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ANGULAR LIMITS OF THE INTEGRALS OF THE CAUCHY TYPE

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Dedicated to Professor Fumi-Yuki Maeda on the occasion of his sixtieth birthday

Abstract. Integrals of the Cauchy type extended over the boundary ∂A of a general compact set A in the complex plane are investigated. Necessary and sufficient conditions on ∂A are established guaranteeing the existence of angular limits of these integrals at a fixed $z \in \partial A$ for all densities satisfying a Hölder-type condition at z.

Keywords: integrals of Cauchy type, angular limits

MSC 1991: 30E20

We shall identify the points (x, y) in the Euclidean plane $\mathbb{R}^2$ with the complex numbers $x + \mathrm{i}y$ in $\mathbb{C}$ (i is the imaginary unit). The scalar product of u, $v \in \mathbb{C}$ will be denoted by $\langle u, v\rangle := \operatorname{Re} u\overline{v}$ where $\overline{v}$ is the complex conjugate of v. If $U \subset \mathbb{C}$ is open, then $\mathcal{C}_0^{(1)}(U)$ stands for the class of all continuously differentiable real-valued functions φ with a compact support $\operatorname{spt}\varphi \subset U$.

Let now $A \subset \mathbb{R}^2$ be a Lebesgue measurable set with a compact boundary ∂A, $G = \mathbb{R}^2 \setminus A$. The class of all restrictions to ∂A of functions in $\mathcal{C}_0^{(1)}(\mathbb{R}^2)$ will be denoted by

$$\mathcal{C}^{(1)}(\partial A) := \{\varphi|_{\partial A};\ \varphi \in \mathcal{C}_0^{(1)}(\mathbb{R}^2)\}.$$

Given $f \in \mathcal{C}^{(1)}(\partial A)$ and $z \in \mathbb{C} \setminus \partial A$ we choose a $\varphi_f \in \mathcal{C}_0^{(1)}(\mathbb{R}^2)$ such that $\varphi_f = f$ on ∂A, $z \notin \operatorname{spt}\varphi_f$ and define the Cauchy's type integral

$$\mathcal{K}^A f(z) := \frac{1}{\pi \mathrm{i}} \int\limits_G \frac{\overline{\partial}\varphi_f(\xi)}{\xi - z}\, \mathrm{d}\lambda_2(\xi),$$

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where λ_2 is the Lebesgue measure in the plane, $\overline{\partial} = \frac{1}{2}(\partial_1 + \mathrm{i}\partial_2)$ and ∂_j denotes the partial derivative with respect to the j-th variable ($j = 1, 2$). The imaginary and real parts

$$W^A f(z) := \operatorname{Im} \mathcal{K}^A f(z), \quad P^A f(z) := \operatorname{Re} \mathcal{K}^A f(z)$$

will also be investigated. It is not difficult to verify that $\mathcal{K}^A f(z)$, $W^A f(z)$, $P^A f(z)$ do not depend on the choice of φ_f with the properties specified above (compare Lemma 2.1 in [7]). If the boundary ∂G is a properly oriented smooth Jordan curve then $\mathcal{K}^A f(z)$ reduces to the well-known Cauchy's type integral

$$\frac{1}{2\pi} \int\limits_{\partial G} \frac{f(\xi)}{z - \xi} \, \mathrm{d}\xi$$

while $W^A f(z)$ is the value at z of the double layer potential with momentum density f and $P^A f(z)$ is the so-called modified logarithmic potential in the sense of § 12, chap. II in [16] (cf. also [10]). It is easily seen that, for each $f \in \mathcal{C}^{(1)}(\partial A)$, $\mathcal{K}^A f \colon z \mapsto \mathcal{K}^A f(z)$ is a holomorphic function on $\mathbb{C} \setminus \partial A$, whence it follows that $W^A f \colon z \mapsto W^A f(z)$ and $P^A f \colon z \mapsto P^A f(z)$ are harmonic on the same set (compare Lemma 2.4 in [7]). We shall first specify conditions on A guaranteeing natural extendability of $\mathcal{K}^A f$, $W^A f$, $P^A f$ to more general functions f on ∂A.

Writing

$$B(z, r) := \{\xi \in \mathbb{R}^2 ; \, |\xi - z| < r\}$$

we denote by

$$\overline{d}(A, z) := \limsup_{r \downarrow 0} \lambda_2[B(z, r) \cap A] / \lambda_2[B(z, r)]$$

the upper density of A at z and define the so-called essential boundary of A by

$$\partial_e A := \{z \in \mathbb{R}^2 ; \, \overline{d}(A, z) > 0, \overline{d}(G, z) > 0\}.$$

If $U \subset \mathbb{C}$ is open, then $\mathcal{A}(U)$ and $\mathcal{H}(U)$ will denote the space of all holomorphic functions and harmonic functions on U, respectively; both $\mathcal{A}(U)$ and $\mathcal{H}(U)$ are equipped with the topology of uniform convergence on compact subsets of U.

Let $\eta \in \partial A$ be a fixed point and let $q \colon \partial A \to [0, +\infty[$ be a lower-semicontinuous bounded function on ∂A which is strictly positive on $\partial A \setminus \{\eta\}$. $\mathcal{C}(\partial A, q)$ is the space of all continuous functions $f \colon \partial A \to \mathbb{R}$ satisfying

$$f(z) - f(\eta) = o\big(q(z)\big), \quad z \to \eta, \quad z \in \partial A;$$

defining

$$\|f\|_{q,0} = \sup \frac{|f(z) - f(\eta)|}{q(z)}, \quad z \in \partial A \setminus \{\eta\},$$

we introduce the norm

$$\|f\|_q = \|f\|_{q,0} + \sup_{z \in \partial A} |f(z)|$$

in $\mathcal{C}(\partial A, q)$ which turns it into a Banach space; clearly, $\|\dots\|_{q,0}$ is an equivalent norm in the subspace

$$\mathcal{C}_0(\partial A, q) = \{f \in \mathcal{C}(\partial A, q);\ f(\eta) = 0\}.$$

Let λ_1 denote the 1-dimensional Hausdorff measure (= length in the sense of [17], chap. II, §8). Now we are in position to formulate the following result establishing necessary and sufficient condition for continuity of the operators

$$\mathcal{K}^A : f \mapsto \mathcal{K}^A f, \tag{1}$$

$$W^A : f \mapsto W^A f, \tag{2}$$

$$P^A : f \mapsto P^A f. \tag{3}$$

Theorem 1. *The following conditions* (I)–(IV) *are mutually equivalent:*

$$\int_{\partial_c A} q(z)\, \mathrm{d}\lambda_1(z) < +\infty. \tag{I}$$

(II) *The operator* (1) *acts continuously form* $\mathcal{C}(\partial A, q) \cap \mathcal{C}^{(1)}(\partial A)$ *into* $\mathcal{A}(\mathbb{C} \setminus \partial A)$.
(III) *The operator* (2) *is continuous from* $\mathcal{C}(\partial A, q) \cap \mathcal{C}^{(1)}(\partial A)$ *into* $\mathcal{H}(\mathbb{R}^2 \setminus \partial A)$.
(IV) *The operator* (3) *is continuous from* $\mathcal{C}(\partial A, q) \cap \mathcal{C}^{(1)}(\partial A)$ *into* $\mathcal{H}(\mathbb{R}^2 \setminus \partial A)$.

Remark. The above theorem will be proved below.

Assuming (I) and taking into account that $\mathcal{C}(\partial A, q) \cap \mathcal{C}^{(1)}(\partial A)$ is dense in $\mathcal{C}(\partial A, q)$ we extend the operators (1), (2), (3) by continuity to the whole space $\mathcal{C}(\partial A, q)$. For any $f \in \mathcal{C}(\partial A, q)$ we have then

$$\mathcal{K}f \equiv \mathcal{K}^A f \in \mathcal{A}(\mathbb{C} \setminus \partial A),$$
$$Wf \equiv W^A f \in \mathcal{H}(\mathbb{R}^2 \setminus \partial A),$$
$$Pf \equiv P^A f \in \mathcal{H}(\mathbb{R}^2 \setminus \partial A)$$

and we shall be concerned with the existence of angular limits of these functions at η. For this purpose it appears useful to introduce the following geometric quantities characterizing the complexity of the boundary ∂A near $\eta \in \partial A$.

Notation. For $\varrho > 0$ denote by

$$\mathcal{U}^q(\varrho, \eta) := \sum_{\xi} q(\xi), \quad \xi \in \partial_e A, \quad |\xi - \eta| = \varrho, \tag{4}$$

the sum counting, with the weight $q(\xi)$, all the points ξ in the intersection of the essential boundary $\partial_e A$ with the circle $\partial B(\eta, \varrho)$. (Note that this sum equals $+\infty$ if $\partial_e A \cap \partial B(\eta, \varrho)$ is uncountable, because $q(\xi) > 0$ for $\xi \in \partial A \setminus \{\eta\}$.) We shall see below that

$$\varrho \mapsto \mathcal{U}^q(\varrho, \eta)$$

is Lebesgue measurable which permits us to define for any $r \in]0, +\infty]$

$$\mathcal{U}_r^q(\eta) = \int_0^r \varrho^{-1} \mathcal{U}^q(\varrho, \eta)\, \mathrm{d}\varrho, \quad u_r^q(\eta) = \int_0^r \mathcal{U}^q(\varrho, \eta)\, \mathrm{d}\varrho. \tag{5}$$

Denoting by

$$H(\eta, \theta) = \{\eta + t\theta, t > 0\}$$

the half-line issuing at η in the direction of $\theta \in \partial B(0,1)$ we introduce the sum

$$v^q(\theta, \eta) = \sum_{\xi} q(\xi), \quad \xi \in \partial_e A \cap H(\eta, \theta) \tag{6}$$

counting, with the weight $q(\xi)$, all the points ξ in the intersection of the essential boundary $\partial_e A$ with the half-line $H(\eta, \theta)$, and a similar sum

$$\mathcal{V}_r^q(\theta, \eta) = \sum_{\xi} |\xi - \eta| q(\xi), \quad \xi \in \partial_e A \cap H(\eta, \theta) \cap B(\eta, r) \tag{7}$$

extended over all points ξ in the intersection of the essential boundary $\partial_e A$ with the segment $H(\eta, \theta) \cap B(\eta, r)$, where now the weight at ξ is given by $q(\xi)|\xi - \eta|$. Again, the functions

$$\theta \mapsto v^q(\theta, \eta), \quad \theta \mapsto \mathcal{V}_r^q(\theta, \eta) \tag{8}$$

will be shown to be λ_1-measurable on $\partial B(0,1)$ which permits to introduce the quantities

$$v^q(\eta) = \frac{1}{2\pi} \int_{\partial B(0,1)} v^q(\theta, \eta)\, \mathrm{d}\lambda_1(\theta), \tag{9}$$

$$\mathcal{V}_r^q(\eta) = \frac{1}{2\pi} \int_{\partial B(0,1)} \mathcal{V}_r^q(\theta, \eta)\, \mathrm{d}\lambda_1(\theta), \quad r > 0.$$

For $S \subset \mathbb{R}^2$ we denote by $\operatorname{cl} S$ the closure of S, by $\operatorname{contg}(S,\eta)$ the contingent of S at η (cf. [17], chap. IX, §2) consisting of all the half-lines $H(\eta,\theta)$ for which there exists a sequence of points $z_n \in S \setminus \{\eta\}$ such that

$$\lim_{n\to\infty} \frac{z_n - \eta}{|z_n - \eta|} = \theta, \quad \lim_{n\to\infty} z_n = \eta.$$

With this notation we may present the following result.

Theorem 2. *Assume* (I). *Let $S \subset \mathbb{R}^2 \setminus \partial A$ be a connected set, $\eta \in \operatorname{cl} S \cap \partial A$ and denote by*

$$\check{S} = \{2\eta - z\,;\, z \in S\}$$

the reflection of S at η. If

$$\operatorname{contg}(\partial A,\eta) \cap \operatorname{contg}(S,\eta) = \emptyset = \operatorname{contg}(\partial A,\eta) \cap \operatorname{contg}(\check{S},\eta) \tag{10}$$

then the following assertions (P), (W), $(\mathcal{K})$ hold:

(P) The finite limit

$$\lim_{\substack{z\to\eta\\ z\in S}} Pf(z)$$

exists for each $f \in \mathcal{C}(\partial A, q)$ iff

$$\mathcal{U}^q_\infty(\eta) + \sup_{r>0} r^{-1}\mathcal{V}^q_r(\eta) < \infty. \tag{11}$$

(W) The finite limit

$$\lim_{\substack{z\to\eta\\ z\in S}} Wf(z)$$

exists for each $f \in \mathcal{C}(\partial A, q)$ iff

$$v^q(\eta) + \sup_{r>0} r^{-1}u^q_r(\eta) < \infty. \tag{12}$$

$(\mathcal{K})$ The limit

$$\lim_{\substack{z\to\eta\\ z\in S}} \mathcal{K}f(z) \quad (\in \mathbb{C}) \tag{13}$$

exists for each $f \in \mathcal{C}(\partial A, q)$ iff

$$\mathcal{U}^q_\infty(\eta) + v^q(\eta) < \infty. \tag{14}$$

Now we shall return to the *proof of Theorem 1.* Let $f \in \mathcal{C}^{(1)}(\partial A)$, $z \in \mathbb{C} \setminus \partial A$ and suppose that $\varphi \equiv \varphi_f$ has the properties specified in the definition of $\mathcal{K}f(z)$. Writing $z = x + \mathrm{i}y$, $\xi = \xi_1 + \mathrm{i}\xi_2$ we have

$$\mathcal{K}f(z) = \frac{1}{2\pi} \int_G \frac{-\partial_1\varphi(\xi)(\xi_2 - y) + \partial_2\varphi(\xi)(\xi_1 - x)}{|\xi - z|^2} \, \mathrm{d}\lambda_2(\xi)$$
$$- \frac{\mathrm{i}}{2\pi} \int_G \frac{\partial_1\varphi(\xi)(\xi_1 - x) + \partial_2\varphi(\xi)(\xi_2 - y)}{|\xi - z|^2} \, \mathrm{d}\lambda_2(\xi).$$

Defining

$$h_z(\xi) = \frac{1}{2\pi} \ln \frac{1}{|z - \xi|}, \quad \xi \neq z, \tag{15}$$

we obtain

$$Wf(z) \equiv \operatorname{Im} \mathcal{K}f(z) = \int_G \langle \operatorname{grad} \varphi(\xi), \operatorname{grad} h_z(\xi) \rangle \, \mathrm{d}\lambda_2(\xi), \tag{16$_W$}$$

$$Pf(z) \equiv \operatorname{Re} \mathcal{K}f(z) = \int_G \langle \operatorname{grad} \varphi(z), -\mathrm{i} \operatorname{grad} h_z(\xi) \rangle \, \mathrm{d}\lambda_2(\xi). \tag{16$_P$}$$

We see that Wf coincides with the double layer potential as investigated in [11] where the equivalence (I) $\Leftrightarrow$ (III) has been established. Once we prove (I) $\Leftrightarrow$ (IV) the proof of Theorem 1 will be complete, because (II) is equivalent with simultaneous validity of (III) & (IV). We start with

Proof of (I) $\Rightarrow$ (IV). Assuming (I) and noting that $q > 0$ on $\partial G \setminus \{\eta\}$ we observe that $\lambda_1(V \cap \partial_e G) < \infty$ for each open bounded set V with $\operatorname{cl} V \subset \mathbb{R}^2 \setminus \{\eta\}$. This implies that G has locally finite perimeter in $\mathbb{R}^2 \setminus \{\eta\}$ (cf. [5], chap. 4 and [19], section 5.8) which implies that, for each $\mathbb{R}^2$-valued function $v = v_1 + iv_2$ with components $v_j \in \mathcal{C}_0^{(1)}(\mathbb{R}^2 \setminus \{\eta\})$ the divergence formula holds

$$\int_G (\partial_1 v_1 + \partial_2 v_2) \, \mathrm{d}\lambda_2 = \int_{\widehat{\partial G}} \langle n^G, v \rangle \, \mathrm{d}\lambda_1, \tag{17}$$

where $\widehat{\partial G}$ is the reduced boundary of G and $n^G \colon \widehat{\partial G} \to \partial B(0,1)$ is the exterior normal of G in Federer's sense which are defined as follows:

$\widehat{\partial G}$ consists of those $\xi \in \mathbb{R}^2$ for which there is a unit vector $n \in \partial B(0,1)$ such that the half-plane

$$H_n(\xi) := \{z \in \mathbb{R}^2;\ \langle z - \xi, n \rangle < 0\}$$

satisfies

$$\overline{d}(H_n(\xi) \setminus G, \xi) = 0 = \overline{d}(G \setminus H_n(\xi), \xi);$$

such an $n \equiv n^G(\xi)$ is uniquely determined and is then called the exterior normal of G at ξ in Federer's sense.

$\widehat{\partial G}$ is a Borel set (cf. [4]) contained in $\partial_e G$ and the fact that G has locally finite perimeter in $\mathbb{R}^2 \setminus \{\eta\}$ implies that

$$\lambda_1(\partial_e G \setminus \widehat{\partial G}) = 0. \tag{18}$$

Assume first that $f \in \mathcal{C}^{(1)}(\partial A)$ vanishes in some neighbourhood of η in ∂A; the corresponding $\varphi_f \equiv \varphi$ can then be chosen in $\mathcal{C}_0^{(1)}(\mathbb{R}^2 \setminus \{\eta\})$. Applying the divergence formula (17) to $v(\xi) = \mathrm{i}\varphi(\xi)\frac{\xi - z}{|\xi - z|^2}$ (which vanishes in some neighbourhood of z together with φ) we transform (16_P) into

$$Pf(z) = \frac{1}{2\pi} \int_{\widehat{\partial G}} f(\xi)\Big\langle n^G(\xi), \mathrm{i}\frac{\xi - z}{|\xi - z|^2}\Big\rangle \,\mathrm{d}\lambda_1(\xi). \tag{19}$$

Thanks to (I), validity of (19) extends to all f in $\mathcal{C}^{(1)}(\partial A) \cap \mathcal{C}_0(\partial A, q)$ (compare the reasoning in the proof of Lemma 2.1 in [7]).

It follows from the definition of $\mathcal{K}^A f$ that, given $f \in \mathcal{C}^{(1)}(\partial A)$,

$$\mathcal{K}^A f(z) + \mathcal{K}^G f(z) = \frac{1}{\pi \mathrm{i}} \int_{\mathbb{R}^2} \frac{\overline{\partial}\varphi_f(\xi)}{\xi - z} \,\mathrm{d}\lambda_2(\xi) = 0.$$

We may thus suppose without loss of generality that G is bounded (replacing A by G would only change the sign of $\mathcal{K}f \equiv \mathcal{K}^A f$). Doing so we have for $z \in \operatorname{int} A$ (= the interior of A) and any g constant on ∂A

$$\mathcal{K}g(z) = 0;$$

if $z \in \operatorname{int} G$, $g(\partial A) = \{c\}$, then we may choose $\varphi_g \in \mathcal{C}_0^{(1)}(\mathbb{R}^2)$ and $B(z,r) \subset \operatorname{int} G$ with sufficiently small $r > 0$ such that $\varphi_g = c$ on $\operatorname{cl} G \setminus B(z,r)$ and $\varphi_g = 0$ in some neighbourhood of z, which results in

$$\mathcal{K}g(z) = \frac{1}{\pi \mathrm{i}} \int_{B(z,r)} \frac{\overline{\partial}\varphi_g(\xi)}{\xi - z} \,\mathrm{d}\lambda_2(\xi) = \frac{1}{2\pi} \int_{\partial B(z,r)} \frac{\varphi_g(\xi)}{z - \xi} \,\mathrm{d}\xi = -\mathrm{i}c.$$

We see that for any g constant on ∂A

$$Pg(z) \equiv \operatorname{Re} \mathcal{K}g(z) = 0,$$

whence we get by (19) for any $f \in \mathcal{C}^{(1)}(\partial A)$

$$(20) \qquad Pf(z) = \frac{1}{2\pi} \int_{\widehat{\partial G}} [f(\xi) - f(\eta)] \left\langle n^G(\xi), \mathrm{i} \frac{\xi - z}{|\xi - z|^2} \right\rangle \mathrm{d}\lambda_1(\xi).$$

Writing

$$\operatorname{dist}(z, M) := \inf\{|z - \xi|;\ \xi \in M\}$$

for the distance from $z \in \mathbb{R}^2$ to $M \subset \mathbb{R}^2$ we arrive at

$$(21) \qquad |Pf(z)| \leqslant \frac{1}{2\pi} [\operatorname{dist}(z, \widehat{\partial G})]^{-1} \|f\|_q \int_{\widehat{\partial G}} q \,\mathrm{d}\lambda_1,$$

$$f \in \mathcal{C}^{(1)}(\partial A) \cap \mathcal{C}(\partial A, q), \quad z \in \mathbb{C} \setminus \partial A,$$

which proves (IV). □

Proof of (IV) $\Rightarrow$ (I). Assuming (IV) we shall first prove that G has locally finite perimeter in $\mathbb{R}^2 \setminus \{\eta\}$. Denoting by $\partial_\theta \ldots = \langle \theta, \operatorname{grad} \ldots \rangle$ the derivative in the direction of $\theta \in \partial B(0,1)$ we have to verify that

$$(22) \qquad \sup \left\{ \int_G \partial_\theta \psi \,\mathrm{d}\lambda_2;\ \psi \in \mathcal{C}_0^{(1)}(V), |\psi| \leqslant 1 \right\} < +\infty$$

for any bounded open set V with $\operatorname{cl} V \subset \mathbb{R}^2 \setminus \{\eta\}$ and any $\theta \in \partial B(0,1)$.

Fix such a V and θ. As in Lemma 1 from [11] we shall employ the argument from the proof of Theorem 2.12 in [7].

Choose points z^1, z^2, $z^3 \in \mathbb{R}^2 \setminus \partial G$ which are not situated on a single straight line. The assumption (IV) guarantees the existence of a $c \in [0, +\infty[$ such that

$$(23) \qquad f \in \mathcal{C}^{(1)}(\partial A) \cap \mathcal{C}_0(\partial A, q) \Rightarrow |Pf(z^k)| \leqslant c\|f\|_{q,0}, \quad 1 \leqslant k \leqslant 3.$$

Put $B_j = \{1,2,3\} \setminus \{j\}$ and denote by Π_j the straight line containing the points in $\{z^k;\ k \in B_j\}$. Since

$$\bigcup_{j=1}^{3} (\mathbb{R}^2 \setminus \Pi_j) = \mathbb{R}^2$$

we may choose

$$\alpha_j \in \mathcal{C}_0^{(1)}(\mathbb{R}^2 \setminus [\{\eta\} \cup \Pi_j]) \quad (j = 1,2,3)$$

such that

$$\alpha := \sum_{j=1}^{3} \alpha_j$$

is identically equal to 1 in some neighbourhood of $\operatorname{cl} V$.

If $\psi \in \mathcal{C}_0^{(1)}(V)$ then

$$\int_G \partial_\theta \psi \, d\lambda_2 = \int_G \alpha(\xi) \partial_\theta \psi(\xi) \, d\lambda_2(\xi).$$

Thus (22) will be verified if we show that

$$\text{(24)} \qquad \sup \left\{ \int_G \alpha_j(\xi) \partial_\theta \psi(\xi) \, d\lambda_2(\xi) \, ; \, \psi \in \mathcal{C}_0^{(1)}(V), |\psi| \leqslant 1 \right\} < +\infty, \quad j \in \{1,2,3\}.$$

Fix $j \in \{1,2,3\}$ and notice that the two vectors $\xi - z^k$ ($k \in B_j$) are linearly independent for all ξ sufficiently close to $\operatorname{spt} \alpha_j$. This guarantees the existence of infinitely differentiable real-valued functions $a_k(\xi)$ such that (cf. (15) for the notation)

$$\theta = - \sum_{k \in B_j} a_k(\xi) \mathrm{i} \operatorname{grad} h_{z^k}(\xi)$$

for all ξ in some neighbourhood of $\operatorname{spt} \alpha_j$. Hence

$$\int_G \alpha_j \partial_\theta \psi \, d\lambda_2 = \sum_{k \in B_j} \int_G \alpha_j(\xi) a_k(\xi) \langle \operatorname{grad} \psi(\xi), -\mathrm{i} \operatorname{grad} h_{z^k}(\xi) \rangle \, d\lambda_2(\xi).$$

Fix j, k and put $F_{j,k}(\xi) = \alpha_j(\xi) a_k(\xi)$, so that

$$F_{j,k} \in \mathcal{C}_0^{(1)}(\mathbb{R}^2 \setminus \{z^k\})$$

and

$$\int_G F_{j,k} \langle \operatorname{grad} \psi, -\mathrm{i} \operatorname{grad} h_{z^k} \rangle \, d\lambda_2 = \int_G \langle \operatorname{grad}(F_{j,k}\psi), -\mathrm{i} \operatorname{grad} h_{z^k} \rangle \, d\lambda_2 - \int_G \psi \langle \operatorname{grad} F_{j,k}, -\mathrm{i} \operatorname{grad} h_{z^k} \rangle \, d\lambda_2.$$

Clearly, the last integral has a bound independent of ψ ($\in \mathcal{C}_0^{(1)}(V)$, $|\psi| \leqslant 1$):

$$\left| \int_G \psi \langle \operatorname{grad} F_{j,k}, -\mathrm{i} \operatorname{grad} h_{z^k} \rangle \, d\lambda_2 \right| \leqslant \frac{1}{2\pi} \int_G |\operatorname{grad} F_{j,k}(\xi)| \cdot |\xi - z^k|^{-1} \, d\lambda_2(\xi) < +\infty.$$

Noting that $q > 0$ on $\partial G \cap \operatorname{spt} \alpha_j \supset \partial G \cap \operatorname{spt} F_{j,k}$ we fix $a \in [0, +\infty[$ such that

$$|F_{j,k}| \leqslant aq \quad \text{on } \partial G, \quad j \in \{1,2,3\}, \ k \in B_j.$$

Denoting by $f_{j,k}$ the restriction of $F_{j,k}\psi$ to ∂G we get from (16_P)

$$\int_G \langle \operatorname{grad}(F_{j,k}\psi), -\mathrm{i} \operatorname{grad} h_{z^k} \rangle \, \mathrm{d}\lambda_2 = P f_{j,k}(z^k),$$

whence it follows by (23)

$$\left| \int_G \langle \operatorname{grad}(F_{j,k}\psi), -\mathrm{i} \operatorname{grad} h_{z^k} \rangle \, \mathrm{d}\lambda_2 \right| \leqslant ac, \quad j \in \{1,2,3\}, \ k \in B_j$$

and (24) is verified.

Now when we know that G has locally finite perimeter in $\mathbb{R}^2 \setminus \{\eta\}$ we have for each $f \in \mathcal{C}^{(1)}(\partial A)$ vanishing in some neighbourhood of η in ∂G and any $z \in \mathbb{R}^2 \setminus \partial G$ the formula (19) which implies that

$$\begin{aligned}
&\int_{\widehat{\partial G}} q |\langle n^q, -\mathrm{i} \operatorname{grad} h_z \rangle| \, \mathrm{d}\lambda_1 \\
&\qquad = \sup\{Pf(z);\ f \in \mathcal{C}^{(1)}(\partial A) \cap \mathcal{C}_0(\partial A, q), \eta \notin \operatorname{spt} f, \|f\|_{q,0} \leqslant 1\}, \\
&\qquad\qquad z \in \mathbb{R}^2 \setminus \partial A.
\end{aligned}$$

Setting $z = z^k$ we get by (23)

$$\int_{\widehat{\partial G}} q \left| \langle n^G, -\mathrm{i} \operatorname{grad} h_{z^k} \rangle \right| \mathrm{d}\lambda_1 \leqslant c, \quad k \in \{1,2,3\}.$$

Since the points z^1, z^2, z^3 are affinely independent we have

$$\xi \in \widehat{\partial G} \Rightarrow \sum_{k=1}^{3} \left| \langle n^G(\xi), -\mathrm{i} \operatorname{grad} h_{z^k}(\xi) \rangle \right| \geqslant b$$

for suitable $b \in]0, +\infty[$, whence

$$\int_{\widehat{\partial G}} q \, \mathrm{d}\lambda_1 \leqslant b^{-1} \sum_{k=1}^{3} \int_{\widehat{\partial G}} q \left| \langle n^G, -\mathrm{i} \operatorname{grad} h_{z^k} \rangle \right| \mathrm{d}\lambda_1 \leqslant 3b^{-1}c$$

which proves (I). This also completes the proof of Theorem 1. □

Remark 1. In what follows we always assume (I). As verified in the course of the proof of Theorem 1, the operator (3) extends continuously to $\mathcal{C}(\partial A, q)$; for any $f \in \mathcal{C}(\partial A, q)$, the value of $Pf \in \mathcal{H}(\mathbb{R}^2 \setminus \partial A)$ at $z \in \mathbb{R}^2 \setminus \partial A$ is given by the formula (20).

As we know from [11], also the operator (2) extends by continuity to $\mathcal{C}(\partial A, q)$ and, for any $f \in \mathcal{C}(\partial A, q)$, the value of $Wf \in \mathcal{H}(\mathbb{R}^2 \setminus \partial A)$ at $z \in \mathbb{R}^2 \setminus \partial A$ is given by the formula

$$(25) \qquad Wf(z) = -f(\eta)\chi_G(z) + \int\limits_{\widehat{\partial G}} [f(\xi) - f(\eta)] \left\langle n^G(\xi), \operatorname{grad} h_z(\xi) \right\rangle \, \mathrm{d}\lambda_1(\xi),$$

where χ_G denotes the indicator function of G (cf. Remark 1 in [11]).

Combining these results we conclude that the operator (1) extends by continuity to $\mathcal{C}(\partial A, q)$; for any $f \in \mathcal{C}(\partial A, q)$, the value of $\mathcal{K}f \in \mathcal{A}(\mathbb{C} \setminus \partial A)$ at any $z \in \mathbb{C} \setminus \partial A$ is given by the formula

$$(26) \qquad \mathcal{K}f(z) = -\mathrm{i}f(\eta)\chi_G(z) + \frac{1}{2\pi} \int\limits_{\widehat{\partial G}} \frac{f(\xi) - f(\eta)}{z - \xi} \tau^G(\xi) \, \mathrm{d}\lambda_1(\xi),$$

where $\tau^G(\xi) = \mathrm{i}n^G(\xi)$, $\xi \in \widehat{\partial G}$.

The weight q has so far been defined on ∂A only; we extend it to $\mathbb{R}^2$ defining $q(z) = \sup q(\partial A)$, $z \in \mathbb{R}^2 \setminus \partial A$; thus $q\colon \mathbb{R}^2 \to [0, \infty[$ remains bounded and lower semicontinuous on $\mathbb{R}^2$, $q > 0$ on $\mathbb{R}^2 \setminus \{\eta\}$.

Next we shall prove several auxiliary results needed for the proof of Theorem 2.

Lemma 1. *The function $\varrho \mapsto \mathcal{U}^q(\varrho, \eta)$ defined by (4) is Lebesgue measurable on $]0, \infty[$ and if $u^q_r(\eta)$, $\mathcal{U}^q_r(\eta)$ are given by (5), then we have for any $r \in]0, \infty]$*

$$(27) \qquad \begin{aligned} u^q_r(\eta) &= \sup \Big\{ \int\limits_G \Big\langle \operatorname{grad} \varphi(\xi), i\frac{\xi - \eta}{|\xi - \eta|} \Big\rangle \, \mathrm{d}\lambda_2(\xi); \\ &\qquad \varphi \in \mathcal{C}^{(1)}_0(B(\eta, r) \setminus \{\eta\}) \text{ on } \operatorname{spt} \varphi, |\varphi| < q \Big\} \\ &= \int\limits_{\widehat{\partial G} \cap B(\eta, r)} q(x) \Big| \Big\langle n^G(x), \mathrm{i}\frac{x - \eta}{|x - \eta|} \Big\rangle \Big| \, \mathrm{d}\lambda_1(x), \end{aligned}$$

$$(28) \qquad \mathcal{U}^q_r(\eta) = \int\limits_{\widehat{\partial G} \cap B(\eta, r)} q(\xi) \Big| \Big\langle n^G(\xi), \mathrm{i}\frac{\xi - \eta}{|\xi - \eta|^2} \Big\rangle \Big| \, \mathrm{d}\lambda_1(\xi),$$

where we put $B(\eta, \infty) := \mathbb{R}^2$.

Proof. Fix $r \in]0,\infty]$ and put for $j \in \{1,2,3,4\}$

$$U_j = \left\{\eta + \varrho e^{i\theta}\,;\, 0 < \varrho < r, (j-2)\frac{\pi}{2} < \theta < j\frac{\pi}{2}\right\},$$

so that

$$\bigcup_{j=1}^{4} U_j = B(\eta, r) \setminus \{\eta\} \equiv U.$$

Choose continuously differentiable functions f_j on U such that

$$\sum_{j=1}^{4} f_j = 1, \quad f_j \geqslant 0, \quad f_j = 0 \quad \text{on } U \setminus U_j \ (1 \leqslant j \leqslant 4).$$

Define for $\varphi \in \mathcal{C}_0^{(1)}(U)$

$$L(\varphi) = \int_G \left\langle \operatorname{grad}\varphi(\xi), \mathrm{i}\frac{\xi - \eta}{|\xi - \eta|}\right\rangle \mathrm{d}\lambda_2(\xi).$$

It is not difficult to show that

$$\begin{aligned} (29) \qquad & \sup\{L(\varphi)\,;\, \varphi \in \mathcal{C}_0^{(1)}(U), |\varphi| < q \text{ on } \operatorname{spt}\varphi\} \\ &= \sum_{j=1}^{4} \sup\{L(\varphi_j)\,;\, \varphi_j \in \mathcal{C}_0^{(1)}(U), |\varphi_j| < f_j q \text{ on } \operatorname{spt}\varphi_j\}. \end{aligned}$$

Consider $\varphi_1 \in \mathcal{C}_0^{(1)}(U)$ such that $|\varphi_1| < f_1 q$ on $\operatorname{spt}\varphi_1$ (whence $\operatorname{spt}\varphi_1 \subset U_1$). Employing the diffeomorphism

$$\Psi\colon]0, r[\times\left] -\frac{\pi}{2}, \frac{\pi}{2}\right[\to U_1$$

introducing the polar coordinates ϱ, θ by

$$\Psi(\varrho, \theta) = \eta + \varrho e^{i\theta}, \quad 0 < \varrho < r, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

we get

$$L(\varphi_1) = \int_{G \cap U_1} \left\langle \operatorname{grad}\varphi_1(\xi), \mathrm{i}\frac{\xi - \eta}{|\xi - \eta|}\right\rangle \mathrm{d}\lambda_2(\xi) = \int_{\tilde{G}} \partial_\theta \tilde{\varphi}_1(\varrho, \theta)\, \mathrm{d}\varrho,$$

where we have put

$$\tilde{G} = \Psi^{-1}(U_1 \cap G), \quad \tilde{\varphi}_1(\varrho, \theta) = \varphi_1\big(\Psi(\varrho, \theta)\big)$$

and ∂_θ denotes the derivative $w.r.$ to the variable θ. Denoting by $\tilde{q}_1 \equiv f_1(\Psi)q(\Psi)$ the composition of Ψ and $f_1 q$ we have clearly $\tilde{\varphi}_1 < \tilde{q}_1$ on $\operatorname{spt}\tilde{\varphi}_1 \subset]0,r[\times]-\frac{\pi}{2},\frac{\pi}{2}[$. If φ_1 runs over $\{\varphi_1 \in C_0^{(1)}(U);\, |\varphi_1| < f_1 q \text{ on } \operatorname{spt}\varphi_1\}$, then the corresponding $\tilde{\varphi}_1 = \varphi_1(\Psi)$ runs over the class

$$\mathcal{A}_1 \equiv \left\{\tilde{\varphi}_1 \in C_0^{(1)}\left(]0,r[\times\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right);\, |\tilde{\varphi}_1| < \tilde{q}_1 \text{ on } \operatorname{spt}\tilde{\varphi}_1\right\}.$$

Referring to lemma 5 in [11] we obtain that the function

$$\tilde{n}_1 \colon \varrho \mapsto \sum_\theta \tilde{q}_1(\varrho,\theta), \quad (\varrho,\theta) \in \partial_e\tilde{G} \cap \left(\{\varrho\}\times\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$$

is Lebesgue measurable and

$$\sup\left\{\int_{\tilde{G}} \partial_\theta\tilde{\varphi}_1(\varrho,\theta)\,\mathrm{d}\varrho;\, \tilde{\varphi}_1 \in \mathcal{A}_1\right\} = \int_0^r \tilde{n}_1(\varrho)\,\mathrm{d}\varrho.$$

Since Ψ is a diffeomorphism, it is easy to see that for $x \in]0,r[\times\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$ the following equivalences are true:

$$\begin{aligned}&\overline{d}_G(\Psi(x)) = 0 \Leftrightarrow \overline{d}_{\tilde{G}}(x) = 0,\\ &\overline{d}_A(\Psi(x)) = 0 \Leftrightarrow \overline{d}_{\tilde{A}}(x) = 0 \quad (\text{here } \tilde{A} = \Psi^{-1}(A));\end{aligned}$$

hence

$$\Psi(x) \in \partial_e G \Leftrightarrow x \in \partial_e\tilde{G},$$

so that

$$\tilde{n}_1(\varrho) = \sum_\xi f_1(\xi)q(\xi), \quad \xi \in \partial_e G \cap \left\{\eta + \varrho e^{i\theta}, -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right\}.$$

Noting that f_1 vanishes on $U \setminus U_1$, we have

$$\tilde{n}_1(\varrho) = \sum_\xi f_1(\xi)q(\xi), \quad \xi \in \partial_e G \cap \{\xi;\, |\xi - \eta| = \varrho\}.$$

We have thus proved that

$$\sup\{L(\varphi_1);\, \varphi_1 \in C_0^{(1)}(U), |\varphi_1| < f_1 q \text{ on } \operatorname{spt}\varphi_1\} = \int_0^r \tilde{n}_1(\varrho)\,\mathrm{d}\varrho.$$

Defining

$$\tilde{n}_j(\varrho) = \sum f_j(\xi)q(\xi), \quad \xi \in \partial_e G \cap \{\xi;\, |\xi - \eta| = \varrho\}$$

we get similarly

$$\sup\{L(\varphi_j);\, \varphi_j \in \mathcal{C}_0^{(1)}(U), |\varphi_j| < f_j q \text{ on } \operatorname{spt}\varphi_j\} = \int_0^r \tilde{n}_j(\varrho)\,\mathrm{d}\varrho \qquad \text{for } j \in \{1,2,3,4\},$$

which together with (29) proves

$$\sup\{L(\varphi);\, \varphi \in \mathcal{C}_0^{(1)}(U), |\varphi| < q \text{ on } \operatorname{spt}\varphi\} = u_r^q(\eta).$$

We have seen in the course of proof of theorem 1 that (I) implies that G has locally finite perimeter in $\mathbb{R}^2 \setminus \{\eta\}$. Using the divergence formula we have thus for any $\varphi \in \mathcal{C}_0^{(1)}(U)$

$$\int_G \Big\langle \operatorname{grad}\varphi(\xi), \mathrm{i}\frac{\xi-\eta}{|\xi-\eta|}\Big\rangle \,\mathrm{d}\lambda_2(\xi) = \int_{\widehat{\partial G}} \varphi(\xi)\Big\langle \mathrm{i}\frac{\xi-\eta}{|\xi-\eta|}, n^G(\xi)\Big\rangle \,\mathrm{d}\lambda_1(\xi),$$

whence

$$\sup\{L(\varphi);\, \varphi \in \mathcal{C}_0^{(1)}(U), |\varphi| < q \text{ on } \operatorname{spt}\varphi\} = \int_{\widehat{\partial G}\cap B(\eta,r)} q(\xi)\Big|\Big\langle \mathrm{i}\frac{\xi-\eta}{|\xi-\eta|}, n^G(\xi)\Big\rangle\Big| \,\mathrm{d}\lambda_1(\xi)$$

and the proof of (27) is complete. □

It is easy to observe that boundedness of $q\colon \mathbb{R}^2 \to [0,\infty[$ is irrelevant for validity of (27). Defining

$$Q(\xi) = \begin{cases} q(\xi)/|\xi-\eta|, & \xi \neq \eta \\ 0, & \xi = \eta \end{cases}$$

and applying (27) to Q instead of q we obtain

$$\mathcal{U}_r^q(\eta) \equiv u_r^Q(\eta) = \int_{\widehat{\partial G}\cap B(\eta,r)} q(\xi)/|\xi-\eta| \Big|\Big\langle n^G(\xi), \mathrm{i}\frac{\xi-\eta}{|\xi-\eta|}\Big\rangle\Big| \,\mathrm{d}\lambda_1(\xi)$$

which is just (28).

Lemma 2. *Define*

$$\mu^q(M) = \int_{M\cap\partial_e G} q\,\mathrm{d}\lambda_1$$

for any Borel set $M \subset \mathbb{R}^2$. Then

$$\sup_{r>0} r^{-1}\mu^q\big(B(\eta,r)\big) < \infty \Rightarrow \sup_{r>0} r^{-1}u_r^q(\eta) < \infty \tag{30}$$

and, in case $v^q(\eta) < \infty$, also conversely

$$\sup_{r>0} r^{-1}u_r^q(\eta) < \infty \Rightarrow \sup_{r>0} r^{-1}\mu^q\big(B(\eta,r)\big) < \infty. \tag{31}$$

Proof. Using Lemma 1 we get

$$\begin{aligned} u_r^q(\eta) &= \int\limits_{\widehat{\partial G}\cap B(\eta,r)} q(\xi)\Big|\Big\langle n^G(\xi), \mathrm{i}\frac{\xi-\eta}{|\xi-\eta|}\Big\rangle\Big|\,\mathrm{d}\lambda_1(\xi) \\ &\leqslant \int\limits_{\widehat{\partial G}\cap B(\eta,r)} q(\xi)\,\mathrm{d}\lambda_1(\xi) = \mu^q\big(B(\eta,r)\big), \end{aligned}$$

which implies (30). Since the unit vectors

$$\frac{\xi-\eta}{|\xi-\eta|},\quad \mathrm{i}\frac{\xi-\eta}{|\xi-\eta|}\quad (\xi\neq\eta)$$

are orthogonal, we have

$$\Big|\Big\langle n^G(\xi), \frac{\xi-\eta}{|\xi-\eta|}\Big\rangle\Big| + \Big|\Big\langle n^G(\xi), \mathrm{i}\frac{\xi-\eta}{|\xi-\eta|}\Big\rangle\Big| \geqslant 1,\quad \xi\in\widehat{\partial G}\setminus\{\eta\}. \tag{32}$$

Integrating over $\widehat{\partial G}\cap B(\eta,r)$ and using (18) we obtain

$$\begin{aligned} \mu^q\big(B(\eta,r)\big) &= \int\limits_{\widehat{\partial G}\cap B(\eta,r)} q\,\mathrm{d}\lambda_1 \leqslant \int\limits_{\widehat{\partial G}\cap B(\eta,r)} q(\xi)\Big|\Big\langle n^G(\xi), \mathrm{i}\frac{\xi-\eta}{|\xi-\eta|}\Big\rangle\Big|\,\mathrm{d}\lambda_1(\xi) \\ &\quad + \int\limits_{\widehat{\partial G}\cap B(\eta,r)} q(\xi)\Big|\Big\langle n^G(\xi), \frac{\xi-\eta}{|\xi-\eta|}\Big\rangle\Big|\,\mathrm{d}\lambda_1(\xi). \end{aligned}$$

The first integral in the last sum equals $u_r^q(\eta)$ by Lemma 1. The second integral can be estimated with help of (27) from Lemma 3 in [11] as follows:

$$\begin{aligned} &\int\limits_{\widehat{\partial G}\cap B(\eta,r)} q(\xi)\Big|\Big\langle n^G(\xi), \frac{\xi-\eta}{|\xi-\eta|}\Big\rangle\Big|\,\mathrm{d}\lambda_1(\xi) \\ &\qquad = 2\pi \int\limits_{\widehat{\partial G}\cap B(\eta,r)} q(\xi)|\xi-\eta|\cdot|\langle n^G(\xi), \operatorname{grad} h_\eta(\xi)\rangle|\,\mathrm{d}\lambda_1(\xi) \\ &\qquad \leqslant 2\pi r\int\limits_{\widehat{\partial G}} q|\langle n^G, \operatorname{grad} h_\eta\rangle|\,\mathrm{d}\lambda_1 = 2\pi r v^q(\eta). \end{aligned}$$

We thus arrive at

$$\mu^q\big(B(\eta,r)\big) \leqslant u_r^q(\eta) + 2\pi r v^q(\eta)$$

which yields (31) in case $v^q(\eta) < \infty$. □

Proof of the assertion (W) in Theorem 2. Combine Lemma 2 and Theorem 3 in [11]. □

Combining Lemma 2 with Theorem 2 in [11] we obtain

Theorem 3. *Suppose that $S_j \subset \mathbb{R}^2 \setminus \partial A$ are connected sets such that*

$$\eta \in \operatorname{cl} S_j \cap \partial A, \quad \lim_{\substack{z\to\eta \\ z\in S_j}} \frac{z-\eta}{|z-\eta|} = \theta_j \quad (j = 1,2).$$

If the vectors θ_1, θ_2 are linearly independent, then (12) *is necessary for the existence of the finite limits*

$$\lim_{\substack{z\to\eta \\ z\in S_j}} Wf(z) \quad (j = 1,2)$$

for all $f \in \mathcal{C}(\partial A, q)$; if, besides that,

$$\operatorname{contg}(\partial A, \eta) \cap \operatorname{contg}(S_j, \eta) = \emptyset \quad (j = 1,2),$$

then (12) *is also sufficient.*

Lemma 3. *Let $S \subset \mathbb{R}^2 \setminus \partial A$, $\eta \in \operatorname{cl} S \cap \partial A$. Then the finite limit*

$$\lim_{\substack{z\to\eta \\ z\in S}} P^A f(z) \tag{33}$$

exists for each $f \in \mathcal{C}(\partial A, q)$ iff

$$\sup_{z\in S} \int_{\widehat{\partial G}} q(\xi)\left|\left\langle n^G(\xi), \mathrm{i}\frac{\xi - z}{|\xi - z|^2}\right\rangle\right| \mathrm{d}\lambda_1(\xi) < \infty. \tag{34}$$

Proof. We shall first show that validity of

$$\limsup_{\substack{z\to\eta \\ z\in S}} |P^A f(z)| < \infty$$

for all $f \in \mathcal{C}_0(\partial A, q)$ implies (34). Observe that, for any fixed $z \in \mathbb{R}^2 \setminus \partial A$, the norm of the linear functional

$$L_z \colon f \mapsto P^A f(z) = \frac{1}{2\pi} \int\limits_{\widehat{\partial G}} f(\xi) \Big\langle n^G(\xi), \mathrm{i}\frac{\xi - z}{|\xi - z|^2} \Big\rangle \, \mathrm{d}\lambda_1(\xi)$$

on $\mathcal{C}_0(\partial A, q)$ is given by

$$\sup \Big\{ \frac{1}{2\pi} \int\limits_{\widehat{\partial G}} f(\xi) \Big\langle n^G(\xi), \mathrm{i}\frac{\xi - z}{|\xi - z|^2} \Big\rangle \, \mathrm{d}\lambda_1(\xi) \,;\, |f| \leqslant q, f \in \mathcal{C}_0(\partial A, q) \Big\}$$
$$= \frac{1}{2\pi} \int\limits_{\widehat{\partial G}} q(\xi) \Big| \Big\langle n^G(\xi), \mathrm{i}\frac{\xi - z}{|\xi - z|^2} \Big\rangle \Big| \, \mathrm{d}\lambda_1(\xi).$$

If the values attained by the functionals $\{L_z\}_{z \in S}$ at any $f \in \mathcal{C}_0(\partial A, q)$ remain bounded, then the Banach–Steinhaus principle of uniform boundedness guarantees (34). Conversely, assume (34). Fatou's lemma yields

$$\int\limits_{\widehat{\partial G}} q(\xi) \Big| \Big\langle n^G(\xi), \mathrm{i}\frac{\xi - \eta}{|\xi - \eta|^2} \Big\rangle \Big| \, \mathrm{d}\lambda_1(\xi) \leqslant s, \tag{35}$$

where

$$s := \sup_{z \in S} \int\limits_{\widehat{\partial G}} q(\xi) \Big| \Big\langle n^G(\xi), \mathrm{i}\frac{\xi - z}{|\xi - z|^2} \Big\rangle \Big| \, \mathrm{d}\lambda_1(\xi), \tag{36}$$

so that we may define for any $f \in \mathcal{C}(\partial A, q)$

$$P^A f(\eta) = \int\limits_{\widehat{\partial G}} [f(\xi) - f(\eta)] \Big\langle n^G(\xi), \mathrm{i}\frac{\xi - z}{|\xi - z|^2} \Big\rangle \, \mathrm{d}\lambda_1(\xi).$$

We shall show that

$$\lim_{\substack{z \to \eta \\ z \in S}} P^A f(z) = P^A f(\eta), \quad f \in \mathcal{C}(\partial A, q). \tag{37}$$

It will suffice to verify (37) for $f \in \mathcal{C}_0(\partial A, q)$ only, because $P^A f(z) = 0$ ($z \in S$) for constant f by (20). Fix an arbitrary $f \in \mathcal{C}_0(\partial A, q)$ and $\varepsilon > 0$; choose $\delta > 0$ such that

$$\xi \in \partial A \cap B(\eta, \delta) \Rightarrow |f(\xi)| \leqslant \varepsilon q(\xi).$$

Then

$$|Pf(z) - Pf(\eta)| \leqslant \int\limits_{\widehat{\partial G} \cap B(\eta,\delta)} |f(\xi)| \cdot \left| \left\langle n(\xi), \mathrm{i}\frac{\xi - z}{|\xi - z|^2} \right\rangle \right| \mathrm{d}\lambda_1(\xi)$$
$$+ \int\limits_{\widehat{\partial G} \cap B(\eta,\delta)} |f(\xi)| \cdot \left| \left\langle n(\xi), \mathrm{i}\frac{\xi - \eta}{|\xi - \eta|^2} \right\rangle \right| \mathrm{d}\lambda_1(\xi)$$
$$+ \int\limits_{\widehat{\partial G} \setminus B(\eta,\delta)} |f(\xi)| \cdot \left| \left\langle n(\xi), \mathrm{i}\frac{\xi - z}{|\xi - z|^2} - \mathrm{i}\frac{\xi - \eta}{|\xi - \eta|^2} \right\rangle \right| \mathrm{d}\lambda_1(\xi)$$
$$\leqslant \varepsilon \int\limits_{\widehat{\partial G}} q(\xi) \left| \left\langle n(\xi), \mathrm{i}\frac{\xi - z}{|\xi - z|^2} \right\rangle \right| \mathrm{d}\lambda_1(\xi)$$
$$+ \varepsilon \int\limits_{\widehat{\partial G}} q(\xi) \left| \left\langle n(\xi), \mathrm{i}\frac{\xi - \eta}{|\xi - \eta|^2} \right\rangle \right| \mathrm{d}\lambda_1(\xi) + J_\delta f(z),$$

where

$$J_\delta f(z) = \int\limits_{\widehat{\partial G} \setminus B(\eta,\delta)} |f(\xi)| \cdot \left| \left\langle n(\xi), \mathrm{i}\frac{\xi - \eta}{|\xi - \eta|^2} - \mathrm{i}\frac{\xi - z}{|\xi - z|^2} \right\rangle \right| \mathrm{d}\lambda_1(\xi).$$

Using the notation from (36) we have

$$|Pf(z) - Pf(\eta)| \leqslant 2\varepsilon s + J_\delta f(z).$$

Noting that $\mathrm{i}\frac{\xi - z}{|\xi - z|^2} \to \mathrm{i}\frac{\xi - \eta}{|\xi - \eta|^2}$ uniformly *w.r.* to $\xi \in \widehat{\partial G} \setminus B(\eta, \delta)$ as $z \to \eta$ $(z \in S)$, we conclude that

$$\limsup_{\substack{z \to \eta \\ z \in S}} |Pf(z) - Pf(\eta)| \leqslant 2\varepsilon s$$

which proves (37), because $\varepsilon > 0$ was arbitrary. □

Lemma 4. *Let $S \subset \mathbb{R}^2 \setminus \partial A$ be a connected set, $\eta \in \operatorname{cl} S \cap \partial A$ and assume* (10). *Then* (34) *implies validity of the following relations* (38)–(40):

(38) $$\int\limits_{\widehat{\partial G}} q(\xi) \left| \left\langle n^G(\xi), \mathrm{i}\frac{\xi - \eta}{|\xi - \eta|^2} \right\rangle \right| \mathrm{d}\lambda_1(\xi) < \infty,$$

(39) $$\sup_{r>0} r^{-1} \mu^q\big(B(\eta, r)\big) < \infty,$$

(40) $$\sup_{r>0} r^{-1} \int\limits_{\widehat{\partial G} \cap B(\eta,r)} q(\xi) \left| \left\langle n(\xi), \frac{\xi - \eta}{|\xi - \eta|} \right\rangle \right| \mathrm{d}\lambda_1(\xi) < \infty.$$

Proof. Assume (34). In view of (35), (36) we have then

$$\int\limits_{\widehat{\partial G}} q(\xi)\left|\left\langle n^G(\xi), \mathrm{i}\frac{\xi-\eta}{|\xi-\eta|^2}\right\rangle\right| \mathrm{d}\lambda_1(\xi) \leqslant s$$

and (38) is checked. Next we shall use reasoning similar to that in the proof of Proposition 3 in [11]. It follows from (10) that there are constants $\delta \in]0, \pi/4[$ and $\varrho \in]0, \infty[$ such that, for $\xi \in \widehat{\partial G} \cap [B(\eta, \varrho) \setminus \{\eta\}]$ and $z \in S \cap B(\eta, \varrho)$, the angle enclosed by the vectors

$$\frac{\xi-\eta}{|\xi-\eta|}, \quad \frac{z-\eta}{|z-\eta|}$$

exceeds 2δ and the same holds of the vectors

$$\frac{\xi-\eta}{|\xi-\eta|}, \quad -\frac{z-\eta}{|z-\eta|}.$$

Since $\eta \in \operatorname{cl} S$ and S is connected we may assume that $\varrho > 0$ has been fixed small enough to guarantee that

$$0 < r < \varrho \Rightarrow S \cap \partial B(\eta, r) \neq \emptyset.$$

Fix $r \in]0, \varrho[$, $\xi \in \widehat{\partial G} \cap B(\eta, r)$ and choose $z \in S \cap \partial B(\eta, r)$. If the vector $in^G(\xi)$ encloses with one of the vectors

$$\frac{z-\eta}{|z-\eta|}, \quad -\frac{z-\eta}{|z-\eta|} \tag{41}$$

the angle not exceeding $\frac{1}{2}\pi - \delta$, then

$$\begin{aligned} \left|\left\langle in^G(\xi), \xi-\eta\right\rangle\right| + \left|\left\langle in^G(\xi), \xi-z\right\rangle\right| &\geqslant \left|\left\langle in^G(\xi), z-\eta\right\rangle\right| \\ &\geqslant r\cos\left(\frac{1}{2}\pi-\delta\right). \end{aligned}$$

If both vectors (41) enclose with $in^G(\xi)$ the angle exceeding $\frac{1}{2}\pi - \delta$, then one at least of the vectors

$$\frac{\xi-\eta}{|\xi-\eta|}, \quad -\frac{\xi-\eta}{|\xi-\eta|}$$

encloses with $in^G(\xi)$ the angle which is less than

$$\frac{1}{2}\pi - 2\delta + \delta = \frac{1}{2}\pi - \delta,$$

whence

$$\left|\left\langle in^G(\xi), \xi-\eta\right\rangle\right| \geqslant |\xi-\eta|\cos\left(\frac{1}{2}\pi-\delta\right).$$

Since $|\xi - z| \leqslant |\xi - \eta| + |\eta - z| \leqslant 2r$, we have in any case

$$\frac{|\langle \mathrm{i}n^G(\xi), \xi - \eta\rangle|}{|\xi - \eta|^2} + \frac{|\langle \mathrm{i}n^G(\xi), \xi - z\rangle|}{|\xi - z|^2} \geqslant \frac{1}{4} r^{-1} \cos\Big(\frac{1}{2}\pi - \delta\Big).$$

Hence

$$r^{-1}\mu^q\big(B(\eta,r)\big) \leqslant 4\cos^{-1}\Big(\frac{1}{2}\pi - \delta\Big)\Bigg[\int\limits_{\widehat{\partial G}\cap B(\eta,r)} q(\xi)\Big|\Big\langle \mathrm{i}n^G(\xi), \frac{\xi-\eta}{|\xi-\eta|^2}\Big\rangle\Big|\,\mathrm{d}\lambda_1(\xi)$$

$$+ \int\limits_{\widehat{\partial G}\cap B(\eta,r)} q(\xi)\Big|\Big\langle \mathrm{i}n^G(\xi), \frac{\xi-z}{|\xi-z|^2}\Big\rangle\Big|\,\mathrm{d}\lambda_1(\xi)\Bigg] \leqslant 4\cos^{-1}\Big(\frac{1}{2}\pi - \delta\Big)\cdot 2s.$$

If $r \geqslant \varrho$, then $r^{-1}\mu^q\big(B(\eta,r)\big) \leqslant \varrho^{-1}\mu^q(\mathbb{R}^2) < \infty$ and (39) is verified. Clearly, (39) $\Rightarrow$ (40) and the proof is complete. $\square$

Lemma 5. *Assume* (38), (40). *If* $S \subset \mathbb{R}^2 \setminus \partial A$, $\eta \in \operatorname{cl} S \cap \partial A$ *and*

$$\operatorname{contg}(S,\eta) \cap \operatorname{contg}(\widehat{\partial G},\eta) = \emptyset, \tag{42}$$

then

$$\limsup_{\substack{z\to\eta\\ z\in S}} \int\limits_{\widehat{\partial G}} q(\xi)\Big|\Big\langle n^G(\xi), \mathrm{i}\frac{\xi-z}{|\xi-z|^2}\Big\rangle\Big|\,\mathrm{d}\lambda_1(\xi) < \infty. \tag{43}$$

Proof. We shall first show that (39) is a consequence of (38) and (40). Note that

$$\int\limits_{\widehat{\partial G}\cap B(\eta,r)} q(\xi)\Big|\Big\langle n^G(\xi), \mathrm{i}\frac{\xi-\eta}{|\xi-\eta|^2}\Big\rangle\Big|\,\mathrm{d}\lambda_1(\xi)$$

$$\geqslant r^{-1} \int\limits_{\widehat{\partial G}\cap B(\eta,r)} q(\xi)\Big|\Big\langle n^G(\xi), \mathrm{i}\frac{\xi-\eta}{|\xi-\eta|}\Big\rangle\Big|\,\mathrm{d}\lambda_1(\xi)$$

for any $r > 0$. Using this together with (32) and (18) we get

$$r^{-1}\mu^q\big(B(\eta,r)\big) \leqslant r^{-1} \int\limits_{\widehat{\partial G}\cap B(\eta,r)} q(\xi)\Big[\Big|\Big\langle n^G(\xi), \mathrm{i}\frac{\xi-\eta}{|\xi-\eta|}\Big\rangle\Big|$$

$$+ \Big|\Big\langle n^G(\xi), \frac{\xi-\eta}{|\xi-\eta|}\Big\rangle\Big|\Big]\,\mathrm{d}\lambda_1(\xi)$$

$$\leqslant \int\limits_{\widehat{\partial G}} q(\xi)\Big|\Big\langle n^G(\xi), \mathrm{i}\frac{\xi-\eta}{|\xi-\eta|^2}\Big\rangle\Big|\,\mathrm{d}\lambda_1(\xi)$$

$$+ r^{-1} \int\limits_{\widehat{\partial G}\cap B(\eta,r)} q(\xi)\Big|\Big\langle n^G(\xi), \frac{\xi-\eta}{|\xi-\eta|}\Big\rangle\Big|\,\mathrm{d}\lambda_1(\xi).$$

We see that

$$k := \sup_{r>0} r^{-1}\mu^q\big(B(\eta,r)\big)$$

is majorized by the sum of the expressions occurring in (38) and (40). The assumption (42) guarantees the existence of constants a, $\varepsilon \in]0,\infty[$ such that

$$z \in S \cap B(\eta,\varepsilon) \Rightarrow \operatorname{dist}(z,\widehat{\partial G}) \geqslant a|z-\eta| \tag{44}$$

(cf. (0.1) in [3]); we may clearly suppose that $0 < a < 2$. According to formula (45) from [11] there is a constant $c \in]0,\infty[$ such that

$$z \in S \cap B(\eta,\varepsilon) \Rightarrow \sup_{r>0} r^{-1}\mu^q\big(B(z,r)\big) \leqslant ck. \tag{45}$$

Fix now $z \in S \cap B(\eta,\varepsilon)$ and put $F = \widehat{\partial G} \cap B(z,2|z-\eta|)$, $E = \widehat{\partial G} \setminus B(z,2|z-\eta|)$, so that

$$\int\limits_{\widehat{\partial G}} q(\xi)\Big|\Big\langle n^G(\xi),\mathrm{i}\frac{\xi-z}{|\xi-z|^2}\Big\rangle\Big|\,\mathrm{d}\lambda_1(\xi)$$
$$= \Big(\int\limits_F + \int\limits_E\Big) q(\xi)\Big|\Big\langle n^G(\xi),\mathrm{i}\frac{\xi-z}{|\xi-z|^2}\Big\rangle\Big|\,\mathrm{d}\lambda_1(\xi).$$

Clearly,

$$\int\limits_F q(\xi)\Big|\Big\langle n^G(\xi),\mathrm{i}\frac{\xi-z}{|\xi-z|^2}\Big\rangle\Big|\,\mathrm{d}\lambda_1(\xi) \leqslant \int\limits_F q(\xi)|\xi-z|^{-1}\,\mathrm{d}\lambda_1(\xi)$$
$$= \int\limits_0^\infty \mu^q\big(B(z,t^{-1})\cap F\big)\,\mathrm{d}t = \int\limits_0^{(2|z-\eta|)^{-1}} \mu^q(F)\,\mathrm{d}t + \int\limits_{(2|z-\eta|)^{-1}}^\infty \mu^q\big(B(z,t^{-1})\big)\,\mathrm{d}t$$
$$\leqslant \frac{\mu^q\big(B(z,2|z-\eta|)\big)}{2|z-\eta|} + \int\limits_0^{2|z-\eta|} r^{-2}\mu^q\big(B(z,r)\big)\,\mathrm{d}r.$$

In view of (44) we have $\mu^q\big(B(z,r)\big) = 0$ for $0 < r < a|z-\eta|$. Employing (45) we arrive at

$$\int\limits_F q(\xi)\Big|\Big\langle n^G(\xi),\mathrm{i}\frac{\xi-z}{|\xi-z|^2}\Big\rangle\Big|\,\mathrm{d}\lambda_1(\xi) \leqslant ck + ck\int\limits_{a|z-\eta|}^{2|z-\eta|} r^{-1}\,\mathrm{d}r = ck\Big(1+\ln\frac{2}{a}\Big).$$

We proceed to estimate the integral

$$\int_E q(\xi)\left|\left\langle n^G(\xi), \mathrm{i}\frac{\xi - z}{|\xi - z|^2}\right\rangle\right| \mathrm{d}\lambda_1(\xi)$$

$$\leqslant \int_E q(\xi)\left|\left\langle n^G(\xi), \mathrm{i}\frac{\xi - \eta}{|\xi - \eta|^2}\right\rangle\right| \mathrm{d}\lambda_1(\xi) + \int_E q(\xi)\left|\frac{\xi - z}{|\xi - z|^2} - \frac{\xi - \eta}{|\xi - \eta|^2}\right| \mathrm{d}\lambda_1(\xi)$$

$$\leqslant \int_{\widehat{\partial G}} q(\xi)\left|\left\langle n^G(\xi), \mathrm{i}\frac{\xi - \eta}{|\xi - \eta|^2}\right\rangle\right| \mathrm{d}\lambda_1(\xi) + I_1 + I_2,$$

where we have put

$$I_1 = \int_E q(\xi)|\xi - \eta| \cdot \left|\frac{1}{|\xi - z|^2} - \frac{1}{|\xi - \eta|^2}\right| \mathrm{d}\lambda_1(\xi),$$

$$I_2 = |z - \eta| \int_E q(\xi)|\xi - z|^{-2}\, \mathrm{d}\lambda_1(\xi).$$

$$I_1 \leqslant \int_E q(\xi)\frac{\big|\,|\xi - \eta|^2 - |\xi - z|^2\big|}{|\xi - z|^2 \cdot |\xi - \eta|}\, \mathrm{d}\lambda_1(\xi)$$

$$\leqslant |z - \eta| \int_E q(\xi)\left[\frac{1}{|\xi - z|^2} + \frac{1}{|\xi - z| \cdot |\xi - \eta|}\right] \mathrm{d}\lambda_1(\xi).$$

Observe that

$$\xi \in E \Rightarrow |\xi - \eta| > |\xi - z| - |z - \eta| \geqslant |\xi - z| - \frac{1}{2}|\xi - z| = \frac{1}{2}|\xi - z|,$$

whence

$$I_1 \leqslant |z - \eta| \int_E q(\xi)\frac{3}{|\xi - z|^2}\, \mathrm{d}\lambda_1(\xi),$$

$$I_1 + I_2 \leqslant 4I_2.$$

Note that, for any $t > 0$,

$$(\xi \in E, |\xi - z|^{-2} > t) \Rightarrow 2|z - \eta| < |\xi - z| < t^{-\frac{1}{2}} \Rightarrow t < (2|z - \eta|)^{-2}.$$

Hence

$$I_2 \leqslant |z - \eta| \int_0^{(2|z-\eta|)^{-2}} \mu^q\big(B(z, t^{-\frac{1}{2}})\big)\, \mathrm{d}t = 2|z - \eta| \int_{2|z-\eta|}^{\infty} r^{-3}\mu^q\big(B(z, r)\big)\, \mathrm{d}r$$

$$\leqslant 2|z - \eta| \cdot ck \cdot \int_{2|z-\eta|}^{\infty} r^{-2}\, \mathrm{d}r = ck.$$

Summarizing we have for $z \in S \cap B(\eta, \varepsilon)$

$$\int_{\widehat{\partial G}} q(\xi)\left|\left\langle n^G(\xi), \mathrm{i}\frac{\xi - z}{|\xi - z|^2}\right\rangle\right| \mathrm{d}\lambda_1(\xi) \leqslant \int_F \ldots + \int_E \ldots$$
$$\leqslant ck\left(1 + \ln\frac{2}{a}\right) + \int_{\widehat{\partial G}} q(\xi)\left|\left\langle n^G(\xi), \mathrm{i}\frac{\xi - \eta}{|\xi - \eta|^2}\right\rangle\right| \mathrm{d}\lambda_1(\xi) + 4ck$$

which completes the proof of (43). □

Lemma 6. *The functions* (8) *of the variable* $\theta \in \partial B(0,1)$ *defined by* (6) *and* (7) *(where* $r > 0$ *is fixed) are* λ_1*-measurable on* $\partial B(0,1)$ *and*

$$\int_{\partial B(0,1)} v^q(\theta, \eta)\, \mathrm{d}\lambda_1(\theta) = \int_{\widehat{\partial G}} q(\xi)\left|\left\langle n^G(\xi), \frac{\xi - \eta}{|\xi - \eta|^2}\right\rangle\right| \mathrm{d}\lambda_1(\xi), \tag{46}$$

$$\int_{\partial B(0,1)} \mathcal{V}_r^q(\theta, \eta)\, \mathrm{d}\lambda_1(\theta) = \int_{\widehat{\partial G} \cap B(\eta, r)} q(\xi)\left|\left\langle n^G(\xi), \frac{\xi - \eta}{|\xi - \eta|}\right\rangle\right| \mathrm{d}\lambda_1(\xi). \tag{47}$$

Proof. We shall use the following assertion:
Let $q\colon \mathbb{R}^2 \to [0, \infty[$ be lower semicontinuous and suppose that G has locally finite perimeter in $\mathbb{R}^2 \setminus \{\eta\}$. Define $v^q(\theta, \eta)$ by (6) for any $\theta \in \partial B(0,1)$. Then $\theta \mapsto v^q(\theta, \eta)$ is λ_1-measurable on $\partial B(0,1)$ and

$$\frac{1}{2\pi} \int_{\partial B(0,1)} v^q(\theta, \eta)\, \mathrm{d}\lambda_1(\theta) = \int_{\widehat{\partial G}} q(\xi) |\langle n^G(\xi), \operatorname{grad} h_\eta(\xi)\rangle|\, \mathrm{d}\lambda_1(\xi).$$

This assertion was proved in Lemma 3 in [11] (dealing with $\mathbb{R}^m$ for general $m \geqslant 2$) under the additional assumptions that q is bounded and strictly positive off $\{\eta\}$. An easy inspection of the proof reveals that these additional assumptions are superfluous. This gives us the formula (46) and permits us to replace q by the function Q defined by

$$Q(\xi) = \begin{cases} q(\xi)|\xi - \eta|, & \xi \in B(\eta, r), \\ 0, & \xi \in \mathbb{R}^2 \setminus B(\eta, r). \end{cases}$$

Applying (46) to Q instead of q we obtain that

$$\theta \mapsto v^Q(\theta, \eta) \equiv \mathcal{V}_r^q(\theta, \eta)$$

is λ_1-measurable on $\partial B(0,1)$ and (47) holds. □

Now we are in position to prove the rest of Theorem 2.

Proof of the assertion (P) in Theorem 2. It follows from Lemmas 3, 4, 5 that the existence of the finite limit (33) for each $f \in \mathcal{C}(\partial A, q)$ is equivalent with simultaneous validity of (38) and (40) which by Lemma 1 and Lemma 6 amounts the same as (11). □

Proof of the assertion $(\mathcal{K})$ in Theorem 2. It follows from (P) and (W) that the existence of the limit (13) for each $f \in \mathcal{C}(\partial A, q)$ is equivalent with simultaneous validity of (11) and (12) which, in view of the inequalities

$$\mathcal{U}^q_\infty(\eta) \geqslant \mathcal{U}^q_r(\eta) \geqslant r^{-1} u^q_r(\eta),$$
$$v^q(\eta) \geqslant r^{-1} \mathcal{V}^q_r(\eta) \quad (r > 0)$$

amounts the same as (14).

Thus the proof of Theorem 2 is complete. □

This theorem establishes the conjecture presented by J. Král in [6] (see 10.3, pp. 415–417 in Part 1) where, however, in the assertion concerning Wf the quantity $\mathcal{U}^Q_r$ should be replaced by u^Q_r as defined by (5).

Remark 2. The method of defining the Cauchy type integral with a smooth density on the boundary of a general domain $G \subset \mathbb{C}$ based on the transformation of the integral with help of the divergence theorem into the two-dimensional integral extended over G was employed in [1]; the same method was used in [2], [8] (cf. also [7], [15], [18]) for defining double layer potentials on boundaries of general domains $G \subset \mathbb{R}^m$. Additional references concerning angular limits of double layer potentials may be found in [11]. Related investigations of Cauchy's integrals on rectifiable curves appeared in [9], [12], [13], [14].

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