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SEVERAL RESULTS ON CHORDAL BIPARTITE GRAPHS

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Abstract. The question of generalizing results involving chordal graphs to similar concepts for chordal bipartite graphs is addressed. First, it is found that the removal of a bisimplicial edge from a chordal bipartite graph produces a chordal bipartite graph. As consequence, occurance of arithmetic zeros will not terminate perfect Gaussian elimination on sparse matrices having associated a chordal bipartite graph. Next, a property concerning minimal edge separators is presented. Finally, it is shown that, to any vertex of a chordal bipartite graph an edge may be added such that the chordality is maintained.

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1. INTRODUCTION

For terminology and results concerning graph theory we essentially follow the book [4]. An *undirected graph* is a pair $G = (V, E)$ is which V, the *vertex set*, is a finite set (usually $V = \{1, \dots, n\}$), and the *edge set* E is a symmetric binary relation on V. The *adjacency set* of a vertex v is denoted by $Adj(v)$, i.e. $w \in \mathrm{Adj}(v)$ if $vw \in E$. Given a subset $A \subseteq V$, define the *subgraph induced* by A by $G_A = (A, E_A)$, in which $E_A = \{xy \in E \mid x \in A \text{ and } y \in A\}$. The *complete graph* is the graph with the property that every pair of distinct vertices is adjacent. A subset $A \subseteq V$ is a *clique* if the induced graph on A is complete.

A special type of undirected graphs are the bipartite graphs. An undirected graph is called *bipartite* if $V = X + Y$ (the union of two disjoint sets X and Y) and any edge $ij \in E$ has one endpoint in X and the other one in Y. If $G = (X, Y, E)$ is a bipartite graph and $S \subset X + Y$ then S_X and S_Y will denote $S \cap X$ respectively $S \cap Y$.

A *path* $[v_1, \ldots, v_n]$ is a sequence of vertices such that $v_j v_{j+1} \in E$ for $j = 1, \ldots, k-1$. A cycle of length $k > 2$ is a path $[v_1, \ldots, v_k, v_1]$ is which $v_1, \ldots, v_k$ are distinct. A graph G is called *chordal* if every cycle of length greater than 3 possesses a chord, i.e. an edge joining two nonconsecutive vertices of the cycle. A bipartite graph is called *chordal bipartite* if every cycle of length greater than 4 has a chord.

An edge xy of a bipartite graph $G = (X, Y, E)$ is said to be *bisimplicial* if $\mathrm{Adj}(x) + \mathrm{Adj}(y)$ induces a complete bipartite graph. Let $\varphi = [x_{\varphi(1)} y_{\varphi(1)}, \ldots, x_{\varphi(k)} y_{\varphi(k)}]$ be a sequence of pairwise nonadjacent edges of a bipartite graph $G = (X, Y, E)$. Denote

$$S_j = \{x_{\varphi(1)}, \ldots, x_{\varphi(j)}\} \cup \{y_{\varphi(1)}, \ldots, y_{\varphi(j)}\} \tag{1.1}$$

and let $S_0 = \emptyset$. Then φ is said to be a *perfect edge elimination scheme* for G if each edge $x_{\varphi(j)} y_{\varphi(j)}$ is bisimplicial in G_{X+Y-S_j}, for $j = 1, \ldots, k-1$ and G_{X+Y-S_k} has no edges. Bipartite graphs with a perfect edge elimination scheme will be referred as *perfect elimination bipartite graphs.*

A pair of edges $x_a y_a$ and $x_b y_b$ of a bipartite graph $G = (X, Y, E)$ is *separable* if $x_a y_b$, $x_b y_a \notin E$. In this case, a set S of vertices is called an $x_a y_a$, $x_b y_b$ separator if the removal of S from the graph causes $x_a y_a$ and $x_b y_b$ to lie in distinct connected components of the remaining subgraph G_{X+Y-S}. S is called *minimal* if no proper subset of S is an $x_a y_a$, $x_b y_b$ separator. The graph G is said to be *separable* if it contains a pair of separable edges. Otherwise it is said to be *nonseparable.* Bipartite chordality can be characterized in terms of minimal edge separators ([5]).

Theorem 1.1. *A bipartite graph is chordal bipartite if and only if every minimal edge separator is a biclique.*

Let S be a minimal $x_a y_a$, $x_b y_b$ edge separator in the chordal bipartite graph $G = (X, Y, E)$ and let A be the connected component of G_{X+Y-S} containing $x_a y_a$. We prove in Section 3 that there exist $x \in X_A$ such that $S_Y \subset Ad_j(x)$. This generalizes a result mentioned in [4] for chordal graphs.

The next two theorems present important properties of chordal bipartite graphs.

Theorem 1.2. ([5]) *Let $G = (X, Y, E)$ be a chordal bipartite graph. If G is separable, then it has at least two separable bisimplicial edges.*

Theorem 1.3. ([5]) *Every chordal bipartite graph is a perfect elimination bipartite graph.*

Unfortunately, the converse of Theorem 1.3 is false.

Let $M = (m_{ij})_{i,j=1}^n$ be a matrix. The bipartite graph $G = (X, Y, E)$ is said to be the bipartite graph of the nonzero-pattern of M if $X = \{x_1, \ldots, x_n\}$,

$Y = \{y_1, \ldots, y_n\}$ and $m_{ij} \neq 0$ if and only if $(x_i, y_j) \in E$. Perfect elimination bipartite graphs play an important role in matrix theory in connection with the graph-theoretic description of Gaussian elimination on sparse matrices. Let $\varphi = [x_{\varphi(1)}y_{\varphi(1)}, \ldots, x_{\varphi(k)}y_{\varphi(k)}]$ be a perfect edge elimination scheme for the bipartite graph $G = (X, Y, E)$. If G is a bipartite graph of the nonzero-pattern of a matrix M, then M can be reduced by perfect Gaussian elimination ([4]). This means that choosing the entries on the positions $x_{\varphi(1)}y_{\varphi(1)}, \ldots, x_{\varphi(k)}y_{\varphi(k)}$ to act as pivots, M will be reduced to a matrix with at most one nonzero element on each row and column without ever changing (even temporarily) a zero entry to a nonzero. Based on a result in Section 2, we present a property of Gaussian elimination on sparse matrices having associated a chordal bipartite graph.

Given two chordal (not bipartite) graphs $G = (V, E)$ and $G' = (V, E')$ such that $E \subset E'$ and E' contains at least two more edges than E. It is known ([9] and [2]) that there is a chordal graph $G'' = (V, E'')$ such that $E \subset E'' \subset E'$. This property was independently proved in the case G' is complete and used as a key tool in the theory of positive definite completions of partial matrices in [6]. Motivated by a conjecture in [3] concerning minimal rank completions of partial matrices, we prove in Section 4 that it is possible to add an edge to any vertex of a chordal bipartite graph such that the resulting graph retains its chordal bipartite property. This latter problem was suggested to us by Professor H.J. Woerdeman.

2. Arithmetic zeros

We present in this section an algorithmic characterization of chordal bipartite graphs and its application to Gaussian perfect elimination. The following simple result plays a key role.

Proposition 2.1. *A bipartite graph $G = (X, Y, E)$ having a bisimplicial edge $xy \in E$ is chordal bipartite if and only if the graph $G' = (X, Y, E - \{xy\})$ is chordal bipartite.*

Proof. Let G be a chordal bipartite graph with a bisimplicial edge xy. Assume that the removal of this edge creates a chordless cycle in G' (note that this cycle must bo of length 6) which is of the form $\{x, y_1, x_1, y, x_2, y_2, x\}$. Then x_1, $x_2 \in \mathrm{Adj}(y)$ and y_1, $y_2 \in \mathrm{Adj}(x)$. Since xy is bisimplicial, we have x_1y_2, $x_2y_1 \in E$ which both are chords in the latter cycle, thus G' is chordal bipartite.

If the additions of xy to the chordal bipartite graph G' creates a chordless cycle of length $\geqslant 6$, this cycle must be of the form $\{x, y, x_1, y_1, \ldots, x_k, y_k, x\}$. Then $x_1 \in \mathrm{Adj}(y)$, $y_k \in \mathrm{Adj}(x)$ and xy bisimplicial imply that $x_1y_k \in E$, a contradiction. Thus G is chordal bipartite. □

We next examine the applications of the above proposition.

A bipartite graph $G = (X, Y, E)$ is said to be a *complete edge reduction* bipartite graph if the edges in E can be ordered in a sequence $\{x_1y_1, \ldots, x_my_m\}$, where $x_iy_i \neq x_jy_j$ for $i \neq j$ and each x_ky_k is bisimplicial in the bipartite graph induced by G on the set of edges $\{x_ky_k, \ldots, x_my_m\}$, for $k = 1, \ldots, m$.

The following is a consequence of Proposition 2.1.

Theorem 2.2. *A bipartite graph is chordal bipartite if and only if it is a complete edge reduction graph. Moreover, if a bipartite graph $G = (X, Y, E)$ is a complete edge reduction graph and xy is a bisimplicial edge in G, then $(X, Y, E - \{xy\})$ is a complete edge reduction graph also.*

The above theorem leads to simple algorithm that can be used to determine whether or not a bipartite graph is chordal or not. It consists in successively eliminating bisimplicial edges. The bipartite graph is chordal if and only all its edges can be eliminated in this way. Unfortunately this algorithm is not time efficient.

Another consequence of Proposition 2.1 involves coincidental zeros that may appear during Gaussian elimination.

An arithmetic zero in the Gaussian elimination process is a zero entry created as a result of numerical coincidence. Zeros in the original matrix are called generic.

As mentioned in Chapter XII of [4], a matrix A can be reduced by perfect Gaussian elimination to a matrix with at most one nonzero element on each row and column if and only if the bipartite graph associated with the nonzero-pattern of A has a perfect edge elimination scheme. A problem occurs, if an arithmetic zero appears during the process causing the disappearance of a bisimplicial edge in the associated bipartite graph, crucial for the perfect Gaussian elimination. For example, consider the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

The associated bipartite graph with A has the perfect edge elimination scheme $\varphi = [x_1y_2, x_2y_3, x_3y_4, x_4y_5]$. Any perfect Gaussian elimination must use as first pivot the (1,2) or the (2,2) entry. This will create at the next step a situation in which the unique entry corresponding to a bisimplicial edge is 0, so a perfect Gaussian elimination cannot be performed on A.

Let $\varphi = [x_{\varphi(1)}y_{\varphi(1)}, \ldots, x_{\varphi(k)}y_{\varphi(k)}]$ be a perfect edge elimination scheme for the chordal bipartite graph $G = (X, Y, E)$ and let A be matrix having G as associated bipartite graph. Consider the Gaussian elimination process of A subordinated to φ.

Assume that at step j, the entry $x_{\varphi(j)}y_{\varphi(j)}$ is 0 due to an arithmetic coincidence, so it cannot act as a pivot. The bipartite graph G_j obtained by deleting the edge $x_{\varphi(j)}y_{\varphi(j)}$ from G_{X+Y-S_j}, where S_j is given by (1.1) is chordal bipartite by Proposition 2.1. Thus G_j has another bisimplicial edge e. If in the partially reduced matrix the entry corresponding to e at this moment is 0, we can eliminate e and look for another bisimplicial edge. If we cannot find in this way a nonzero entry, the submatrix subordinated to G_{X+Y-S_j} is zero. In conclusion, it is alway possible to carry out perfect Gaussian elimination on matrices with a chordal bipartite associated graph, without worrying about arithmetic zeros.

3. Separable chordal bipartite graphs

In this section we present a property of minimal edge separators of chordal bipartite graphs. It is a bipartite generalization of a result mentioned as Ex.12, Ch.IV in [4].

Theorem 3.1. *Let $G = (X, Y, E)$ be a separable chordal bipartite graph and let S be a minimal $x_a y_a$, $x_b y_b$ separator. Then there exists a vertex x'_a in the connected component of G_{X+Y-S} containing $x_a y_a$ such that $S_Y \subset \mathrm{Adj}(x'_a)$.*

Proof. Let A and B be the connected components of G_{X+Y-S} containing $x_a y_a$, respectively $x_b y_b$. Assume the statement of the theorem is false. Select $x_{\max} \in X_A$ such that $\mathrm{card}\big(\mathrm{Adj}(x_{\max}) \cap Y_S\big) \geqslant \mathrm{card}\big(\mathrm{Adj}(x) \cap Y_S\big)$, for any $x \in X_A$.

Define the set

$$F = \big\{x \in X_A \mid \mathrm{Adj}(x) \cap Y_S \not\subset \mathrm{Adj}(x_{\max}) \cap Y_S\big\}$$

If $F = \emptyset$, by the minimality of S we have $\mathrm{Adj}(x_{\max}) = Y_s$ and the result follows.

Let $\{x'_a, y_1, x_1, \ldots, x_k = x_{\max}\}$ be a shortest length path through A connecting a vertex in F to $x_{\max}$. It is clear that $x_j \notin F$ for any $j = 1, \ldots, k-1$. Since $x'_a \in F$, we can choose $y_S \in \mathrm{Adj}(x_{\max}) \cap Y_S - \mathrm{Adj}(x'_a)$ and $y'_S \in \mathrm{Adj}(x'_a) \cap Y_S - \mathrm{Adj}(x_{\max})$. Since $x'_a y_S \notin E$, the shortest path in A from y_S to y'_S must have length at least four. Since S is minimal, there exists a minimal length path from y_S to y'_S through B. We obtain in this way a cycle $\{y_S, x'_1, \ldots, y'_S, x'_p, \ldots, y_S\}$ of length greater or equal to 6. This cycle is chordless because no vertex in A can be connected to a vertex in B. This completes the proof.

A proof based on induction on the number of vertices of G can also be provided.

□

Motivated by a conjecture in [3] concerning minimum rank completions of partial matrices, we prove in this section that it is possible to add an edge to any vertex of a chordal bipartite graph such that the resulting graph is also chordal bipartite. Similar results for chordal graphs have been successfully used in the theory of positive definite completions in [6] and in many other papers later.

It is known that any nonseparable bipartite graph is chordal bipartite ([4]). In [7] and [1] there is remarked that a bipartite graph $G = (X, Y, E)$ is nonseparable if and only if any matrix having G as associated bipartite graph is permutation equivalent to a block lower triangular matrix. As consequence of the latter result we mention the following.

Proposition 4.1. *Given any nonseparable bipartite graph $G = (X, Y, E)$ and $x \in X$ with $\operatorname{Adj}(x) \neq Y$, there exists $y \in Y$ such that $G' = \big(X, Y, E \cup \{xy\}\big)$ is also nonseparable. A similar statement holds for any $y \in Y$ with $\operatorname{Adj}(y) \neq X$.*

The above result plays a key role in the theory of contractive completion of partial matrices (see e.g. [7] and [1]).

Theorem 4.2. *Let $G = (X, Y, E)$ be a chordal bipartite graph and $x \in X$ with $\operatorname{Adj}(x) \neq Y$. Then there exists $y \in Y$ such that $G' = \big(X, Y, E \cup \{xy\}\big)$ is also chordal bipartite. A similar statement holds for any $y \in Y$ with $\operatorname{Adj}(y) \neq X$.*

Proof. We have already discussed the case when G is nonseparable in Proposition 4.1. We assume that G is separable and prove the theorem by induction on m, the number of edges of G. For $m \leqslant 4$ the result is immediate. Suppose the theorem is true for chordal bipartite graphs with less than m edges and let $G = (X, Y, E)$ be a separable chordal bipartite graph with m edges. By Theorem 1.2, G has the separable bisimplicial edges $x_a y_a$ and $x_b y_b$. By Proposition 2.1, the bipartite graph $G'' = \big(X, Y, E - \{x_a y_a\}\big)$ is chordal bipartite.

Let $x \in X$, $x \neq x_a$ with $\operatorname{Adj}(x) \neq Y$. By our assumption, there exists $y \in Y$ such that $\big(X, Y, (E \cup \{xy\}) - \{x_a y_a\}\big)$ is chordal bipartite.

We first assume that $y \neq y_a$ and prove that in this case $\big(X, Y, E \cup \{xy\}\big)$ is also chordal bipartite. Assuming that G' is not chordal bipartite, G' has a chordless cycle $\{x_0 = x, y_0 = y, x_1, \ldots, x_k = x_a, y_k = y_a, \ldots, y_p, x_{p+1} = x\}$ of length greater than 4. Since $x_a y_a$ is bisimplicial, $y_{k-1} x_{k+1} \in E$, a contradiction Thus G' is chordal bipartite.

We next consider the case when $y = y_a$ and assume that $G' = \big(X, Y, E \cup \{x y_a\}\big)$ is not chordal bipartite. Then there exists a chordless cycle in G', $\{x_0 = x, y_0 =$

$y_a, x_1 = x_a, y_1, \ldots, x_{p+1} = x\}$ of length greater than 4. It is clear that $xy_1 \notin E$. Let in this case G_a be the subgraph of G induced by $X + (Y - \{y_a\})$. By our assumption we can add the edge xy', $y' \neq y_a$ to G_a and obtain a chordal bipartite graph G''_a. Assume that $G'_a = (X, Y, E \cup \{xy'\})$ is also not chordal bipartite. Then there exists a cycle in G'_a, $\{x'_0 = x, y'_0 = y', x'_1, y'_1, \ldots, x'_k, y'_k = y_a, \ldots, x'_{p+1} = x\}$, $k > 1$, of length greater than 4. Then, since $x'_k y_a$, $x'_{k+1} y_a$ and $x_a y_a$ is bisimplicial in G we must have $x'_k y_1$, $x'_{k+1} y_a \in E$. Thus $\{x'_0 = x, y'_0 = y', x'_1, \ldots, x'_k, y_1, x'_{k+1}, \ldots, x'_{p+1} = x\}$ is a cycle in G''_a of length greater than 4. Since G''_a is bipartite chordal, we have $xy_1 \in E$, a contradiction. Thus either G' or G''_a is chordal bipartite and this solves the problem when $x \neq x_a$.

In the case $x = x_a$, we modify the previous proof by replacing x_a by x_b, y_a by y_b and x by x_a. Thus there exists $y \in Y - \mathrm{Adj}(x_a)$ such that $G' = (X, Y, E \cup \{x_a y\})$ is chordal bipartite. This completes the proof. □

We end by stating a possible generalization of Theorem 4.2. Let $G = (X, Y, E)$ and $G' = (X, Y, E')$ be chordal bipartite graphs such that $E \subset E'$. Then there exists an edge $e \in E' - E$ such that $G'' = (X, Y, E \cup \{e\})$ is also chordal bipartite. This latter result is known to be true for chordal graphs (see [9] and [2]) but it is unknown for chordal bipartite graphs.

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