^{(t)}.$$

We form the set $K_{k,0}^{(t)}$ as in $(11)_1$ or $(11)_2$, and then the band $H_{k,0}^{(t)}$ supported by $F_{k,0}^{(t)}$ and $F_{k,1}^{(t)}$ satisfying the conditions $(12)_t$, similar to $(12)_1$ or $(12)_2$. We may choose $H_{k,0}^{(t)}$ in such a way that in addition

$$H_{k,0}^{(t)} \cap \overline{\bigcup \{H_{t-1} \cap D(\varrho p, 1-\varrho) \mid p \in J_{t-1}\}} = \emptyset.$$

We then replace $K_{k,0}^{(t)}$ by a larger compact subset of $\bar{U}$ (denoted also $K_{k,0}^{(t)}$) such that

$$(28)_t \qquad \overline{\bigcup \{H_{t-1} \cap D(\varrho p, 1-\varrho) \mid p \in J_{t-1}\}} \subset K_{k,0}^{(t)}, \quad \text{and still } H_{k,0}^{(t)} \cap K_{k,0}^{(t)} = \emptyset.$$

There also exists a function $g_{k,0}^{(t)}(z) = \sum\limits_{n=0}^{\infty} a_{k,0,n}^{(t)} z^n$, holomorphic in a neighborhood of $\bar{U}$, such that

$$\sum_{n=0}^{\infty} (n+1)^{-2} \left|a_{k,0,n}^{(t)}\right|^2 < 2^{-2(k+t)},$$

for every $p \in J_{k,0}^{(t)}$ we have

$$(13)_t \qquad \int\limits_{D(\varrho p, 1-\varrho) \cap H_{k,0}^{(t)}} \left|g_{k,0}^{(t)}\right|^2 dm > t + 2 + \sum_{l=1}^{t-1} \int\limits_{D(\varrho p, 1-\varrho) \cap H_{k,0}^{(t)}} |g_l|^2 \, dm,$$

and

$$(14)_t \qquad \int\limits_{K_{k,0}^{(t)}} \left|g_{k,0}^{(t)}\right|^2 dm < 2^{-(k+t)}.$$

As before, since the set

$$\overline{\bigcup \{\overline{D(\varrho p, 1-\varrho)} \cap H_{k,0}^{(t)} \mid p \in J_{k,0}^{(t)}\}}$$

is compact in U, the condition $(13)_t$ can be fulfilled.

If the points of $G_k^{(t)} \cap G$ are arbitrarily close to both ends of the arc $G_k^{(t)}$, we choose a two-sided sequence $\{F_{k,m}^{(t)}\}_{m=-\infty}^{\infty}$ of open subarcs of $G_k^{(t)}$, each of them contained with its closure in G_{t+2}, ordered from the left to the right with respect to m, with closures pairwise disjoint, clustering only at the end points of G, and such that between each two sets $\overline{F_{k,m}^{(t)}}$ and $\overline{F_{k,m+1}^{(t)}}$ there exists in $G_k^{(t)}$ an open and non-void subset of G_{t+3}. Then, as before, we define $L_{k,m}^{(t)}, J_{k,m}^{(t)}, K_{k,m}^{(t)}$ (similarly to (16) and (29)) and construct a band $H_{k,m}^{(t)}$ satisfying the conditions $(17)_t$, which are similar to $(17)_1$ or $(17)_2$. We may choose $H_{k,m}^{(t)}$ in such a way that in addition

$$H_{k,m}^{(t)} \cap \overline{\bigcup\{H_{t'} \cap D(\varrho p, 1-\varrho) \mid p \in J_{t'}, t' < t\}} = \emptyset.$$

We then replace $K_{k,m}^{(t)}$ by a larger compact subset of $\bar{U}$ such that

$$(30)_t \qquad \overline{\bigcup\{H_{t'} \cap D(\varrho p, 1-\varrho) \mid p \in J_{t'}, t' < t\}} \subset K_{k,m}^{(t)},$$
$$\text{and such that still } H_{k,m}^{(t)} \cap K_{k,m}^{(t)} = \emptyset.$$

There also exists a function $g_{k,m}^{(t)}(z) = \sum_{n=0}^{\infty} a_{k,m,n}^{(t)} z^n$, holomorphic in a neighborhood of $\bar{U}$, such that it satisfies the conditions $(18)_t$, $(19)_t$, and $(20)_t$ which are similar to the previous ones in the sense that the right-hand sides of $(18)_2$, $(19)_2$, and $(20)_2$ must be replaced respectively by $2^{-2(k+|m|+t+1)}$,

$$t + 2 + \sum_{l=1}^{t-1} \int_{D(\varrho p, 1-\varrho) \cap H_{k,m}^{(t)}} |g_l|^2 \, dm,$$

and $2^{-(k+|m|+t+1)}$, and the integration on the left-hand side of $(20)_t$ ranges over $K_{k,m}^{(t)}$. The condition $(19)_t$ can be satisfied since the set

$$\overline{\bigcup\{\overline{D(\varrho, 1-\varrho)} \cap H_{k,m}^{(t)} \mid p \in J_{k,m}^{(t)}\}}$$

is compact in U.

Similarly as for $t = 2$, we change conveniently the construction, if $G_k^{(t)} \cap G$ is separated only from the left (right) end point of $G_k^{(t)}$.

Moreover, by a condition similar to $(15)_1$ or $(15)_2$, which is assumed to hold for $t-1$, we can assume that for each admissible m, κ, μ,

$$\overline{F_{k,m}^{(t)}} \cap \overline{F_{\kappa,\mu}^{(t-1)}} = \emptyset.$$

In particular, this implies that

$$(32)_t \qquad \text{all arcs } \overline{F_{\kappa,\mu}^{(t-1)}} \text{ which lie in } G_k^{(t)} \cap G_{t+1}, \quad \text{are actually in } \bigcup_m J_{k,m}^{(t)}.$$

Then we form the sets H_t, F_t, and J_t as for $t = 1$ or 2; they satisfy also the assumptions $(21)_t$, $(22)_t$, and $(23)_t$, similar to $(21)_1$, $(22)_1$, and $(23)_1$, or $(21)_2$, $(22)_2$, and $(23)_2$.

If we define

$$(24)_t \qquad g_t(z) = \sum_{k,m} g_{k,m}^{(t)}(z) =: \sum_{n=0}^{\infty} a_n^{(t)} z^n,$$

then g_t is a well-defined holomorphic function in U such that

$$(25)_t \qquad \sum_{n=0}^{\infty} (n+1)^{-2} |a_n^{(t)}|^2 \leqslant 2^{-2t},$$

for every $p \in \partial U \setminus G_t$ we have

$$(26)_t \qquad \int_{D(\varrho p, 1-\varrho)} |g_t|^2 \, \mathrm{d}m < 2^{-t},$$

and for every $p \in J_t$ we have

$$(27)_t \qquad \int_{D(\varrho p, 1-\varrho)\cap H_t} |g_t|^2 \, \mathrm{d}m \geqslant t + 1 + \sum_{l=1}^{t-1} \int_{D(\varrho p, 1-\varrho)\cap H_t} |g_l|^2 \, \mathrm{d}m.$$

Set $g = \sum_{l=0}^{\infty} g_l =: \sum_{n=0}^{\infty} a_n z^n$. Then g is a well-defined holomorphic function in U. Moreover, by $(25)_t$, we have

$$\sum_{n=0}^{\infty} (n+1)^{-2} |a_n|^2 < +\infty.$$

Let $p \in \partial U \setminus G$. Then, for some positive integer t_0 and all $t \geqslant t_0$, $p \notin G_t$. It follows from $(26)_t$ that

$$\left(\int_{D(\varrho p, 1-\varrho)} |g|^2 \, \mathrm{d}m \right)^{\frac{1}{2}} \leqslant \sum_{t=1}^{\infty} \left(\int_{D(\varrho p, 1-\varrho)} |g_t|^2 \, \mathrm{d}m \right)^{\frac{1}{2}} \leqslant$$

$$\leqslant \sum_{t=1}^{t_0 - 1} \left(\int_{D(\varrho p, 1-\varrho)} |g_t|^2 \, \mathrm{d}m \right)^{\frac{1}{2}} + \sum_{t=t_0}^{\infty} 2^{-t} < +\infty.$$

If $p \in G$, then $p \in G_t$ for all t. By construction, $p \in J_t \cup F_t$ for all t. It follows from $(31)_t$ and $(32)_t$ that if $p \in F_t \cap G$, then $p \in J_{t+1}$. Therefore p belongs to infinitely many sets J_t. Consider any t_0 such that $p \in J_{t_0}$. Then, by $(27)_{t_0}$,

$$\int\limits_{D(\varrho p,1-\varrho)\cap H_{t_o}} |g_{t_0}|^2 \, \mathrm{d}m \geqslant t_0 + 1 + \sum_{l=1}^{t_0-1} \int\limits_{D(\varrho p,1-\varrho)\cap H_{t_o}} |g_l|^2 \, \mathrm{d}m. \tag{33}$$

On the other hand, for every $l > t_0$ it follows from $(30)_t, (28)_t, (20)_t, (14)_t$ and $(24)_t$ that

$$\int\limits_{D(\varrho p,1-\varrho)\cap H_{t_o}} |g_l|^2 \, \mathrm{d}m \leqslant 2^{-l}. \tag{34}$$

Then, by (33) and (34),

$$\int\limits_{D(\varrho p,1-\varrho)\cap H_{t_o}} |g|^2 \, \mathrm{d}m \geqslant \int\limits_{D(\varrho p,1-\varrho)\cap H_{t_o}} |g_{t_0}|^2 \, \mathrm{d}m$$
$$- \sum_{l=1}^{t_0-1} \int\limits_{D(\varrho p,1-\varrho)\cap H_{t_o}} |g_l|^2 \, \mathrm{d}m - \sum_{l=t_0+1}^{\infty} \int\limits_{D(\varrho p,1-\varrho)\cap H_{t_o}} |g_l|^2 \, \mathrm{d}m \geqslant t_0.$$

Since t_0 can be arbitrarily large, it follows that

$$\int\limits_{D(\varrho p,1-\varrho)} |g|^2 \, \mathrm{d}m = +\infty.$$

Thus g is the desired function. This completes the proof of the theorem.

Note. It is still an open problem to characterize all exceptional sets for functions from the Bergman space in the unit ball in $\mathbb{C}^2$; e.g. it is of interest to know whether all G_δ-subsets of measure zero in the unit disc U in $\mathbb{C}$ (not necessarily contained in the circles) are exceptional subsets for some functions.

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